Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 422 – 438
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doi:10.3906/mat-1809-3

On product complex Finsler manifolds

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Received: 02.09.2018 • **Accepted/Published Online:** 21.12.2018 • **Final Version:** 18.01.2019

Abstract: Let (M,F) be the product complex Finsler manifold of two strongly pseudoconvex complex Finsler manifolds (M_1,F_1) and (M_2,F_2) with $F=\sqrt{f(K,H)}$ and $K=F_1^2$, $H=F_2^2$. In this paper, we prove that (M,F) is a weakly Kähler–Finsler (resp. weakly complex Berwald) manifold if and only if (M_1,F_1) and (M_2,F_2) are both weakly Kähler–Finsler (resp. weakly complex Berwald) manifolds, which is independent of the choice of function f. Meanwhile, we prove that (M,F) is a complex Landsberg manifold if and only if either (M_1,F_1) and (M_2,F_2) are both complex Landsberg manifolds and $f=c_1K+c_2H$ with c_1,c_2 positive constants, or (M_1,F_1) and (M_2,F_2) are both Kähler–Finsler manifolds.

Key words: Product manifold, weakly complex Berwald manifold, weakly Kähler–Finsler manifold, complex Landsberg manifold

1. Introduction and main results

Complex Finsler geometry generalizes Hermitian geometry in the same sense that Banach spaces generalize Hilbert spaces. A complex Finsler manifold is a complex manifold endowed with a complex Finsler metric. As is well known, complex Finsler metrics have become a very useful tool in geometric function theory of holomorphic mappings [3]. The first fundamental examples of complex Finsler metrics are undoubtedly the Kobayashi metrics and Carathéodory metrics. These two classes of holomorphic invariant metrics coincide and are smooth strongly convex weakly Kähler–Finsler metrics with constant holomorphic curvature -4 in bounded strictly convex domains with smooth boundaries in $\mathbb{C}^n$ [1, 2, 12, 13]. Hence, the analysis on the manifold is intimately tied to the geometry [8]. However, in general, we do not have the explicit formulae for the Kobayashi and Carathéodory metrics. Since 1981, Lempert [12] and Abate and Patrizio [3–5] considered the problem of existence and uniqueness of complex geodesics, which involves three conditions: the weakly Kähler condition, the constancy of holomorphic curvature, and a symmetric property of the curvature operator. However, to our knowledge, there is a lack of concrete examples of (weakly) Kähler–Finsler metrics, as well as complex Finsler metrics of nonzero constant holomorphic curvature in the literature. Most results are obtained based on abstract complex Finsler manifolds. Therefore, we need more approaches to construct special complex Finsler metrics.

Recently, the first author and Zhong systematically investigated the unitary invariant complex Finsler metrics[16, 17, 22], general complex $(\boldsymbol{\alpha},\boldsymbol{\beta})$ metrics[18], modified complex Finlser metrics arising from unitary invariant metrics[20], and showed that there exist lots of strongly pseudoconvex (even strongly convex) complex

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2010 *AMS Mathematics Subject Classification:* 53C60, 53C40

Finsler metrics, weakly complex Berwald metrics, complex Berwald metrics, and complex Finsler metrics with vanishing holomorphic curvature. He and Zhong studied the doubly warped product of complex Finsler manifolds [11]. However, (weakly) Kähler–Finsler metrics are still inadequate. One may wonder whether there are some other ways to construct special complex Finsler metrics.

In [15], Wu and Zhong initiated a study on product complex Finsler manifolds, which are defined as follows. Let (M_1, F_1) and (M_2, F_2) be a pair of complex Finsler manifolds, and $f(s,t)$ be a 1-homogeneous function on s and t. Denote $K = F_1^2$, $H = F_2^2$, one can define the fundamental function on product manifold $M = M_1 \times M_2$ by

$$F = \sqrt{f(K,H)}. \tag{1.1}$$

Under some necessary conditions of f, one can ensure that the complex Finsler metric (1.1) is strongly pseudoconvex. It is worth mentioning that for a given product complex manifold $M = M_1 \times M_2$, the product complex Finsler metric given by (1.1) is different from the warped product complex Finsler metric considered by He and Zhong in [11], the latter is defined by

$$F = \sqrt{f_2^2 F_1^2 + f_1^2 F_2^2}, \tag{1.2}$$

where f_1 and f_2 are positive smooth functions on M_1 and M_2, respectively. These two classes of metrics (1.1) and (1.2) coincide only when f_1 and f_2 are constants, and f is linear in K and H.

Wu and Zhong [15] proved that if both (M_1, F_1) and (M_2, F_2) are Kähler–Finsler manifolds (resp. complex Berwald manifolds), then the product complex Finsler manifold (M, F) with F given by (1.1) is a Kähler–Finsler manifold (resp. complex Berwald manifold). Inspired by the work of Wu and Zhong, one may wonder whether the converse is correct or whether similar conclusions for weakly Kähler–Finsler manifolds (resp. weakly complex Berwald manifolds or complex Landsberg manifolds) still hold. This paper focuses on the invariant property of product method in constructing special complex Finsler manifolds. As our first main result, we prove the following.

Theorem 1.1 *Let (M_1, F_1) and (M_2, F_2) be two strongly pseudoconvex complex Finsler manifolds, and (M, F) be the product complex Finsler manifold of (M_1, F_1) and (M_2, F_2) with F given by (1.1). Then*

(i) (M, F) *is a Kähler–Finsler manifold if and only if* (M_1, F_1) *and* (M_2, F_2) *are both Kähler–Finsler manifolds;*

(ii) (M, F) *is a complex Berwald manifold if and only if* (M_1, F_1) *and* (M_2, F_2) *are both complex Berwald manifolds;*

(iii) (M, F) *is a weakly Kähler–Finsler manifold if and only if* (M_1, F_1) *and* (M_2, F_2) *are both weakly Kähler–Finsler manifolds;*

(iv) (M, F) *is a weakly complex Berwald manifold if and only if* (M_1, F_1) *and* (M_2, F_2) *are both weakly complex Berwald manifolds.*

Theorem 1.1 shows that the characterization of product complex Finsler manifold (M, F) to be a (weakly) Kähler–Finsler manifold or (weakly) Berwald manifold is independent of the choice of function f.

Note that Hermitian manifolds are trivial complex Berwald manifolds, Kähler manifolds are trivial Kähler–Finsler manifolds, thus the product method provides an effective approach to construct nontrivial complex Berwald manifolds (resp. Kähler–Finsler manifolds) via Hermitian manifolds (resp. Kähler manifolds).

As we know, every Kähler–Finsler manifold is a complex Landsberg manifold, one naturally wonders whether the converse is true or whether the characterization of product complex Finsler manifold (M,F) to be a complex Landsberg manifold is independent of the choice of function f? We obtain the following rigid result as our second main result.

Theorem 1.2 *Let (M_1,F_1) and (M_2,F_2) be two strongly pseudoconvex complex Finsler manifolds, and (M,F) be the product complex Finsler manifold of (M_1,F_1) and (M_2,F_2) with F given by (1.1). Then (M,F) is a complex Landsberg manifold if and only if either one of the following two conditions holds:*

(i) $f(K,H)=c_1K+c_2H$, *and (M_1,F_1) and (M_2,F_2) are both complex Landsberg manifolds, where c_1 and c_2 are positive constants;*

(ii) (M_1,F_1) *and* (M_2,F_2) *are both Kähler–Finsler manifolds.*

By Theorems 1.1 and 1.2, we easily obtain the following corollary.

Corollary 1.1 *Let (M_1,F_1) and (M_2,F_2) be two strongly pseudoconvex complex Finsler manifolds, and (M,F) be the product complex Finsler manifold of (M_1,F_1) and (M_2,F_2) with F given by (1.1) and f not linear in K and H. Then (M,F) is a complex Landsberg manifold if and only if it is a Kähler–Finsler manifold.*

The remainder of this paper is organized as follows. In Section 2, we introduce some necessary definitions and notions. In Section 3, simplified characterizations for the product complex Finsler metric F given by (1.1) to be strongly pseudoconvex and strongly convex are obtained, respectively. In Section 4, we deduce some different types of connection coefficients, which shall serve the proofs of our main results. In Sections 5 and 6, we present the proof of Theorems 1.1 and 1.2, respectively.

2. Preliminaries

We refer to [3] to recall some necessary notations and definitions. A complex Finsler manifold is a complex manifold endowed with a complex Finsler metric. Let (M_1,F_1) and (M_2,F_2) be two complex Finsler manifolds of complex dimensions m and n, respectively. Denote by $M=M_1\times M_2$ the product complex manifold of M_1 and M_2. Let $\{z^1,\cdots,z^m\}$ and $\{z^{m+1},\cdots,z^{m+n}\}$ be a set of local complex coordinates on M_1 and M_2, respectively, $\{z^1,\cdots,z^m,\ v^1,\cdots,v^m\}$ and $\{z^{m+1},\cdots,z^{m+n},\ v^{m+1},\cdots,v^{m+n}\}$ be the induced local complex coordinates on the holomorphic tangent bundles $T^{1,0}M_1$ and $T^{1,0}M_2$, respectively. Then $\{z^1,\cdots,z^{m+n}\}$ are the local complex coordinates on M, and $\{z^1,\cdots,z^{m+n},v^1,\cdots,v^{m+n}\}$ are the naturally induced local complex coordinates on the holomorphic tangent bundles $T^{1,0}M$ of M. Note that there is a natural isomorphism $T^{1,0}M\cong T^{1,0}M_1\oplus T^{1,0}M_2$.

Below, we denote $\tilde{M}$ as the complement of the zero section in $T^{1,0}M$. And the Einstein summation convention is assumed throughout this study. For functions defined on $\tilde{M}$, we denote by lower indices such as i and j the derivatives with respect to the z-coordinates or v-coordinates, and use a semicolon to distinguish derivatives with respect to z and v, for example,

$$G_i = \frac{\partial G}{\partial v^i}, \quad G_{;i} = \frac{\partial G}{\partial z^i}, \quad G_{i;\bar{j}} = \frac{\partial^2 G}{\partial v^i \partial \overline{z^j}}.$$

For simplicity, we set

$$\partial_i := \frac{\partial}{\partial z^i}, \quad \partial_{\bar{i}} := \frac{\partial}{\partial \overline{z^i}}, \quad \dot{\partial}_i := \frac{\partial}{\partial v^i}, \quad \dot{\partial}_{\bar{i}} := \frac{\partial}{\partial \overline{v^i}}.$$

Definition 2.1 ([3]) *A complex Finsler metric F on a complex manifold M is a continuous function $F: T^{1,0}M \to \mathbb{R}^+$ that satisfies:*

(i) $G = F^2$ is smooth on $\tilde{M}$;

(ii) $F(p,v) > 0$ for all $(p,v) \in \tilde{M}$;

(iii) $F(p,\zeta v) = |\zeta| F(p,v)$ for all $(p,v) \in T^{1,0}M$ and $\zeta \in \mathbb{C}$.

Definition 2.2 ([3]) *A complex Finsler metric F is called strongly pseudoconvex if the Levi matrix (or complex Hessian matrix)*

$$(G_{i\bar{j}}) = \left(\frac{\partial^2 G}{\partial v^i \partial \overline{v^j}} \right)$$

is positive definite on $\tilde{M}$.

Below, we denote $G^{\bar{j}k}$ such that $G^{\bar{j}k} G_{i\bar{j}} = \delta_i^k$. The key point to study complex Finsler geometry is to linearize the geometry of $\tilde{M}$ by introducing a complex nonlinear connection, which is characterized by its connection coefficients. There are two complex nonlinear connections associated to a given strongly pseudoconvex complex Finsler metric F: the Chern–Finsler nonlinear connection and the complex Berwald nonlinear connection. We denote $\mathbb{G}^i$ as the complex spray coefficients associated to F, where

$$\mathbb{G}^i = \frac{1}{2} \varGamma^i_{;j} v^j, \qquad \varGamma^i_{;j} = G^{\overline{k}i} G_{\overline{k};j}, \tag{2.1}$$

and $\varGamma^i_{;j}$ are called the Chern–Finsler nonlinear connection coefficients. $\mathbb{G}^i_j = \dot{\partial}_j \mathbb{G}^i$ are called the complex Berwald nonlinear connection coefficients. The corresponding horizontal frames $\{\delta_i\}$ and $\{\mathscr{X}_i\}$ associated to the two nonlinear connections are respectively given by

$$\delta_i = \partial_i - \varGamma^j_{;i} \dot{\partial}_j, \qquad \mathscr{X}_i = \partial_i - \mathbb{G}^j_i \dot{\partial}_j.$$

In general, $\delta_i \neq \mathscr{X}_i$; however, for Kähler Finsler metrics, we have[19]

$$\delta_i = \mathscr{X}_i. \tag{2.2}$$

We mention here that the horizontal radial vector fields $\chi = v^i \delta_i$ and $\mathscr{X} = v^i \mathscr{X}_i$ associated to the two nonlinear connections coincide[23], i.e.

$$\chi = \mathscr{X}. \tag{2.3}$$

There are several complex Finsler connections associated to a given complex Finsler manifold (M, F). The most often used are the Chern–Finsler connection [3] and the complex Berwald connection [14]. These connections are appropriate for considering different problems in complex Finsler geometry. The horizontal Chern–Finsler connection coefficients $\varGamma^i_{j;k}$ and complex Berwald connection coefficients $\mathbb{G}^i_{jk}$ are respectively related to $\varGamma^i_{;k}$ and $\mathbb{G}^i_k$ by

$$\varGamma^i_{j;k} = \dot{\partial}_j \varGamma^i_{;k}, \quad \mathbb{G}^i_{jk} = \dot{\partial}_j \mathbb{G}^i_k, \quad \varGamma^i_{j;k} v^j = \varGamma^i_{;k}, \quad \mathbb{G}^i_{jk} v^j = \mathbb{G}^i_k. \tag{2.4}$$

In general, $\varGamma^i_{j;k}$ and $\mathbb{G}^i_{jk}$ are locally functions of (z, v) and they are 0-homogeneous with respect to the fiber coordinate v. It is clear that $\mathbb{G}^i_{jk} = \mathbb{G}^i_{kj}$. In general, however, $\varGamma^i_{j;k} \neq \varGamma^i_{k;j}$.

Definition 2.3 ([3, 9]) *Let F be a strongly pseudoconvex complex Finsler metric on a complex manifold M. F is called a Kähler–Finsler metric if in local coordinates, $\varGamma^i_{j;k} - \varGamma^i_{k;j} = 0$; called a weakly Kähler Finsler metric if $G_i(\varGamma^i_{j;k} - \varGamma^i_{k;j})v^j = 0$.*

For the Chern–Finsler nonlinear connection, the following formulas hold [3]:

$$\delta_i(G) = \delta_{\bar{i}}(G) = \delta_i(G_{\bar{j}}) = 0. \tag{2.5}$$

For the complex Berwald nonlinear connection, we have

Proposition 2.1 ([23]) *Let F be a strongly pseudoconvex complex Finsler metric on a complex manifold. Then*

(i) F is a weakly Kähler–Finsler metric if and only if $\mathscr{X}_i(G) = 0$;

(ii) F is a Kähler–Finsler metric if and only if $\mathscr{X}_i(G_{\bar{j}}) = 0$.

Definition 2.4 ([6, 23]) *Let F be a strongly pseudoconvex complex Finsler metric on a complex manifold M. F is called a complex Berwald metric if locally $\varGamma^i_{j;k}$ depend only on the base manifold coordinates z; F is called a weakly complex Berwald metric if locally $\mathbb{G}^i_{jk}$ depend only on z.*

A complex Berwald metric is a weakly complex Berwald metric; the converse, however, is not true[23].

Definition 2.5 ([7]) *Let F be a strongly pseudoconvex complex Finsler metric on a complex manifold M. F is called a complex Landsberg metric if the complex Berwald connection coefficients $\mathbb{G}^i_{jk}$ coincide with the horizontal connection coefficients $\mathbb{L}^i_{jk}$ of the Rund type complex linear connection in the sense of Munteanu[14], i.e.*

$$\mathbb{G}^i_{jk} = \mathbb{L}^i_{jk},$$

where

$$\mathbb{L}^i_{jk} = \frac{1}{2} G^{\bar{l}i} \left[\mathscr{X}_j(G_{k\bar{l}}) + \mathscr{X}_k(G_{j\bar{l}}) \right].$$

By the above definition and formula (2.2), it is easy to check that every Kähler–Finsler metric is necessarily a complex Landsberg metric.

3. Characterizations of strong pseudoconvexity and strong convexity

Now we introduce the definition of product complex Finsler manifold[15]. Let (M_1, F_1) and (M_2, F_2) be two complex Finsler manifolds of complex dimensions m and n, respectively, and $M = M_1 \times M_2$ be the product complex manifold of M_1 and M_2. Below and throughout this paper, we assume the following ranges of indices:

$$1 \leq a, b, c, d \leq m, \quad m+1 \leq \alpha, \beta, \gamma, \sigma \leq m+n, \quad 1 \leq i, j, k, l \leq m+n.$$

Denote $K = F_1^2$, $H = F_2^2$, if one defines $F : T^{1,0}M \to [0, +\infty)$ by

$$F(z, v) = \sqrt{f\left(K(z^a, v^a), H(z^\alpha, v^\alpha)\right)}, \tag{3.1}$$

where $(z, v) \in T^{1,0}M$, $(z^a, v^a) \in T^{1,0}M_1$, $(z^\alpha, v^\alpha) \in T^{1,0}M_2$ with $z = (z^a, z^\alpha)$, $v = (v^a, v^\alpha)$, and $f : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ be a continuous function satisfying

(a) $f(s, t) = 0$ if and only if $(s, t) = (0, 0)$;

(b) $f(\lambda s, \lambda t) = \lambda f(s, t)$ for $\lambda \in [0, +\infty)$;

(c) f is smooth on $(0, +\infty) \times (0, +\infty)$;

(d) $\frac{\partial f}{\partial s} \neq 0$, $\frac{\partial f}{\partial t} \neq 0$, $\frac{\partial f}{\partial s}\frac{\partial f}{\partial t} - f\frac{\partial^2 f}{\partial s \partial t} \neq 0$ for $(s, t) \in (0, +\infty) \times (0, +\infty)$.

It is easy to see that F given by (3.1) is a complex Finsler metric on the product manifold M. We call (M, F) a product complex Finsler manifold.

By the 1-homogeneity of f, it is easy to see that

$$f_K K + f_H H = f, \tag{3.2}$$

$$f_{KK}K + f_{KH}H = f_{HK}K + f_{HH}H = 0, \quad f_{KH}^2 = f_{KK}f_{HH}, \tag{3.3}$$

$$f_{KKK}K + f_{KKH}H = -f_{KK},\ f_{KHK}K + f_{KHH}H = -f_{KH},\ f_{HHK}K + f_{HHH}H = -f_{HH} \tag{3.4}$$

for $K \neq 0$ and $H \neq 0$. Here, we denote $f_K = \frac{\partial f}{\partial K}$, $f_{KH} = \frac{\partial^2 f}{\partial K \partial H}$ and so on.

In [15], Wu and Zhong showed that F is strongly pseudoconvex on $\tilde{M}_1 \times \tilde{M}_2$ if and only if

$$f_K > 0, \quad f_H > 0, \quad f_K + Kf_{KK} > 0, \quad f_H + Hf_{HH} > 0, \quad f_K f_H - f f_{KH} > 0.$$

We mention here that the above characterization of strong pseudoconvexity of F can be simplified. And we have the following proposition.

Proposition 3.1 *Let (M_1, F_1) and (M_2, F_2) be two strongly pseudoconvex complex Finsler manifolds, and (M, F) be the corresponding product complex Finsler manifold with F defined by (3.1). Then F is strongly pseudoconvex on $\tilde{M}_1 \times \tilde{M}_2 \subset \tilde{M}$ if and only if*

$$f_K > 0, \quad f_H > 0, \quad \Delta := f_K f_H - f f_{KH} > 0. \tag{3.5}$$

Proof We only need to show that (3.5) implies

$$f_K + K f_{KK} > 0, \quad f_H + H f_{HH} > 0.$$

In fact, if (3.5) holds, by the third inequality in (3.5) and using (3.2) and (3.3), we obtain

$$\begin{aligned} 0 &< f_K f_H H - f f_{KH} H \\ &= f_K(f - f_K K) - f(-f_{KK} K) \\ &= f(f_K + K f_{KK}) - f_K^2 K, \end{aligned}$$

which yields that $f_K + K f_{KK} > 0$. Similarly, we obtain $f_H + H f_{HH} > 0$. This completes the proof. □

In the rest of this paper, complex Finsler manifolds that emerged are all assumed to be strongly pseudoconvex. Roughly speaking, a strongly pseudoconvex complex Finsler metric F on a complex manifold M is called strongly convex if F is also a real Finsler metric. We refer to [3, 19, 21] for more details.

Proposition 3.2 *Let (M_1, F_1) and (M_2, F_2) be two strongly convex complex Finsler manifolds, and (M, F) be the corresponding product complex Finsler manifold with F defined by (3.1). Then F is strongly convex on $\tilde{M}_1 \times \tilde{M}_2 \subset \tilde{M}$ if and only if*

$$f_K > 0, \quad f_H > 0, \quad f_K f_H - 2 f f_{KH} > 0. \tag{3.6}$$

Proof Since F_1 and F_2 are strongly convex complex Finsler metrics, they are naturally real Finsler metrics, by equalities (1.20) and (1.21) in [9, page 14], F is strongly convex if and only if

$$f_K > 0, \quad f_H > 0, \quad f_K + 2K f_{KK} > 0, \quad f_H + 2H f_{HH} > 0, \quad f_K f_H - 2 f f_{KH} > 0. \tag{3.7}$$

A similar argument to that in the proof of Proposition 3.1 yields that (3.7) is equivalent to (3.6). This completes the proof. □

Since Hermitian metrics are naturally strongly convex complex Finsler metrics, Proposition 3.2 provides an effective approach to construct strongly convex complex Finsler metrics via Hermitian metrics.

Example 3.1 *(Szabó metric [10, 19]) Suppose that $(M, \boldsymbol{\alpha})$ and $(N, \boldsymbol{\beta})$ are two Hermitian manifolds, and F_ε is a product complex Finsler metric on the product manifold $M \times N$ with F_ε defined as follows:*

$$F_\varepsilon = \sqrt{\boldsymbol{\alpha}^2 + \boldsymbol{\beta}^2 + \varepsilon(\boldsymbol{\alpha}^{2p} + \boldsymbol{\beta}^{2p})^{\frac{1}{p}}},$$

where real numbers $\varepsilon > 0$ and $p > 0$. Then

(i) F_ε is a strongly pseudoconvex complex Finsler metric on $M \times N$ for every $p \in (0, +\infty)$;

(ii) F_ε is a strongly convex complex Finsler metric on $M \times N$ for every $p \in \left[\frac{1}{2}, +\infty\right)$.

4. Connection coefficients of product complex Finsler manifolds

At the beginning of this section, we list some basic results in [15]. Let (M_1, F_1) and (M_2, F_2) be two complex Finsler manifolds, and (M, F) be the corresponding product complex Finsler manifold as defined in Section 3. We denote $\mathbf{K} = (K_{a\bar{b}})$ and $\mathbf{H} = (H_{\alpha\bar{\beta}})$ the complex Hessian matrices of K and H, respectively, and denote their inverse matrices by $\mathbf{K}^{-1} = (K^{\bar{b}a})$ and $\mathbf{H}^{-1} = (H^{\bar{\beta}\alpha})$, respectively.

Proposition 4.1 ([15]) *The fundamental tensor matrix* $\mathbf{G}$ *and its inverse* $\mathbf{G}^{-1}$ *associated to the product complex Finsler metric* F *are given respectively by*

$$\mathbf{G} = \begin{pmatrix} G_{a\bar{b}} & G_{a\overline{\beta}} \\ G_{\alpha\bar{b}} & G_{\alpha\overline{\beta}} \end{pmatrix}, \quad \mathbf{G}^{-1} = \begin{pmatrix} G^{\bar{b}a} & G^{\bar{b}\alpha} \\ G^{\overline{\beta}a} & G^{\overline{\beta}\alpha} \end{pmatrix},$$

where

$$\begin{aligned} G_{a\bar{b}} &= f_K K_{a\bar{b}} + f_{KK} K_a K_{\bar{b}}, \\ G_{a\overline{\beta}} &= f_{KH} K_a H_{\overline{\beta}}, \\ G_{\alpha\bar{b}} &= f_{KH} H_\alpha K_{\bar{b}}, \\ G_{\alpha\overline{\beta}} &= f_H H_{\alpha\overline{\beta}} + f_{HH} H_\alpha H_{\overline{\beta}}, \\ G^{\bar{b}a} &= \frac{1}{f_K}\left[K^{\bar{b}a} - \frac{f_H f_{KK}}{\Delta} v^a \overline{v^b}\right], \\ G^{\bar{b}\alpha} &= -\frac{1}{\Delta} f_{KH} v^\alpha \overline{v^b}, \\ G^{\overline{\beta}a} &= -\frac{1}{\Delta} f_{KH} v^a \overline{v^\beta}, \\ G^{\overline{\beta}\alpha} &= \frac{1}{f_H}\left[H^{\overline{\beta}\alpha} - \frac{f_K f_{HH}}{\Delta} v^\alpha \overline{v^\beta}\right]. \end{aligned}$$

Here, Δ *is given by* (3.5).

From now on, we use the symbol " $\sim$ " to mark the geometric objects associated to the complex Finsler metric F_1 or F_2, depending on the different ranges of indices, for instance, $\tilde{\varGamma}^a_{;c}$ and $\tilde{\varGamma}^\alpha_{;\gamma}$ denote the Chern–Finsler nonlinear connection coefficients associated to F_1 and F_2, respectively.

Proposition 4.2 ([15]) *The Chern–Finsler nonlinear connection coefficients* $\varGamma^i_{;k}$ *associated to the product complex Finsler metric* F *are*

$$\varGamma^a_{;c} = \tilde{\varGamma}^a_{;c}, \quad \varGamma^\alpha_{;\gamma} = \tilde{\varGamma}^\alpha_{;\gamma}, \quad \varGamma^a_{;\gamma} = \varGamma^\alpha_{;c} = 0. \tag{4.1}$$

By Proposition 4.2 and the first equality in (2.4), one can easily get the following proposition.

Proposition 4.3 ([15]) *The horizontal Chern–Finsler connection coefficients* $\varGamma^i_{j;k}$ *associated to the product complex Finsler metric* F *are*

$$\varGamma^a_{b;c} = \tilde{\varGamma}^a_{b;c}, \quad \varGamma^\alpha_{\beta;\gamma} = \tilde{\varGamma}^\alpha_{\beta;\gamma}, \quad \varGamma^a_{b;\gamma} = \varGamma^a_{\beta;c} = \varGamma^a_{\beta;\gamma} = \varGamma^\alpha_{b;c} = \varGamma^\alpha_{b;\gamma} = \varGamma^\alpha_{\beta;c} = 0.$$

Proposition 4.4 *(i) The complex spray coefficients* $\mathbb{G}^i$ *associated to the product complex Finsler metric* F *are*

$$\mathbb{G}^a = \tilde{\mathbb{G}}^a, \quad \mathbb{G}^\alpha = \tilde{\mathbb{G}}^\alpha;$$

(ii) the complex Berwald nonlinear connection coefficients $\mathbb{G}^i_k$ *associated to the product complex Finsler metric* F *are*

$$\mathbb{G}^a_c = \tilde{\mathbb{G}}^a_c, \quad \mathbb{G}^\alpha_\gamma = \tilde{\mathbb{G}}^\alpha_\gamma, \quad \mathbb{G}^a_\gamma = \mathbb{G}^\alpha_c = 0;$$

(iii) the horizontal frame $\{\mathscr{X}_i\}$ *associated to the product complex Finsler metric* F *are*

$$\mathscr{X}_a = \tilde{\mathscr{X}}_a, \quad \mathscr{X}_\alpha = \tilde{\mathscr{X}}_\alpha;$$

(iv) the complex Berwald connection coefficients $\mathbb{G}^i_{jk}$ *associated to the product complex Finsler metric* F *are*

$$\mathbb{G}^a_{bc} = \tilde{\mathbb{G}}^a_{bc}, \quad \mathbb{G}^\alpha_{\beta\gamma} = \tilde{\mathbb{G}}^\alpha_{\beta\gamma}, \quad \mathbb{G}^a_{b\gamma} = \mathbb{G}^a_{\beta\gamma} = \mathbb{G}^\alpha_{bc} = \mathbb{G}^\alpha_{b\gamma} = 0.$$

Proof

(i) It follows from (2.1) and (4.1) that

$$\mathbb{G}^a = \frac{1}{2}\varGamma^a_{;j}v^j = \frac{1}{2}\left[\varGamma^a_{;b}v^b + \varGamma^a_{;\beta}v^\beta\right] = \frac{1}{2}\varGamma^a_{;b}v^b = \frac{1}{2}\tilde{\varGamma}^a_{;b}v^b = \tilde{\mathbb{G}}^a,$$
$$\mathbb{G}^\alpha = \frac{1}{2}\varGamma^\alpha_{;j}v^j = \frac{1}{2}\left[\varGamma^\alpha_{;b}v^b + \varGamma^\alpha_{;\beta}v^\beta\right] = \frac{1}{2}\varGamma^\alpha_{;\beta}v^\beta = \frac{1}{2}\tilde{\varGamma}^\alpha_{;\beta}v^\beta = \tilde{\mathbb{G}}^\alpha.$$

(ii) It follows form (i) that

$$\mathbb{G}^a_c = \dot{\partial}_c\mathbb{G}^a = \dot{\partial}_c\tilde{\mathbb{G}}^a = \tilde{\mathbb{G}}^a_c, \quad \mathbb{G}^a_\gamma = \dot{\partial}_\gamma\mathbb{G}^a = \dot{\partial}_\gamma\tilde{\mathbb{G}}^a = 0.$$

Similarly, we obtain the rest of the equalities of (ii).

(iii) It follows form (ii) that

$$\mathscr{X}_a = \partial_a - \mathbb{G}^b_a\dot{\partial}_b - \mathbb{G}^\beta_a\dot{\partial}_\beta = \partial_a - \mathbb{G}^b_a\dot{\partial}_b = \partial_a - \tilde{\mathbb{G}}^b_a\dot{\partial}_b = \tilde{\mathscr{X}}_a,$$
$$\mathscr{X}_\alpha = \partial_\alpha - \mathbb{G}^b_\alpha\dot{\partial}_b - \mathbb{G}^\beta_\alpha\dot{\partial}_\beta = \partial_\alpha - \mathbb{G}^\beta_\alpha\dot{\partial}_\beta = \partial_\alpha - \tilde{\mathbb{G}}^\beta_\alpha\dot{\partial}_\beta = \tilde{\mathscr{X}}_\alpha.$$

(iv) It follows by a similar argument to that in the proof of (ii).

□

Remark 4.1 *The three propositions above indicate that the six geometric objects* $\mathbb{G}^i$, $\mathscr{X}_i$, $\varGamma^i_{;k}$, $\mathbb{G}^i_k$, $\varGamma^i_{j;k}$, *and* $\mathbb{G}^i_{jk}$ *associated to the product complex Finsler metric* F *are all independent of the choice of function* f, *and coincide with those of* F_1 *and* F_2. *Hence, in the following computation, we need not to note whether the mentioned six geometric objects are marked with the symbol "* $\sim$ *".*

Below, we shall derive the horizontal connection coefficients $\mathbb{L}^i_{jk}$ of the Rund-type complex linear connection associated to the product complex Finsler metric F. Note that $\mathbb{L}^i_{jk}$ is symmetric with respect to the lower indices j and k, we actually only need to compute six tensors: $\mathbb{L}^a_{bc}$, $\mathbb{L}^a_{b\gamma}$, $\mathbb{L}^a_{\beta\gamma}$, $\mathbb{L}^\alpha_{bc}$, $\mathbb{L}^\alpha_{b\gamma}$, and $\mathbb{L}^\alpha_{\beta\gamma}$. For simplicity, the derivatives with respect to the horizontal frame $\{\mathscr{X}_a, \mathscr{X}_\alpha\}$ shall be denoted by lower indices behind the symbol " | ", for example

$$K_{\bar{d}|c} = \mathscr{X}_a(K_{\bar{d}}), \quad H_{|\alpha} = \mathscr{X}_\alpha(H).$$

By Proposition 4.1 and a series of contractions, we obtain the following lemma.

Lemma 4.1

$$G^{\overline{d}a}K_{b\overline{d}} = \frac{1}{f_K}\left[\delta_b^a - \frac{1}{\Delta}f_H f_{KK} v^a K_b\right], \tag{4.2}$$

$$G^{\overline{d}a}K_{\overline{d}} = \frac{1}{\Delta}(f_H - f_{KH}K)v^a = \frac{1}{\Delta}(f_H + f_{HH}H)v^a, \tag{4.3}$$

$$G^{\overline{\gamma}a}H_{\overline{\gamma}} = -\frac{1}{\Delta}f_{KH}Hv^a, \tag{4.4}$$

$$G^{\overline{\gamma}a}K_a = -\frac{1}{\Delta}f_{KH}K\overline{v^\gamma} = \frac{1}{\Delta}f_{HH}H\overline{v^\gamma}. \tag{4.5}$$

Proposition 4.5 *The horizontal connection coefficients* $\mathbb{L}^i_{jk}$ *of the Rund-type connection associated to the product complex Finsler metric* F *are*

$$\begin{aligned}\mathbb{L}^a_{bc} =&\tilde{\mathbb{L}}^a_{bc} + \frac{1}{2}f_{KK}G^{\overline{d}a}(K_{\overline{d}|b}K_c + K_{\overline{d}|c}K_b) + \frac{f_{KK}}{2f_K}(\delta^a_c K_{|b} + \delta^a_b K_{|c})\\ &+ \frac{1}{2\Delta}\left[f_H f_{KKK} + f_{KH}f_{KK} - \frac{1}{f_K}f_H f^2_{KK}\right]v^a(K_bK_{|c} + K_cK_{|b}),\end{aligned} \tag{4.6}$$

$$\begin{aligned}\mathbb{L}^a_{b\gamma} =&\frac{1}{2}G^{\overline{d}a}\left[(f_{KH}K_{\overline{d}|b} + f_{KKH}K_{\overline{d}}K_{|b})H_\gamma + (f_{KH}K_{b\overline{d}} + f_{KKH}K_bK_{\overline{d}})H_{|\gamma}\right]\\ &- \frac{1}{2\Delta}f_{KH}(f_{KH} + f_{KHH}H)v^a(K_bH_{|\gamma} + K_{|b}H_\gamma),\end{aligned} \tag{4.7}$$

$$\mathbb{L}^a_{\beta\gamma} =\frac{1}{2\Delta}f^2_H\big[\ln(f_H)\big]_{KH}v^a\left(H_\beta H_{|\gamma} + H_\gamma H_{|\beta}\right), \tag{4.8}$$

$$\mathbb{L}^\alpha_{bc} =\frac{1}{2\Delta}f^2_K\big[\ln(f_K)\big]_{KH}v^\alpha\left(K_bK_{|c} + K_cK_{|b}\right), \tag{4.9}$$

$$\begin{aligned}\mathbb{L}^\alpha_{b\gamma} =&\frac{1}{2}G^{\overline{\sigma}\alpha}\left[(f_{KH}H_{\overline{\sigma}|\gamma} + f_{KHH}H_{\overline{\sigma}}H_{|\gamma})K_b + (f_{KH}H_{\gamma\overline{\sigma}} + f_{KHH}H_\gamma H_{\overline{\sigma}})K_{|b}\right]\\ &- \frac{1}{2\Delta}f_{KH}(f_{KH} + f_{KKH}K)v^\alpha(K_bH_{|\gamma} + K_{|b}H_\gamma),\end{aligned} \tag{4.10}$$

$$\begin{aligned}\mathbb{L}^\alpha_{\beta\gamma} =&\tilde{\mathbb{L}}^\alpha_{\beta\gamma} + \frac{1}{2}f_{HH}G^{\overline{\sigma}\alpha}(H_{\overline{\sigma}|\beta}H_\gamma + H_{\overline{\sigma}|\gamma}H_\beta) + \frac{f_{HH}}{2f_H}(\delta^\alpha_\gamma H_{|\beta} + \delta^\alpha_\beta H_{|\gamma})\\ &+ \frac{1}{2\Delta}\left[f_K f_{HHH} + f_{KH}f_{HH} - \frac{1}{f_H}f_K f^2_{HH}\right]v^\alpha(H_\beta H_{|\gamma} + H_\gamma H_{|\beta}).\end{aligned} \tag{4.11}$$

Proof Firstly, we calculate $\mathbb{L}^a_{bc}$. According to the definition of $\mathbb{L}^i_{jk}$, we have

$$\begin{aligned}\mathbb{L}^a_{bc} &= \frac{1}{2}G^{\overline{l}a}\left[\mathscr{X}_b(G_{c\overline{l}}) + \mathscr{X}_c(G_{b\overline{l}})\right]\\ &= \frac{1}{2}G^{\overline{d}a}\left[\mathscr{X}_b(G_{c\overline{d}}) + \mathscr{X}_c(G_{b\overline{d}})\right] + \frac{1}{2}G^{\overline{\gamma}a}\left[\mathscr{X}_b(G_{c\overline{\gamma}}) + \mathscr{X}_c(G_{b\overline{\gamma}})\right].\end{aligned} \tag{4.12}$$

By Proposition 4.1, we have

$$G_{c\overline{d}} = f_K K_{c\overline{d}} + f_{KK}K_cK_{\overline{d}}, \quad G^{\overline{d}a} = \frac{1}{f_K}\left[K^{\overline{d}a} - \frac{f_H f_{KK}}{\Delta}v^a\overline{v^d}\right].$$

Furthermore

$$\mathscr{X}_b(G_{c\bar{d}}) = f_K \mathscr{X}_b(K_{c\bar{d}}) + f_{KK}K_{c\bar{d}}K_{|b} + \left(f_{KK}K_{c|b} + f_{KKK}K_cK_{|b}\right)K_{\bar{d}} + f_{KK}K_cK_{\bar{d}|b},$$

which together with (4.2) and (4.3) yields

$$\begin{aligned}
G^{\bar{d}a}\mathscr{X}_b(G_{c\bar{d}}) =& K^{\bar{d}a}\mathscr{X}_b(K_{c\bar{d}}) - \frac{f_Hf_{KK}}{\Delta}v^a\overline{v^d}\mathscr{X}_b(K_{c\bar{d}}) + f_{KK}G^{\bar{d}a}K_{c\bar{d}}K_{|b} \\
&+ G^{\bar{d}a}K_{\bar{d}}\left(f_{KK}K_{c|b} + f_{KKK}K_cK_{|b}\right) + f_{KK}G^{\bar{d}a}K_{\bar{d}|b}K_c \\
=& K^{\bar{d}a}\mathscr{X}_b(K_{c\bar{d}}) + \left(G^{\bar{d}a}K_{\bar{d}} - \frac{1}{\Delta}f_Hv^a\right)f_{KK}K_{c|b} \\
&+ \left(f_{KK}G^{\bar{d}a}K_{c\bar{d}} + G^{\bar{d}a}K_{\bar{d}}f_{KKK}K_c\right)K_{|b} + f_{KK}G^{\bar{d}a}K_{\bar{d}|b}K_c \\
=& K^{\bar{d}a}\mathscr{X}_b(K_{c\bar{d}}) + f_{KK}G^{\bar{d}a}K_{\bar{d}|b}K_c + \frac{1}{\Delta}Hf_{HH}f_{KK}v^aK_{c|b} \\
&+ \frac{f_{KK}}{f_K}\delta_c^aK_{|b} + \frac{1}{\Delta}\left[(f_H - Kf_{KH})f_{KKK} - \frac{1}{f_K}f_Hf_{KK}^2\right]v^aK_cK_{|b}.
\end{aligned} \tag{4.13}$$

Again by Proposition 4.1 and using (4.4), we get

$$\begin{aligned}
G^{\bar{\gamma}a}\mathscr{X}_b(G_{c\bar{\gamma}}) =& G^{\bar{\gamma}a}\mathscr{X}_b(f_{KH}K_cH_{\bar{\gamma}}) = G^{\bar{\gamma}a}H_{\bar{\gamma}}\mathscr{X}_b(f_{KH}K_c) \\
=& -\frac{1}{\Delta}f_{KH}Hv^a(f_{KKH}K_cK_{|b} + f_{KH}K_{c|b}) \\
=& -\frac{1}{\Delta}f_{KKH}f_{KH}Hv^aK_cK_{|b} - \frac{1}{\Delta}Hf_{KH}^2v^aK_{c|b}.
\end{aligned} \tag{4.14}$$

(4.13)+(4.14), and using the third equality in (3.3) and first equality in (3.4) implies

$$\begin{aligned}
&G^{\bar{d}a}\mathscr{X}_b(G_{c\bar{d}}) + G^{\bar{\gamma}a}\mathscr{X}_b(G_{c\bar{\gamma}}) \\
&= K^{\bar{d}a}\mathscr{X}_b(K_{c\bar{d}}) + f_{KK}G^{\bar{d}a}K_{\bar{d}|b}K_c + \frac{f_{KK}}{f_K}\delta_c^aK_{|b} \\
&\quad + \frac{1}{\Delta}\left[f_Hf_{KKK} + f_{KK}f_{KH} - \frac{1}{f_K}f_Hf_{KK}^2\right]v^aK_cK_{|b}.
\end{aligned} \tag{4.15}$$

By plunging (4.15) into (4.12), we obtain (4.6). Similarly, we obtain (4.11).

Secondly, we calculate $\mathbb{L}^a_{b\gamma}$. According to the definition of $\mathbb{L}^i_{jk}$, we have

$$\begin{aligned}
\mathbb{L}^a_{b\gamma} &= \frac{1}{2}G^{\bar{l}a}\left[\mathscr{X}_b(G_{\gamma\bar{l}}) + \mathscr{X}_\gamma(G_{b\bar{l}})\right] \\
&= \frac{1}{2}G^{\bar{d}a}\left[\mathscr{X}_b(G_{\gamma\bar{d}}) + \mathscr{X}_\gamma(G_{b\bar{d}})\right] + \frac{1}{2}G^{\bar{\sigma}a}\left[\mathscr{X}_b(G_{\gamma\bar{\sigma}}) + \mathscr{X}_\gamma(G_{b\bar{\sigma}})\right] \\
&= \frac{1}{2}G^{\bar{d}a}\left[\mathscr{X}_b(f_{KH}H_\gamma K_{\bar{d}}) + \mathscr{X}_\gamma(f_K K_{b\bar{d}} + f_{KK}K_b K_{\bar{d}})\right] \\
&\quad + \frac{1}{2}G^{\bar{\sigma}a}\left[\mathscr{X}_\gamma(f_{KH}K_b H_{\bar{\sigma}}) + \mathscr{X}_b(f_H H_{\gamma\bar{\sigma}} + f_{HH}H_\gamma H_{\bar{\sigma}})\right] \\
&= \frac{1}{2}G^{\bar{d}a}\left[(f_{KH}K_{\bar{d}|b} + f_{KKH}K_{\bar{d}}K_{|b})H_\gamma + (f_{KH}K_{b\bar{d}} + f_{KKH}K_b K_{\bar{d}})H_{|\gamma}\right] \\
&\quad + \frac{1}{2}G^{\bar{\sigma}a}\left[(f_{KH}H_{\bar{\sigma}|\gamma} + f_{KHH}H_{\bar{\sigma}}H_{|\gamma})K_b + (f_{KH}H_{\gamma\bar{\sigma}} + f_{KHH}H_\gamma H_{\bar{\sigma}})K_{|b}\right]. \qquad (4.16)
\end{aligned}$$

A simple calculation yields

$$G^{\bar{\sigma}a}H_{\bar{\sigma}|\gamma} = -\frac{1}{\Delta}f_{KH}v^a H_{|\gamma}, \qquad (4.17)$$

$$G^{\bar{\sigma}a}H_{\bar{\sigma}} = -\frac{1}{\Delta}f_{KH}Hv^a, \qquad (4.18)$$

$$G^{\bar{\sigma}a}H_{\gamma\bar{\sigma}} = -\frac{1}{\Delta}f_{KH}v^a H_\gamma. \qquad (4.19)$$

Plunging (4.17)–(4.19) into (4.16) implies (4.7). Similarly, we get (4.10).

Lastly, we compute $\mathbb{L}^a_{\beta\gamma}$. By the definition of $\mathbb{L}^i_{jk}$, we have

$$\begin{aligned}
\mathbb{L}^a_{\beta\gamma} &= \frac{1}{2}G^{\bar{l}a}\left[\mathscr{X}_\beta(G_{\gamma\bar{l}}) + \mathscr{X}_\gamma(G_{\beta\bar{l}})\right] \\
&= \frac{1}{2}G^{\bar{d}a}\left[\mathscr{X}_\beta(G_{\gamma\bar{d}}) + \mathscr{X}_\gamma(G_{\beta\bar{d}})\right] + \frac{1}{2}G^{\bar{\sigma}a}\left[\mathscr{X}_\beta(G_{\gamma\bar{\sigma}}) + \mathscr{X}_\gamma(G_{\beta\bar{\sigma}})\right]. \qquad (4.20)
\end{aligned}$$

By Proposition 4.1 and using (4.3), we have

$$\begin{aligned}
G^{\bar{d}a}\mathscr{X}_\beta(G_{\gamma\bar{d}}) &= G^{\bar{d}a}\mathscr{X}_\beta(f_{KH}H_\gamma K_{\bar{d}}) \\
&= G^{\bar{d}a}K_{\bar{d}}\mathscr{X}_\beta(f_{KH}H_\gamma) \\
&= \frac{1}{\Delta}(f_H + f_{HH}H)v^a\left(f_{KH}H_{\gamma|\beta} + f_{KHH}H_{|\beta}H_\gamma\right), \qquad (4.21)
\end{aligned}$$

$$\begin{aligned}
G^{\bar{\sigma}a}\mathscr{X}_\beta(G_{\gamma\bar{\sigma}}) &= -\frac{1}{\Delta}f_{KH}v^a\overline{v^\sigma}\mathscr{X}_\beta\left(G_{\gamma\bar{\sigma}}\right) \\
&= -\frac{1}{\Delta}f_{KH}v^a\mathscr{X}_\beta\left(G_{\gamma\bar{\sigma}}\overline{v^\sigma}\right) \\
&= -\frac{1}{\Delta}f_{KH}v^a\mathscr{X}_\beta\left[(f_H + Hf_{HH})H_\gamma\right] \\
&= -\frac{1}{\Delta}f_{KH}v^a\left[(f_H + Hf_{HH})H_{\gamma|\beta} + (f_H + Hf_{HH})_H H_{|\beta}H_\gamma\right] \\
&= -\frac{1}{\Delta}f_{KH}v^a\left[(f_H + Hf_{HH})H_{\gamma|\beta} + (f_{HH} - Kf_{KHH})H_{|\beta}H_\gamma\right], \qquad (4.22)
\end{aligned}$$

where, in the last equality, we used $Hf_{HH} = -Kf_{KH}$. Furthermore, (4.21)+(4.22) yields

$$\begin{aligned}
&G^{\overline{d}a}\mathscr{X}_\beta(G_{\gamma\overline{d}}) + G^{\overline{\sigma}a}\mathscr{X}_\beta(G_{\gamma\overline{\sigma}})\\
&= \frac{1}{\Delta}\left[(f_H + f_{HH}H)f_{KHH} - f_{KH}(f_{HH} - Kf_{KHH})\right] v^a H_{|\beta}H_\gamma\\
&= \frac{1}{\Delta}\left(f_H f_{KHH} - f_{KH}f_{HH}\right) v^a H_{|\beta}H_\gamma\\
&= \frac{1}{\Delta}f_H^2\big[\ln(f_H)\big]_{KH} v^a H_{|\beta}H_\gamma. \qquad (4.23)
\end{aligned}$$

By substituting (4.23) into (4.20), we get (4.8). By a similar calculation, we obtain (4.9). This completes the proof. □

Proposition 4.6 *Let $\mathbb{L}^i_{jk}$ be the horizontal connection coefficients of the Rund-type connection associated to the product complex Finsler metric F. Then the following two equalities hold:*

$$\mathbb{L}^a_{b\gamma}K_a = -\frac{1}{2\Delta}Hf_H^2\big[\ln(f_H)\big]_{KH}\left(K_b H_{|\gamma} + K_{|b}H_\gamma\right), \qquad (4.24)$$

$$\mathbb{L}^\alpha_{b\gamma}H_\alpha = -\frac{1}{2\Delta}Kf_K^2\big[\ln(f_K)\big]_{KH}\left(K_b H_{|\gamma} + K_{|b}H_\gamma\right). \qquad (4.25)$$

Proof

By (4.3), we obtain

$$G^{\overline{d}a}K_a = \frac{1}{\Delta}(f_H + f_{HH}H)\overline{v^d},$$

which together with equalities

$$\begin{aligned}
&K_{\overline{d}|b}\overline{v^d} = K_{|b}, \quad K_{b\overline{d}}\overline{v^d} = K_b, \quad K_{\overline{d}}\overline{v^d} = K,\\
&-f_{KH}K = f_{HH}H, \quad f_{KH} + f_{KKH}K = -f_{KHH}H
\end{aligned}$$

yields

$$\begin{aligned}
\mathbb{L}^a_{b\gamma}K_a &= \frac{1}{2\Delta}(f_H + f_{HH}H)\overline{v^d}\Big[(f_{KH}K_{\overline{d}|b} + f_{KKH}K_{\overline{d}}K_{|b})H_\gamma + (f_{KH}K_{b\overline{d}} + f_{KKH}K_bK_{\overline{d}})H_{|\gamma}\Big]\\
&\quad - \frac{1}{2\Delta}f_{KH}K(f_{KH} + f_{KHH}H)(K_bH_{|\gamma} + K_{|b}H_\gamma)\\
&= \frac{1}{2\Delta}\big[(f_H + f_{HH}H)(f_{KH} + f_{KKH}K) + f_{HH}H(f_{KH} + f_{KHH}H)\big]\left(K_bH_{|\gamma} + K_{|b}H_\gamma\right)\\
&= \frac{1}{2\Delta}\big[(f_H + f_{HH}H)(-f_{KHH}H) + f_{HH}H(f_{KH} + f_{KHH}H)\big]\left(K_bH_{|\gamma} + K_{|b}H_\gamma\right)\\
&= -\frac{1}{2\Delta}H\big(f_Hf_{KHH} - f_{HH}f_{KH}\big)\left(K_bH_{|\gamma} + K_{|b}H_\gamma\right)\\
&= -\frac{1}{2\Delta}Hf_H^2\big[\ln(f_H)\big]_{KH}\left(K_bH_{|\gamma} + K_{|b}H_\gamma\right),
\end{aligned}$$

which is (4.24); we obtain (4.25) similarly. □

5. Proof of Theorem 1.1

(i) It follows from Proposition 4.3.

(ii) It follows from Proposition 4.3.

(iii) By Definition 2.3, F is a weakly Kähler–Finsler metric if and only if

$$G_i(\Gamma^i_{j;k} - \Gamma^i_{k;j})v^j = 0.$$

By Proposition 4.3, we obtain

$$\begin{aligned}
&G_i(\Gamma^i_{j;c} - \Gamma^i_{c;j})v^j \\
=&G_a(\Gamma^a_{b;c} - \Gamma^a_{c;b})v^b + G_a(\Gamma^a_{\beta;c} - \Gamma^a_{c;\beta})v^\beta + G_\alpha(\Gamma^\alpha_{b;c} - \Gamma^\alpha_{c;b})v^b + G_\alpha(\Gamma^\alpha_{\beta;c} - \Gamma^\alpha_{c;\beta})v^\beta \\
=&G_a(\Gamma^a_{b;c} - \Gamma^a_{c;b})v^b \\
=&f_K K_a(\tilde{\Gamma}^a_{b;c} - \tilde{\Gamma}^a_{c;b})v^b. \qquad (5.1)
\end{aligned}$$

Similarly, we get

$$G_i(\Gamma^i_{j;\gamma} - \Gamma^i_{\gamma;j})v^j = f_H H_\alpha(\tilde{\Gamma}^\alpha_{\beta;\gamma} - \tilde{\Gamma}^\beta_{\gamma;\beta})v^\beta. \qquad (5.2)$$

It follows from (5.1), (5.2), and the fact that $f_K > 0, f_H > 0$ that

$$G_i(\Gamma^i_{j;k} - \Gamma^i_{k;j})v^j = 0$$

if and only if

$$K_a(\tilde{\Gamma}^a_{b;c} - \tilde{\Gamma}^a_{c;b})v^b = 0 \quad \text{and} \quad H_\alpha(\tilde{\Gamma}^\alpha_{\beta;\gamma} - \tilde{\Gamma}^\beta_{\gamma;\beta})v^\beta = 0,$$

or equivalently, F is a weakly Kähler–Finsler metric if and only if F_1 and F_2 are both weakly Kähler–Finsler metrics.

(iv) It follows from the fourth conclusion of Proposition 4.4.

6. Proof of Theorem 1.2

By symmetric property $\mathbb{L}^i_{jk} = \mathbb{L}^i_{kj}$, the horizontal connection coefficients $\mathbb{L}^i_{jk}$ are decided by six tensors, namely, $\mathbb{L}^a_{bc}$, $\mathbb{L}^a_{b\gamma}$, $\mathbb{L}^a_{\beta\gamma}$, $\mathbb{L}^\alpha_{bc}$, $\mathbb{L}^\alpha_{b\gamma}$, and $\mathbb{L}^\alpha_{\beta\gamma}$, which are just the objects that shall be considered in this section. By the last conclusion in Proposition 4.4, the complex Berwald connection coefficients $\mathbb{G}^i_{jk}$ associated to the product complex Finsler metric F are

$$\mathbb{G}^a_{bc} = \tilde{\mathbb{G}}^a_{bc}, \quad \mathbb{G}^\alpha_{\beta\gamma} = \tilde{\mathbb{G}}^\alpha_{\beta\gamma}, \quad \mathbb{G}^a_{b\gamma} = \mathbb{G}^a_{\beta\gamma} = \mathbb{G}^\alpha_{bc} = \mathbb{G}^\alpha_{b\gamma} = 0.$$

Hence, (M, F) is a Landsberg manifold, i.e. F is a Landsberg metric if and only if

$$\mathbb{L}^a_{bc} = \tilde{\mathbb{G}}^a_{bc}, \quad \mathbb{L}^\alpha_{\beta\gamma} = \tilde{\mathbb{G}}^\alpha_{\beta\gamma}, \quad \mathbb{L}^a_{b\gamma} = \mathbb{L}^a_{\beta\gamma} = \mathbb{L}^\alpha_{bc} = \mathbb{L}^\alpha_{b\gamma} = 0. \qquad (6.1)$$

Proof of the sufficiency

Case 1. Suppose that F_1 and F_2 are both complex Landsberg metrics, and $f = c_1K + c_2H$ with c_1, c_2 positive constants, which apparently yield

$$\tilde{\mathbb{L}}^a_{bc} = \tilde{\mathbb{G}}^a_{bc}, \quad \tilde{\mathbb{L}}^\alpha_{\beta\gamma} = \tilde{\mathbb{G}}^\alpha_{\beta\gamma}, \tag{6.2}$$

$$f_{KK} = f_{KH} = f_{HH} = f_{KKK} = f_{KKH} = f_{KHH} = f_{HHH} = \big[\ln(f_K)\big]_{KH} = \big[\ln(f_H)\big]_{KH} = 0. \tag{6.3}$$

By plunging (6.2) and (6.3) into the expressions of $\mathbb{L}^i_{jk}$ (4.6)–(4.11), we obtain (6.1). Hence, the product complex Finsler metric F is a complex Landsberg metric.

Case 2. Suppose that F_1 and F_2 are both Kähler–Finsler metrics, then by conclusion (i) in Theorem 1.1, the product complex Finsler metric F is a Kähler–Finsler metric, which implies that F is a complex Landsberg metric.

Proof of the necessity

Suppose that the product complex Finsler metric F is a complex Landsberg metric, then equalities in (6.1) hold, and we have

$$\mathbb{L}^a_{\beta\gamma} = 0, \quad \mathbb{L}^\alpha_{bc} = 0, \quad \mathbb{L}^a_{b\gamma}K_a = 0, \quad \mathbb{L}^\alpha_{b\gamma}H_\alpha = 0. \tag{6.4}$$

Substituting expressions (4.8), (4.9), (4.24), and (4.25) into the four equalities in (6.4), respectively, yields

$$\big[\ln(f_H)\big]_{KH}\left(H_\beta H_{|\gamma} + H_\gamma H_{|\beta}\right) = 0, \tag{6.5}$$

$$\big[\ln(f_K)\big]_{KH}\left(K_b K_{|c} + K_c K_{|b}\right) = 0, \tag{6.6}$$

$$\big[\ln(f_H)\big]_{KH}\left(K_b H_{|\gamma} + K_{|b} H_\gamma\right) = 0, \tag{6.7}$$

$$\big[\ln(f_K)\big]_{KH}\left(K_b H_{|\gamma} + K_{|b} H_\gamma\right) = 0. \tag{6.8}$$

Now we analyze (6.5)-(6.8), and divide our discussion into two cases.

Case 1. Either one of $[\ln(f_H)]_{KH}$ and $[\ln(f_K)]_{KH}$ does not vanish. Without loss of generality, we suppose

$$\big[\ln(f_H)\big]_{KH} \neq 0,$$

which together with (6.5) and (6.7) implies

$$H_\beta H_{|\gamma} + H_\gamma H_{|\beta} = 0, \tag{6.9}$$

$$K_b H_{|\gamma} + K_{|b} H_\gamma = 0. \tag{6.10}$$

Noting that by (2.3) and (2.5), we get

$$H_{|\beta} v^\beta = \mathscr{X}(H) = \chi(H) = \delta_\beta(H) v^\beta = 0.$$

Then, contracting (6.9) with v^β in both sides yields

$$H_{|\gamma} = 0. \tag{6.11}$$

A similar process on (6.10) implies

$$K_{|b} = 0. \tag{6.12}$$

By Proposition 2.1, (6.11) and (6.12) indicate that F_1 and F_2 are both weakly Kähler–Finsler metrics. Below, we shall show that F_1 and F_2 are actually both Kähler–Finsler metrics. Plunging (6.11) and (6.12) into (4.7) yields

$$0 = 2\mathbb{L}^a_{b\gamma} = f_{KH} G^{\overline{d}a} K_{\overline{d}|b} H_\gamma. \tag{6.13}$$

It follows from

$$0 \neq \big[\ln(f_H)\big]_{KH} = \left(\frac{f_{KH}}{f_H}\right)_H$$

that $f_{KH} \neq 0$, and note that $G^{\overline{d}a}$ is an invertible tensor, we deduce from (6.13) that

$$K_{\overline{d}|b} = 0. \tag{6.14}$$

Similarly, by plunging (6.11) and (6.12) into (4.10), we obtain

$$H_{\overline{\sigma}|\gamma} = 0. \tag{6.15}$$

It follows from (6.14), (6.15), and Proposition 2.1 that F_1 and F_2 are both Kähler–Finsler metrics.

Case 2. Both $[\ln(f_H)]_{KH}$ and $[\ln(f_K)]_{KH}$ vanish, i.e.

$$\big[\ln(f_H)\big]_{KH} = 0, \tag{6.16}$$

$$\big[\ln(f_K)\big]_{KH} = 0. \tag{6.17}$$

Integrating (6.16) yields that

$$f_H = \exp(\phi(K) + \varphi(H))$$

for some one-variable functions $\phi(K)$ and $\varphi(H)$. Since f_H is 0-homogenous, it results that both $\phi(K)$ and $\varphi(H)$ are 0-homogenous, this implies that $\phi(K) =$ constant and $\varphi(H) =$ constant. Hence, we obtain

$$f_H = constant. \tag{6.18}$$

Similarly, by (6.17), we deduce that

$$f_K = constant. \tag{6.19}$$

Noting that f is a 1-homogeneous function, it follows from (6.18) and (6.19) that

$$f = c_1 K + c_2 H$$

for some positive constant c_1 and c_2. In this case, the higher partial derivatives of f with order greater than 1 all vanish, i.e.

$$f_{KK} = f_{KH} = f_{HH} = f_{KKK} = f_{HHH} = 0. \tag{6.20}$$

By a direct substitution of (6.20) into (4.6) and (4.11), respectively, we obtain

$$\mathbb{L}^a_{bc} = \tilde{\mathbb{L}}^a_{bc}, \quad \mathbb{L}^\alpha_{\beta\gamma} = \tilde{\mathbb{L}}^\alpha_{\beta\gamma},$$

which together with the first two equalities in (6.1) yield that

$$\tilde{\mathbb{L}}^a_{bc} = \tilde{\mathbb{G}}^a_{bc}, \quad \tilde{\mathbb{L}}^\alpha_{\beta\gamma} = \tilde{\mathbb{G}}^\alpha_{\beta\gamma},$$

or equivalently, F_1 and F_2 are both complex Landsberg metrics. This completes the proof.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant Nos. 11701494, 11671330, 11571288) and the Nanhu Scholars Program for Young Scholars of XYNU.

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