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On essential cohomology of powerful p-groups

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Abstract: For an odd prime p, we prove that a finite powerful p-group having rank two Frattini quotient has nonzero essential cohomology. We also provide some examples and applications.

Key words: Cohomology of groups, essential cohomology, powerful p-groups

1. Introduction

Let G be a finite group and k be a field of characteristic p. A cohomology class x in the mod p-cohomology ring $H^*(G,k)$ is called essential if it is in the intersection of kernels of restrictions to proper subgroups ($x \in \cap_{H<G} \ker \operatorname{res}^G_H$). Essential classes in $H^*(G,k)$ generate an ideal called the essential cohomology of G and it is denoted by $\mathrm{Ess}^*(G)$. For a finite group, the kernel of the restriction to a Sylow p-subgroup is trivial, so nonzero essential cohomology only exists in the case of p-groups. Essential cohomology plays a crucial role for calculation methods in [2, 7]. If $\mathrm{Ess}^*(G)$ is nonzero, then one cannot use the process in [2]. Finding a group theoretic characterization of finite p-groups having nonzero essential cohomology will be a valuable step for the methods in [2, 7]. This is an open problem introduced in [1] (see problem 4 on page 438). For groups having Cohen–Macaulay cohomology rings, the characterization is given in [3]. It is rather difficult to find nonzero essential classes, but when possible, it is also involved in determining the depth of the cohomology ring $H^*(G,k)$ (see [6]).

It is well known that for a finite p-group G, the Frattini quotient $G/\Phi(G)$ (here $\Phi(G)$ is the Frattini subgroup of G) is an elementary abelian p-group. For elementary abelian p-groups, a characterization of essential cohomology is given completely in [5]. By using this, we define the inflated essential cohomology of a p-group as the ideal generated by $\inf^G_{G/\Phi(G)}(\mathrm{Ess}^*(G/\Phi(G)))$ and denote it by $\mathrm{InfEss}^*(G)$. For $p>2$, we consider the inflated essential cohomology of powerful p-groups: groups satisfying the condition $[G,G] \leq G^p$ (here $G =< x^p : x \in G >$ the subgroup generated by all pth powers). For the rest of the paper we assume that $k=\mathbb{F}_p$, unless otherwise stated. In this paper, we prove the following:

Theorem 1.1 *Let $p>2$ and G be a finite p-group. If $\mathrm{InfEss}^*(G)$ is nonzero, then G is a powerful p-group.*

For a restricted family of powerful p-groups, we give a characterization as follows.

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Theorem 1.2 *Let $p > 2$ and G be a p-group such that the Frattini quotient has rank two.* $\mathrm{InfEss}^*(G)$ *is nonzero if and only if G is a powerful p-group.*

Theorem 1.3 *If G is a powerful p-group with a rank two Frattini quotient, then* $\mathrm{Ess}^*(G)$ *is nonzero.*

There is another distinguishable algebraic structure of essential cohomology of G if G is not an elementary abelian p-group. By Quillen [15], if G is not an elementary abelian p-group, $\mathrm{Ess}^*(G)$ is a nilpotent ideal. By Mùi (in his unpublished essay: The mod p cohomology algebra of the extra-special group $E(p^3)$ 1982) and Marx [13], it was conjectured that the nilpotency degree is 2. In [10], Green disproved the conjecture. We consider the nilpotency degree of $\mathrm{InfEss}^*(G)$ and prove the following theorem.

Theorem 1.4 *Let G be a finite p-group such that* $\mathrm{InfEss}^*(G)$ *is nonzero. Then the nilpotency degree of* $\mathrm{InfEss}^*(G)$ *is 2.*

The structure of the paper is as follows. In Section 2, we introduce the inflated essential cohomology of a finite p-group and then we give some properties and examples. We briefly explain the characterization of 2-groups with nonzero inflated essential classes and investigate the case $p > 2$. In Section 3, we give the proofs of Theorem 1.1, Theorem 1.2, and Theorem 1.3. In the last section, we prove Theorem 1.4.

2. Inflated essential cohomology

For a finite p-group, the Frattini quotient $G/\Phi(G)$ is an elementary abelian p-group and the essential cohomology of it is completely determined in [5]. We define the inflated essential cohomology of G as follows. Consider the ring homomorphism

$$\mathrm{inf}^G_{G/\Phi(G)} : H^*(G/\Phi(G), k) \to H^*(G, k).$$

Definition 2.1 *The inflated essential cohomology of a finite p-group is the ideal generated by the image* $\mathrm{inf}^G_{G/\Phi(G)}(\mathrm{Ess}^*(G/\Phi(G)))$.

We denote this ideal by $\mathrm{InfEss}^*(G)$. The relation between restriction and inflation explain the reason why we call this ideal inflated essential cohomology.

Lemma 2.2 *Let G be a p-group and N be a normal subgroup of G, which is contained in all maximal subgroups. Then*

$$\mathrm{inf}^G_{G/N}(\mathrm{Ess}^*(G/N)) \subseteq \mathrm{Ess}(G).$$

Proof Let $x \in \mathrm{Ess}^*(G/N)$ and H be a maximal subgroup of G. We have

$$\mathrm{res}^G_H(\mathrm{inf}^G_{G/N}(x)) = \mathrm{inf}^H_{H/N}(\mathrm{res}^{G/N}_{H/N}(x))$$

and $\mathrm{res}^{G/N}_{H/N}(x) = 0$ as x is essential. (For the commutativity of restriction and inflation, see page 70 in [18].) □

Thus, for a p-group G, we have $\mathrm{InfEss}^*(G) \subseteq \mathrm{Ess}^*(G)$.

We consider a particular form of the open problem in [1]. Our main interest is to make a classification of p-groups with nonzero inflated essential classes.

The classification has different results for odd primes and prime 2. For $p = 2$ the classification is determined completely. For odd primes, the classification is much more difficult. For completeness we include the case $p = 2$, which mainly follows from the results in [19].

2.1. The case $p = 2$

Let V be an elementary abelian 2-group. The cohomology ring of V is $H^*(V, k) = k[x_1, ..., x_n]$ where $x_i \in H^1(V, k)$. The essential cohomology of V is given in [5].

Lemma 2.3 ([5], Lemma 2.2) *The essential cohomology of V is the principal ideal generated by $L_n(V) = \lambda \prod_{[x]\in\mathbb{P}H^1(V,k)} x$, where λ is a nonzero scalar in k.*

For $p = 2$, the classification of 2-groups with nonzero inflated essential cohomology follows from a result in [19].

Theorem 2.4 ([19]) *If G is a non-Abelian 2-group, $\sigma_G = \prod_{x\in H^1(G,k)-\{0\}} x = 0$.*

Corollary 2.5 *If G is a non-Abelian 2-group, then* $\mathrm{InfEss}^*(G) = 0$.

Proof Let V denote the elementary Abelian quotient $G/\Phi(G)$. By Lemma 2.3 $\mathrm{Ess}^*(V)$ is generated by $L_n(V)$. We have $\mathrm{inf}^G_{G/\Phi(G)}(L_n(V)) = \sigma_G$, as $\mathrm{inf}^G_{G/\Phi(G)} : H^1(G/\Phi(G), k) \to H^1(G, k)$ is bijective. Then $\mathrm{InfEss}^*(G) = 0$ by Theorem 2.4. □

For some Abelian 2-groups, we have $\sigma_G = 0$. For example, $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$. The following theorem is a characterization of 2-groups with nonzero σ_G.

Theorem 2.6 ([19]) *Let G be a 2-group. Then $\sigma_G \neq 0$ if and only if $G \cong \mathbb{Z}/2^s \times (\mathbb{Z}/2)^n$ for some $n \geq 0$ and $s \geq 1$.*

The characterization of 2-groups having nonzero inflated essential cohomology easily follows.

Corollary 2.7 *Let G be a 2-group. Then* $\mathrm{InfEss}^*(G) \neq 0$ *if and only if $G \cong \mathbb{Z}/2^s \times (\mathbb{Z}/2)^n$ for some $n \geq 0$ and $s \geq 1$.*

For $p = 2$, the characterization is complete.

2.2. The case $p > 2$

The classification of p-groups with nonzero inflated essential cohomology is much more complicated for $p > 2$. Let V be an elementary Abelian p-group of rank n. The cohomology ring of V is

$$H^*(V, k) = k[x_1, ..., x_n] \otimes \wedge(a_1, ..., a_n),$$

where $a_i \in H^1(V, k)$, $x_i = \beta(a_i)$, and here β is Bockstein homomorphism. In [5], the essential ideal of V is characterized by the action of Steenrod algebra.

Theorem 2.8 ([5], Theorem 1.1) *The essential cohomology* $\mathrm{Ess}^*(V)$ *is the Steenrod closure of the product* $a_1 \cdots a_n$.

We can easily conclude the following corollary.

Corollary 2.9 *Let* G *be a* p*-group. The inflated essential cohomology* $\mathrm{InfEss}^*(G)$ *is zero if and only if* $\inf^G_{G/\Phi(G)}(a_1 \cdots a_n) = 0$.

Proof This follows from the fact that the essential cohomology of the elementary Abelian p-group is the Steenrod closure of $a_1 \cdots a_n$ and Steenrod operations commute with inflation by naturality. □

The problem of finding p-groups with nonzero inflated essential classes (i.e. nonzero essential classes) is reduced to the problem of finding finite p-groups such that $\inf^G_{G/\Phi(G)}(a_1 \cdots a_n) \neq 0$.

First of all we consider extraspecial p-groups. We will explain later the background motivation for this.

Definition 2.10 *A finite* p*-group* G *is called extraspecial if the center* $Z(G)$ *is cyclic of order* p *and* $Z(G) = G' = \Phi(G)$. *A finite* p*-group* G *is called almost extraspecial if the center is cyclic of order* p^2 *and* $G' = \Phi(G)$ *is cyclic of order* p.

There are two types of extraspecial p-groups of order p^3:

$$\mathbb{M} \cong \langle x, y \mid x^p = y^{p^2} = 1, xyx^{-1} = y^{p+1} \rangle,$$

$$\mathbb{E} \cong \langle x, y, z \mid x^p = y^p = z^p = [x, z] = [y, z] = 1, [x, y] = z \rangle.$$

An extraspecial p-group of order p^{2n+1} is isomorphic to one of the following central products:

$$\mathbb{E}_n = \mathbb{E} * \cdots * \mathbb{E} \ (n \text{ times}), \qquad \mathbb{M}_{\mathrm{n}} = \mathbb{M} * \mathbb{E}_{\mathrm{n}-1}.$$

If we consider the central product $C_{p^2} * \mathbb{M}_n$ or $C_{p^2} * \mathbb{E}_n$, we get almost extraspecial p-group of order p^{2n+1}. In fact, these groups fit into an extension of the form

$$1 \to C_p \to G \to V \to 1,$$

where V is an elementary Abelian p-group that is isomorphic to $G/\Phi(G)$ and this extension corresponds to a cohomoloy class $\alpha \in H^2(V, \mathbb{F}_p)$.

For details about extraspecial p-groups, see [8].

We consider the inflated essential classes of extraspecial p-groups. For notation, let $H^*(V, \mathbb{F}_p) = \mathbb{F}_p[x_1, ..., x_s] \otimes \wedge(a_1, ..., a_s)$ where $s = 2n$ if G is extraspecial and $s = 2n + 1$ if G is almost extraspecial.

Lemma 2.11 (Lemma 7.6.1 in [8]) *The cohomology class of the extension of* V *by* C_p *is the class*

$$\alpha = \begin{cases} a_1a_2 + ... + a_{2n-1}a_{2n}, & \textit{if } G = \mathbb{E}_n \\ a_1a_2 + ... + a_{2n-1}a_{2n} + x_{2n}, & \textit{if } G = \mathbb{M}_n \\ a_1a_2 + ... + a_{2n-1}a_{2n} + x_{2n+1}, & \textit{if } G \textit{ is almost extraspecial} \end{cases}$$

in $H^2(V, \mathbb{F}_p)$. *For each case,* α *is in the kernel of inflation* $\inf^G_V$.

Lemma 2.12 *Let G be an extraspecial p-group of exponent p. Then* $\mathrm{InfEss}^*(G)=0$.

Proof The product $\alpha a_3 \cdot a_4 \cdots a_{2n} = a_1 \cdots a_{2n}$ is in the kernel of $\inf^G_{G/\Phi(G)}$ as $\alpha = a_1a_2 + ... + a_{2n-1}a_{2n}$, in the kernel of $\inf^G_{G/\Phi(G)}$ by Lemma 2.11. Thus, $\mathrm{InfEss}^*(G)=0$. □

Remark 2.13 By Proposition 4 and Proposition 5 in [14], we can conclude $\mathrm{InfEss}^*(\mathbb{M}_n) \neq 0$ and $\mathrm{InfEss}^*(G) \neq 0$ where G is an almost extraspecial p-group. In [14], there is a question (see question on page 1945) about extraspecial p-groups that are not isomorphic to $\mathbb{E}$. For $G \ncong \mathbb{E}$, is it true that $\mathrm{Ess}^*(G) \cap \mathrm{Im}\, \inf^G_V \neq \{0\}$? We do not have a complete answer but we can say that $\mathrm{Ess}^*(G) \cap \inf^G_V(\mathrm{Ess}^*(V)) = \{0\}$ when G is an extraspecial p-group of exponent p by Lemma 2.12.

The motivation for considering extraspecial p-groups comes from the following. Many of the theorems such as Serre's theorem [16] can be proved by reducing them to the extraspecial case and then by using induction. In fact, this is because of the following lemma.

Lemma 2.14 *Let G be a non-Abelian p-group and let H be a maximal element in the collection of normal subgroups of G that do not contain the Frattini subgroup of G. Then the quotient $Q=G/H$ is an extraspecial or almost extraspecial p-group.*

Proof For details see [8], page 154 . □

Lemma 2.15 *Let G be a p-group. If* $\mathrm{InfEss}^*(G) \neq 0$, *then* $\mathrm{InfEss}^*(G/N) \neq 0$ *for any proper quotient G/N.*

Proof We prove that $\mathrm{InfEss}^*(G/N)=0$, and then $\mathrm{InfEss}^*(G)=0$. By transitivity of inflation, we have a commutative diagram:

$$\begin{array}{ccc} H^1(G/N/\Phi(G/N),k) & \xrightarrow{\inf} & H^1(G/\Phi(G),k) \\ {\scriptstyle\inf}\downarrow & & {\scriptstyle\inf}\downarrow \\ H^1(G/N,k) & \xrightarrow{\inf} & H^1(G,k) \end{array}$$

$\mathrm{InfEss}^*(G/N)=0$ if and only if $\inf^{G/N}_{G/N/\Phi(G/N)}(a_1\cdots a_t)=0$ where $a_1,...a_t$ are the generators of $H^1(G/N/\Phi(G/N),k)$. Thus, $\inf^G_{G/N}\inf^{G/N}_{G/N/\Phi(G/N)}(a_1\cdots a_t)=0$. On the other hand, let $e_1,...,e_n$ be the generators of $H^1(G/\Phi(G),k)$. It is clear that $t \le n$, so we can view $e_i = \inf^{G/\Phi(G)}_{G/N/\Phi(G/N)}(a_i)$ for $1 \le i \le k$. By commutativity of the diagram $\inf^G_{G/\Phi(G)}(e_1\cdots e_k)=0$. Then $\inf^G_{G/\Phi(G)}(e_1\cdots e_n)=0$, which means $\mathrm{InfEss}^*(G)=0$. □

With the same notation as in Lemma 2.14, we have the following proposition.

Proposition 2.16 *Let G be a non-Abelian p-group such that the quotient $Q=G/H$ is extraspecial of exponent p. Then* $\mathrm{InfEss}^*(G)=0$.

Proof By Lemma 2.12 $\mathrm{InfEss}^*(Q)=0$. Since Q is a proper quotient of G, by Lemma 2.15 $\mathrm{InfEss}^*(G)=0$. □

For non-Abelian p-groups having an extraspecial p-group of exponent p as a quotient, inflated essential classes are zero.

Contrary to 2-groups, if G is an Abelian p-group, then $\mathrm{InfEss}^*(G)$ is nonzero. This follows from the fact that the cohomology ring of an Abelian p-group is the tensor product of the cohomology rings of the cyclic p-groups, and the cohomology ring of a cyclic p-group is $k[a,x]/(a^2)$ where $\deg\ a = 1$ and $\deg\ x = 2$.

Proposition 2.17 *Let G and H be p-groups such that $\mathrm{InfEss}^*(G) \neq 0$ and $\mathrm{InfEss}^*(H) \neq 0$. Then $\mathrm{InfEss}^*(G \times H)$ is nonzero.*

Proof Since $\mathrm{InfEss}^*(G) \neq 0$ and $\mathrm{InfEss}^*(H) \neq 0$ we have $\inf^G_{G/\Phi(G)}(a_1 \cdots a_k) \neq 0$ where $a_i \in H^1(G/\Phi(G), k)$ and $\inf^H_{H/\Phi(H)}(e_1 \cdots e_l) \neq 0$ where $e_i \in H^1(H/\Phi(H), k)$. Consider $\tilde{a_i} = \inf^G_{G/\Phi(G)}(a_i)$ and $\tilde{e_i} = \inf^H_{H/\Phi(H)}(e_i)$. $\mathrm{InfEss}^*(G \times H)$ is nonzero, because $\tilde{a_1} \cdots \tilde{a_k} \cdot \tilde{e_1} \cdots \tilde{e_l} \neq 0$ in $H^*(G \times H, k) \cong H^*(G,k) \otimes H^*(H,k)$. □

There is no information on $\mathrm{Ess}^*(G \times H)$ in general. For a restricted family we have the following result.

Corollary 2.18 *Let G be a p-group such that $\mathrm{InfEss}^*(G) \neq 0$. If H is an Abelian p-group, then $\mathrm{Ess}^*(G \times H)$ is nonzero.*

If G is an extraspecial of exponent p^2 or an almost extraspecial p-group, then $\mathrm{InfEss}^*(G)$ is nonzero (see [14], Proposition 4 and Proposition 5). Thus, any direct product of G with an Abelian p-group has nonzero essential classes.

With the same notation as in Lemma 2.14, we have a question:

Question 2.19 *If $Q = G/H$ is an extraspecial p-group of exponent p^2 or almost extraspecial p-group, then is it true that $\mathrm{InfEss}^*(G) \neq \{0\}$?*

This is not true in general.

Example 2.20 *By definition of the central product, we can consider $\mathbb{M}_n$ in the extension*

$$0 \to C_p \to \mathbb{E}_{n-1} \times \mathbb{M} \to \mathbb{M}_n \to 0.$$

$Q = \mathbb{M}_n$ and we know that $\mathrm{InfEss}^*(\mathbb{M}_n) \neq 0$, but $\mathrm{InfEss}^*(\mathbb{E}_{n-1} \times \mathbb{M}) = 0$ as $\mathrm{InfEss}^*(\mathbb{E}_{n-1}) = 0$.

Example 2.21 *Since an almost extraspecial p-group Γ_n of order p^{2n+2} is the central product $C_{p^2} * \mathbb{E}_n$, we can consider the extension*

$$0 \to C_p \to \mathbb{E}_n \times C_{p^2} \to \Gamma_n \to 0.$$

We know that $\mathrm{InfEss}^*(\Gamma_n) \neq 0$, but $\mathrm{InfEss}^*(\mathbb{E}_n \times C_{p^2}) = 0$ as $\mathrm{InfEss}^*(\mathbb{E}_n)$.

Proposition 2.22 *Let G be a non-Abelian p-group of exponent p. Then $\mathrm{InfEss}^*(G) = 0$.*

Proof If G is of exponent p, then any proper quotient is also of exponent p. Thus, the extraspecial quotient Q is also of exponent p, so by Lemma 2.16, $\mathrm{InfEss}^*(G) = 0$ as $\mathrm{InfEss}^*(Q) = 0$. □

3. Powerful p-groups

Proposition 2.22 leads us to consider powerful p-groups, which are introduced in [11] and are used to study the structure of the Schur multiplier of a p-group. They are also used for the study of analytic pro-p-groups [9]. These groups are also important because every finite p-group can be expressed as a section of a powerful p-group (see [12]). We consider powerful p-groups in the case of odd primes.

Definition 3.1 *Let $p > 2$. A finite p-group G is said to be powerful if $[G, G] \leqslant G^p$.*

Proof [Proof of Theorem 1.1] If G is an Abelian p-group then there is nothing to do. Assume G is a non-Abelian p-group. $[G, G] \leqslant G^p$ if and only if $\Phi(G) = G^p$. Now assume that $G^p < \Phi(G)$. Then the quotient G/G^p is a non-Abelian p-group of exponent p. By Proposition 2.22 $\mathrm{InfEss}^*(G/G^p) = 0$ and by Lemma 2.15 $\mathrm{InfEss}^*(G) = 0$. □

Proof [Proof of Theorem 1.2] If $\mathrm{InfEss}^*(G) \neq 0$ then by Theorem 1.1 G is powerful. Assume G is powerful. As we have $\dim(G/\Phi(G)) = 2$, the cohomology group $H^1(G/\Phi(G), k)$ has two generators a_1, a_2, so we need to show that $\inf^G_{G/\Phi(G)}(a_1 \cdot a_2) \neq 0$. This follows from the following theorem as every p-group is a pro-p group. □

Theorem 3.2 ([17], Theorem 5.1.6) *Let p be odd prime and let P be a finitely generated pro-p group. The canonical mapping $H^1(P, \mathbb{F}_p) \wedge H^1(P, \mathbb{F}_p) \to H^2(P, \mathbb{F}_p)$ is injective if and only if P is powerful.*

Theorem 1.3 easily follows as a corollary.

Corollary 3.3 *If G is a powerful p-group such that the Frattini quotient has rank 2, then $\mathrm{Ess}^*(G) \neq 0$.*

Corollary 3.4 *Let P be a powerful p-group such that the Frattini quotient has rank 2 and H be an Abelian p-group. Then, for $G = P \times H$, $\mathrm{Ess}^*(G) \neq 0$*

Remark 3.5 All Abelian p-groups are powerful and those for which the minimal number of generators is 2 are the powerful p-groups having rank two Frattini quotient. For the non-Abelian case, the modular p-groups

$$\mathrm{Mod}_n(p) =< x, y : x^{p^{n-1}} = y^p = 1, \quad x^y = x^{1+p^{n-2}} >$$

for $n \geq 4$ are the example of powerful p-groups having rank 2 Frattini quotient. Note that the family $\mathrm{Mod}_n(p)$ does not satisfy the pc condition mentioned in [3].

4. Nilpotency degree

In [15], it was proved that $\mathrm{Ess}^*(G)$ is nilpotent whenever G is not elementary Abelian. Marx [13] and Mùi (in his unpublished essay: The mod p cohomology algebra of the extra-special group $E(p^3)$, 1982) conjectured that the nilpotency degree of $\mathrm{Ess}^*(G)$ is two and it was proved that it is not two in [10]. Contrary to essential cohomology, we prove that the nilpotency degree of inflated essential cohomology is 2.

Theorem 4.1 *Let G be a finite p-group such that $\mathrm{InfEss}^*(G)$ is nonzero. Then the nilpotency degree of $\mathrm{InfEss}^*(G)$ is 2.*

Proof Let V be the Frattini quotient of G of rank n. By definition $\mathrm{InfEss}^*(G)$ is generated by $\inf_V^G(\mathrm{Ess}^*(V))$, so it is enough to show $\inf_V^G(\mathrm{Ess}^*(V))^2 = 0$.

We know that the essential cohomology of V satisfies $\mathrm{Ess}^*(V)^2 = L_n(V) \cdot \mathrm{Ess}^*(V)$ by Lemma 3.2. in [5]. Applying inflation to the equality we get $\inf_V^G(\mathrm{Ess}^*(V))^2 = \inf_V^G(L_n(V)) \cdot \inf_V^G(\mathrm{Ess}^*(V))$. By Lemma 2.1 in [5], we have

$$L_n(V) = \lambda \prod_{[x]\in \mathbb{P}H^1(V,k)} \beta(x).$$

Inflation commutes with Bockstein homomorphism, so we have

$$\inf_V^G(L_n(V)) = \lambda \prod_{[x]\in \mathbb{P}H^1(V,k)} \beta(\inf_V^G(x)) = \lambda \prod_{[\bar{x}]\in \mathbb{P}H^1(G,k)} \beta(\bar{x}),$$

where $\bar{x} = \inf_V^G(x)$. By the following celebrated theorem of Serre, we get $\inf_V^G(L_n(V)) = 0$. □

Theorem 4.2 (Theorem 1.3 in [16]) *Let S be a subset of $H^1(G,k)$, which does not contain 0 and contains exactly one point from each line in $H^1(G,k)$. If G is not elementary Abelian then*

$$\prod_{x\in \mathbb{P}H^1(G,k)} \beta(x) = 0 \quad \text{in} \quad H^{even}(G,k).$$

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