



Young tableaux and Arf partitions

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Abstract: The aim of this work is to exhibit some relations between partitions of natural numbers and Arf semigroups. We also give characterizations of Arf semigroups via the hook-sets of Young tableaux of partitions.

Key words: Partition, Young tableau, numerical set, numerical semigroup, Arf semigroup, Arf closure

1. Introduction

Numerical semigroups have many applications in several branches of mathematics such as algebraic geometry and coding theory. They play an important role in the theory of algebraic geometric codes. The computation of the order bound on the minimum distance of such a code involves computations in some Weierstrass semigroup. Some families of numerical semigroups have been deeply studied from this point of view. When the Weierstrass semigroup at a point Q is an Arf semigroup, better results are developed for the order bound; see [8] and [3].

Partitions of positive integers can be graphically visualized with Young tableaux. They occur in several branches of mathematics and physics, including the study of symmetric polynomials and representations of the symmetric group. The combinatorial properties of partitions have been investigated up to now and we have quite a lot of knowledge. A connection with numerical semigroups is given in [4] and [10]. The hook-set of a partition encodes information about the other combinatorial objects related to that partition, the most famous being the hook-length formula, which gives the degree of the corresponding irreducible representation of the symmetric group and also counts the number of standard Young tableaux that have the shape of that partition (see, for instance, [6]).

We denote the set of integers by \mathbb{Z} and the set of positive integers by \mathbb{N} . We put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The cardinality of any set K will be denoted by $|K|$. For two subsets U, V of \mathbb{Z} and $z \in \mathbb{Z}$, we set

$$U + V = \{u + v : u \in U, v \in V\}, \quad U - V = \{x \in \mathbb{Z} : x + v \in U \text{ for all } v \in V\};$$

we also set $z + U = \{z\} + U$.

A numerical set S is a subset of \mathbb{N}_0 that contains 0 and has a finite complement $G(S) = \mathbb{N}_0 \setminus S$. \mathbb{N}_0 itself is a numerical set with $G(\mathbb{N}_0) = \emptyset$. A numerical set S is said to be *proper* if $S \neq \mathbb{N}_0$. If S is a proper numerical set, the elements of $G(S)$ are called *gaps* of S . The number of gaps is called the *genus* of S and it is denoted by $g(S)$. The largest gap of S is called the *Frobenius number* of S . The Frobenius number of S

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is denoted by $F(S)$, and $C(S) = F(S) + 1$ is called the *conductor* of S . The conductor of S is the smallest element of S such that every subsequent integer is an element of S . The Frobenius number of \mathbb{N}_0 is defined to be -1 , so that the conductor of \mathbb{N}_0 is 0 . The elements of S that are smaller than $C(S)$ are called the *small elements* of S . If a numerical set has n small elements, it is customary to list them as $s_0 = 0 < s_1 < \dots < s_{n-1}$ and write

$$S = \{s_0 = 0, s_1, \dots, s_{n-1}, s_n = C(S), \rightarrow\},$$

the arrow at the end meaning that all subsequent integers belong to S .

For an example, the numerical set $S = \{0, 3, 5, 6, 9, 11, \rightarrow\}$ has the complement $G(S) = \{1, 2, 4, 7, 8, 10\}$. For this reason, $g(S) = 6$, $F(S) = 10$ and $C(S) = 11$.

Given a numerical set $S = \{s_0 = 0, s_1, \dots, s_{n-1}, s_n = C(S), \rightarrow\}$, for each $i \geq 0$ we define

$$S_i = \{x \in S : x \geq s_i\}, \quad S(i) = S - S_i.$$

For each $i = 0, \dots, n - 1$, the set $-s_i + S_i$ is a numerical set whose Frobenius number $F(-s_i + S_i) = F(S) - s_i$ and $G(-s_i + S_i) = -s_i + \{b \in G(S) : b > s_i\}$.

A numerical set S is called a *numerical semigroup* if $x + y \in S$ for all $x, y \in S$. If A is a subset of \mathbb{N}_0 , we will denote by $\langle A \rangle$ the submonoid of \mathbb{N}_0 generated by A . If $S = \langle A \rangle$, A is called a *set of generators* for S . If $A = \{a_1, \dots, a_r\}$, we write $\langle A \rangle = \langle a_1, \dots, a_r \rangle$. The monoid $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

If $S = \{s_0 = 0, s_1, \dots, s_{n-1}, s_n = C(S), \rightarrow\}$ is a numerical semigroup, then we see that $S(i)$ is a numerical semigroup for each $i = 1, \dots, n$ and we have

$$\dots \subset S_k \subset \dots \subset S_1 \subset S_0 = S = S(0) \subset S(1) \subset \dots \subset S(n) = \mathbb{N}_0.$$

For each $i = 1, \dots, n$, the set $T_i(S) = S(i) \setminus S(i - 1)$ is called the *i th type set* of S , and the sequence $\{t_i(S) = |T_i(S)| : 1 \leq i \leq n\}$ is called the *type sequence* of S .

For general concepts and notations about numerical semigroups, we refer to [11].

In this work, we consider Young tableaux of numerical sets and we obtain some new characterizations of Arf semigroups via their Young tableaux. These characterizations allow us to give a procedure for determining the smallest Arf semigroup, called the Arf closure, containing a given numerical set. Additionally, we define the Arf partition of a positive integer, which seems to deserve further investigation.

2. Partitions, Young tableaux, and numerical semigroups

Given a positive integer N , a *partition* $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n]$ of N is a nonincreasing finite sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq \lambda_n$ such that $\lambda_1 + \lambda_2 + \dots + \lambda_n = N$. For each $i = 1, 2, \dots, n$, the number λ_i is called a *part* of the partition, and the number n of parts is called the *length* of the partition. If $\lambda_i \neq \lambda_{i+1}$ for each $i = 1, 2, \dots, n - 1$, then λ is called a *strict dominant partition*. If $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n]$ is a partition of N , we write

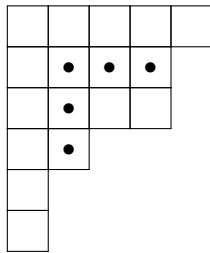
$$\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n] \vdash N.$$

A *Young tableau* is a series of top-aligned columns of boxes such that the number of boxes in each column is not less than the number of boxes in the column immediately to the right of it. The number of boxes in a column (or a row) is called the *length* of that column (or, respectively, that row). Given a box of a Young

tableau, the shape formed by the boxes directly to the right of it, the boxes directly below it, and the box itself is called the *hook* of that box. The boxes to the right form the *arm* and the boxes below form the *leg* of the hook. The hook of a box is a column if it has no arm, it is a row if it has no leg, and it consists of the box itself if it has no arm and no leg. The number of boxes in the hook of a box is called the *hook-length* of that box.

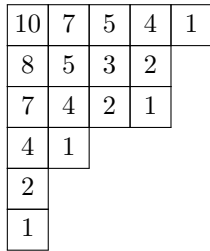
Let us note here that some authors define a Young tableau as a series of left-aligned rows of boxes such that the number of boxes in each row is not less than the number of boxes in the row immediately below it. It is clear that discussions below could also be carried out with this row-based definition.

Example 1 Here is an example of a Young tableau with 5 columns, where we show the hook of the box lying in the second row and the second column. The hook-length of that box is 5.



□

Every partition of a positive integer can be represented by a Young tableau. Given a partition $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n] \vdash N$, the Young tableau Y_λ corresponding to λ consists of n columns of boxes with lengths $\lambda_1, \lambda_2, \dots, \lambda_n$. The Young tableau in Example 1 corresponds to the partition $[6, 4, 3, 3, 1] \vdash 17$. The hook-length of each box in Y_λ is exhibited below.



Clearly, every Young tableau represents a uniquely determined partition. The correspondence $\lambda \rightarrow Y_\lambda$ is a bijection between the set of partitions of positive integers and the set of Young tableaux.

In a Young tableau, different rows may have the same length. In other words, there may be more than one row with the same length. Clearly the length of a row is at most the number of columns. Let us assume that there are n columns in a Young tableau and there are u_i rows of length i for each $i = 1, 2, \dots, n$. Then we denote such a Young tableau by $Y = 1^{u_1} 2^{u_2} \dots n^{u_n}$. If $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n]$ is the partition corresponding to $Y = 1^{u_1} 2^{u_2} \dots n^{u_n}$, then

$$\lambda_j = \sum_{i=j}^n u_i, \quad 1 \leq j \leq n.$$

Note that

$$\lambda_j - \lambda_{j+1} = u_j \quad \text{for each } j = 1, \dots, n - 1 \quad \text{and} \quad \lambda_n = u_n.$$

Numerical sets also can be represented by Young tableaux. Given a proper numerical set S , we construct a uniquely determined Young tableau and thus a uniquely determined partition as follows. We use the first quadrant of the Cartesian xy -plane for the construction by drawing a continuous polygonal path that starts from the origin. Starting with $x = 0$,

- we draw a line segment of unit length to the right if $x \in S$,
- we draw a line segment of unit length up if $x \notin S$,
- we repeat for $x + 1$.

We continue this until $x = F(S)$. Then a path with n horizontal and $g(S)$ vertical segments will be obtained. The lattice lying above this path and below the horizontal line that is $g(S)$ units above the origin defines a Young tableau, which is denoted by Y_S . It is clear that every Young tableau corresponds to a unique proper numerical set. Thus, the correspondence $S \rightarrow Y_S$ is a bijection between the set of proper numerical sets and the set of Young tableaux. For instance, the Young tableau in Example 1 corresponds to the numerical set $S = \{0, 3, 5, 6, 9, 11, \rightarrow\}$.

Let $S = \{s_0 = 0, s_1, \dots, s_{n-1}, s_n = C, \rightarrow\}$. The construction of Y_S implies that the number of columns of Y_S is n and the number of rows is $g(S)$. We denote the columns of Y_S by $G_0(S), G_1(S), \dots, G_{n-1}(S)$. It is clear that for each $j = 0, 1, \dots, n - 1$, the j th column $G_j(S)$ corresponds to s_j and the length of $G_j(S)$ is $g(S) - s_j + j$. We identify each column with the set of hook-lengths of boxes in it. The i th row of Y_S from the bottom corresponds to the i th gap of S ; the hook-length of the box of that row in the first column is the i th gap of S . Thus, $G_0(S)$ consists of the gaps of S ; that is, $G_0(S) = G(S)$.

If $Y_S = 1^{u_1} 2^{u_2} \dots n^{u_n}$, then the sequence $\{u_1, \dots, u_n\}$ is called the *Young sequence* of S . There are u_1 gaps of S less than s_1 , and $s_1 = u_1 + 1$. There are u_2 gaps between s_1 and s_2 , and we have $s_2 = u_1 + u_2 + 2$. Continuing this argument we see that there are u_j gaps between s_{j-1} and s_j , for each $j = 1, \dots, n$, and we have $s_j = u_1 + \dots + u_j + j$. We also note that $u_j = s_j - s_{j-1} - 1$ for each $j = 1, \dots, n$. This proves the first and also the second statements in the following lemma.

Lemma 2 *Let S be a proper numerical set having the Young sequence $\{u_1, \dots, u_n\}$. Then:*

- (i) $S = \{0, u_1 + 1, u_1 + u_2 + 2, \dots, u_1 + u_2 + \dots + u_n + n, \rightarrow\}$,
- (ii) $-s_j + S_j = \{0, u_{j+1} + 1, u_{j+1} + u_{j+2} + 2, \dots, u_{j+1} + \dots + u_n + n - j, \rightarrow\}$ for each $j = 0, \dots, n - 1$,
- (iii) $G_j(S) = G(-s_j + S_j)$ for each $j = 0, \dots, n - 1$,
- (iv) $-s_j + S_j = \mathbb{N}_0 \setminus G_j(S)$ for each $j = 0, \dots, n - 1$.

Proof (iii) The assertion is true for $j = 0$. Assume $j \geq 1$ and let β be a box in $G_j(S)$. Let us denote the box in $G_0(S)$ in the same row as β by β_0 and let us denote the hook-length of that box by b_0 . Then $b_0 > s_j$. Assume that $s_{j+k} < b_0 < s_{j+k+1}$, $0 \leq k \leq n - j - 1$. The length of the row containing β is $j + k$. The number of boxes in the arm of β is j less than the number of boxes in the arm of β_0 while the number of boxes in the leg of the box β is $s_j - j$ less than the number of boxes in the leg of β_0 . Therefore, the hook-length of β is $b_0 - j - (s_j - j) = b_0 - s_j$. Hence,

$$G_j(S) = -s_j + \{b \in G_0(S) : b > s_j\} = \mathbb{N}_0 \setminus (-s_j + S_j) = G(-s_j + S_j),$$

proving the assertion. The fourth assertion is a restatement of the third. □

Corollary 3 Let $S = \{s_0 = 0, s_1, \dots, s_{n-1}, s_n = C, \rightarrow\}$ be a proper numerical set with the Young tableau $Y_S = 1^{u_1} \dots n^{u_n}$ and partition $\lambda_S = [\lambda_1, \dots, \lambda_n] \vdash N$. Then:

- (i) $G_0(S) = G(S)$,
- (ii) For each $j = 0, 1, \dots, n - 1$, the hook-length of the top box in $G_j(S)$ is $F(S) - s_j$,
- (iii) For each $j = 0, 1, \dots, n - 1$, the hook-length of the bottom box in $G_j(S)$ is $\min \{b \in G(S) : b > s_j\} - s_j$,
- (iv) $C = \lambda_1 + n, \lambda_j = g(S) - s_{j-1} + (j - 1)$ for each $j = 1, \dots, n$ and

$$N = n(g(S)) + \frac{n(n - 1)}{2} - \sum_{j=0}^{n-1} s_j.$$

Numerical semigroups can be characterized by their Young tableaux.

Lemma 4 Let $S = \{s_0 = 0, s_1, \dots, s_{n-1}, s_n = C, \rightarrow\}$ be a proper numerical set with the Young tableau Y_S . The following are equivalent:

- (i) S is a numerical semigroup,
- (ii) $x + y \notin G_0(S)$ for all $x, y \in S$,
- (iii) $G_j(S) \subset G_0(S)$ for all $j = 1, \dots, n - 1$.

Proof (i) \Leftrightarrow (ii) This follows from the definition and Corollary 3.

(ii) \Rightarrow (iii) Suppose that $G_j(S) \not\subset G_0(S)$ for some $j = 1, \dots, n - 1$. Consider $x \in G_j(S) \setminus G_0(S)$. Then $x \in S$ and $x = -s_j + y$ for some $y \in G_0(S)$ by Lemma 2(iii). It follows that $x + s_j = y \in G_0(S)$, which is a contradiction.

(iii) \Rightarrow (i) It is enough to show that $s_j + s_k \in S$ for $1 \leq j \leq k \leq n - 1$. Suppose $s_j + s_k \notin S$ for some j, k with $1 \leq j \leq k \leq n - 1$. Then $s_j + s_k \in G_0(S)$, and $s_k = -s_j + (s_j + s_k) \in G(-s_j + S_j) = G_j(S)$ by Lemma 2(iii), which is a contradiction. \square

Lemma 5 Let $S = \{s_0 = 0, s_1, \dots, s_{n-1}, s_n = C, \rightarrow\}$ be a proper numerical semigroup with the Young tableau Y_S . Then:

- (i) $S(i) = \bigcap_{j=i}^{n-1} (-s_j + S_j) = \mathbb{N}_0 \setminus \bigcup_{j=i}^{n-1} G_j(S), 1 \leq i \leq n - 1$,
- (ii) $T_n(S) = G_{n-1}(S)$ and $T_i(S) = G_{i-1}(S) \setminus \bigcup_{j=i}^{n-1} G_j(S), 1 \leq i \leq n - 1$.

Proof (i) We have

$$\begin{aligned} S(i) &= \{z \in \mathbb{N}_0 : z + s_j \in S \text{ for all } j = i, \dots, n - 1\} \\ &= \{z \in \mathbb{N}_0 : z \in -s_j + S_j \text{ for all } j = i, \dots, n - 1\} \\ &= \bigcap_{j=i}^{n-1} (-s_j + S_j) = \bigcap_{j=i}^{n-1} (\mathbb{N}_0 \setminus G_j(S)) = \mathbb{N}_0 \setminus \bigcup_{j=i}^{n-1} G_j(S). \end{aligned}$$

(ii) Since $S(i) = \mathbb{N}_0 \setminus \bigcup_{j=i}^{n-1} G_j(S)$ and $S(i-1) = \mathbb{N}_0 \setminus \bigcup_{j=i-1}^{n-1} G_j(S)$ by (i),

$$\begin{aligned} T_i(S) &= (\mathbb{N}_0 \setminus \bigcup_{j=i}^{n-1} G_j(S)) \setminus (\mathbb{N}_0 \setminus \bigcup_{j=i-1}^{n-1} G_j(S)) \\ &= G_{i-1}(S) \setminus \bigcup_{j=i}^{n-1} G_j(S) \end{aligned}$$

for each $i = 1, \dots, n-1$. Moreover,

$$T_n(S) = S(n) \setminus S(n-1) = \mathbb{N}_0 \setminus S(n-1) = G_{n-1}(S). \quad \square$$

Corollary 6 ([5] Proposition 1.5 and Proposition 1.7) *Let S be a proper numerical semigroup with the Young tableau $Y_S = 1^{u_1}2^{u_2} \dots n^{u_n}$ and t_n denote the n th term of the type sequence of S . Then:*

- (i) $t_n = u_n$,
- (ii) $t_{n-1} = \begin{cases} u_{n-1} + 1 & \text{if } u_{n-1} < u_n, \\ u_{n-1} & \text{if } u_{n-1} \geq u_n. \end{cases}$

Proof (i) $G_{n-1}(S) = \{1, 2, \dots, u_n\}$. Hence, $t_n = u_n$.

(ii) If $u_{n-1} < u_n$, then

$$G_{n-2}(S) = \{1, 2, \dots, u_{n-1}, \underline{u_{n-1} + 2, \dots, u_n, u_n + 1, \dots, u_{n-1} + 1 + u_n}\}.$$

Thus, $t_{n-1} = |G_{n-2}(S) \setminus G_{n-1}(S)| = u_{n-1} + 1$ if $u_{n-1} < u_n$. If $u_{n-1} \geq u_n$, then

$$G_{n-2}(S) = \{1, \dots, u_n, \underline{u_n + 1, u_n + 2, \dots, u_{n-1}, u_{n-1} + 2, \dots, u_{n-1} + 1 + u_n}\}.$$

Hence, $t_{n-1} = |G_{n-2}(S) \setminus G_{n-1}(S)| = u_{n-1}$ if $u_{n-1} \geq u_n$. □

3. Arf semigroups

An *Arf semigroup* is a numerical set that satisfies

$$x, y, z \in S, x \geq y \geq z \implies x + y - z \in S.$$

This is the definition given in [1] and the above condition is known as the *Arf condition*. \mathbb{N}_0 is an Arf numerical semigroup. Since every numerical set contains 0, the Arf condition implies that every Arf semigroup is a numerical semigroup. It is not difficult to see that a numerical set $S = \{s_0 = 0, s_1, \dots, s_{n-1}, s_n = C, \rightarrow\}$ satisfies the Arf condition if and only if the small elements of S satisfy it. In other words, S is an Arf numerical semigroup if and only if $s_i + s_j - s_k \in S$ for all $1 \leq k \leq j \leq i \leq n-1$. We also have the following lemma.

Lemma 7 *Let S be a numerical set and $a \in S$. Then S is an Arf semigroup if and only if $(a + S) \cup \{0\}$ is an Arf semigroup.*

Proof Assume that S is an Arf semigroup and $a \in S$. Let $x, y, z \in a + S$ with $x \geq y \geq z$. Then $x = a + \alpha, y = a + \beta, z = a + \gamma$, where $\alpha, \beta, \gamma \in S$ and $\alpha \geq \beta \geq \gamma$. Thus, $\alpha + \beta - \gamma \in S$ by definition. It follows that

$$x + y - z = (a + \alpha) + (a + \beta) - (a + \gamma) = a + (\alpha + \beta - \gamma) \in a + S,$$

proving the necessity. As for the sufficiency, assume that $(a + S) \cup \{0\}$ is an Arf semigroup and consider elements $x, y, z \in S$ with $x \geq y \geq z$. Then $a + (x + y - z) = (a + x) + (a + y) - (a + z) \in a + S$ and therefore $x + y - z \in S$, which proves that S is an Arf semigroup. \square

In [2], fifteen equivalent conditions are given for the Arf condition, among which the Arf condition as stated in [1] does not appear. The following theorem contains some of these conditions. We give the whole proof for the completeness and also for the convenience of the reader.

Theorem 8 *For a proper numerical set $S = \{s_0 = 0, s_1, \dots, s_{n-1}, s_n = C, \rightarrow\}$ with the Young tableau $Y_S = 1^{u_1}2^{u_2} \dots n^{u_n}$, the following are equivalent:*

- (i) S is an Arf semigroup,
- (ii) $-s_j + S_j$ is a numerical semigroup for all $j = 0, \dots, n - 1$,
- (iii) $-s_j + S_j = S(j)$ for all $j = 0, \dots, n$,
- (iv) $G_j(S) = G(S(j))$ for all $j = 0, \dots, n$,
- (v) $G_j(S) \subset G_{j-1}(S)$ for all $j = 1, \dots, n$,
- (vi) $u_j + 1 \in -s_j + S_j$ for all $j = 1, \dots, n - 1$.

Proof (i) \Rightarrow (ii) $S = -s_0 + S_0$ is a numerical semigroup by the hypothesis. Let $j \in \{1, \dots, n - 1\}$ and let $-s_j + a, -s_j + b \in -s_j + S_j$, where $a \geq s_j$ and $b \geq s_j$. Since S is an Arf semigroup, $(a + b - s_j) \in S$. In fact, $(a + b - s_j) \in S_j$. Then $(-s_j + a) + (-s_j + b) = -s_j + (a + b - s_j) \in -s_j + S_j$. This proves that the numerical set $-s_j + S_j$ is a numerical semigroup.

(ii) \Rightarrow (iii) Note that $-s_0 + S_0 = S = S(0)$. Let $j \in \{1, \dots, n\}$. The inclusion $S(j) \subseteq -s_j + S_j$ is trivial. As for the reverse inclusion, note that for any $a, b \in S_j$, there exists $c \in S_j$ such that $(-s_j + a) + (-s_j + b) = -s_j + c$. Thus, $(-s_j + a) + b = (-s_j + a) + (-s_j + b) + s_j = -s_j + c + s_j = c \in S_j$. Hence, $-s_j + a \in S(j)$, proving that $-s_j + S_j \subseteq S(j)$. So $-s_j + S_j = S(j)$.

(iii) \Rightarrow (iv) This follows from Lemma 2(iii).

(iv) \Rightarrow (v) For any $j = 1, \dots, n$, we have

$$G_j(S) = G(S(j)) = \mathbb{N}_0 \setminus S(j) \subset \mathbb{N}_0 \setminus S(j - 1) = G(S(j - 1)) = G_{j-1}(S).$$

(v) \Rightarrow (vi) For each $j = 1, \dots, n - 1$, we have

$$-s_j + S_j = \{0, u_{j+1} + 1, u_{j+1} + u_{j+2} + 2, \dots, u_{j+1} + \dots + u_n + n - j, \rightarrow\},$$

and by Lemma 2(iii)

$$-s_{j-1} + S_{j-1} = \mathbb{N}_0 \setminus G_{j-1}(S) \subset \mathbb{N}_0 \setminus G_j(S) = -s_j + S_j.$$

It follows that $u_j + 1 \in -s_j + S_j$.

(vi) \Rightarrow (i) Put $u_j + 1 = x_j$, $1 \leq j \leq n$. Then Lemma 2(i) implies

$$\begin{aligned} S &= \{0, x_1, x_1 + x_2, \dots, x_1 + x_2 + \dots + x_{n-1}, x_1 + \dots + x_{n-1} + x_n, \rightarrow\} \\ &= (x_1 + (x_2 + (\dots + (x_{n-1} + ((x_n + \mathbb{N}_0) \cup \{0\}) \cup \{0\}))) \dots) \cup \{0\} \cup \{0\}. \end{aligned}$$

Since \mathbb{N}_0 is an Arf semigroup, applying Lemma 7 repeatedly, we see that S is an Arf semigroup. \square

The last condition (vi) in Theorem 8 states that $\{u_1 + 1, \dots, u_n + 1\}$ is an Arf sequence in the sense of [7]. Therefore, determining the Young sequences $\{u_1, \dots, u_n\}$ for which $\{u_1 + 1, \dots, u_n + 1\}$ are Arf sequences contributes to determining Arf semigroups of genus $g = u_1 + \dots + u_n$.

Type sequences also characterize Arf semigroups.

Corollary 9 *A proper numerical semigroup is an Arf semigroup if and only if its Young sequence and type sequence are identical.*

Proof Let S be a numerical semigroup with the Young sequence $\{u_1, \dots, u_n\}$ and the type sequence $\{t_1, \dots, t_n\}$. Assume that S is an Arf semigroup. For each $j = 1, \dots, n - 1$, we have $G_j(S) \subset G_{j-1}(S)$ by Theorem 8(v). Thus, applying Lemma 5(ii), we get $T_i(S) = G_{i-1}(S) \setminus \bigcup_{j=i}^{n-1} G_j(S) = G_{i-1}(S) \setminus G_i(S)$ for $i = 1, \dots, n - 1$ and $T_n(S) = G_{n-1}(S)$. Therefore, $t_i = u_i$ for $i = 1, \dots, n$. As for the converse, suppose that $t_i = u_i$ for $i = 1, \dots, n$. We prove by backward induction that $G_{n-i}(S) \subset G_{n-i-1}(S)$ for all $i = 1, \dots, n - 1$. $G_{n-1} \subset G_{n-2}$, because otherwise we would have $t_{n-1} > u_{n-1}$ by Corollary 6. Now assume we have shown that $G_{n-k} \subset G_{n-k-1}$, $0 \leq k < n - 1$. If $G_{n-k-1}(S) \not\subset G_{n-k-2}(S)$, then Lemma 5(ii) would imply

$$T_{n-k-1}(S) = G_{n-k-2}(S) \setminus \bigcup_{j=n-k-1}^{n-1} G_j(S) = G_{n-k-2}(S) \setminus G_{n-k-1}(S),$$

and consequently, we would have $t_{n-k-1}(S) > u_{n-k-1}$, contradicting the hypothesis. Therefore, $G_j(S) \subset G_{j-1}(S)$ for all $j = 1, \dots, n - 1$ and S is an Arf semigroup by Theorem 8. \square

4. Arf closure and Arf partitions

The intersection of two Arf semigroups is an Arf semigroup and there are only a finite number of Arf semigroups (one of them being \mathbb{N}_0) containing a given numerical set. The smallest Arf semigroup containing a numerical set S is called the *Arf closure* of S and it is denoted by $\text{Arf}(S)$. The works in [1], [9], and [12] give procedures for finding the Arf closure of a given numerical semigroup. We give below a procedure for finding the Arf closure of a numerical set in terms of its Young sequence.

For the construction of the Arf closure, we will use Theorem 8 and the next four lemmas.

Lemma 10 ([12] Lemma 11) *If S is an Arf semigroup and $x, x + 1 \in S$, then the conductor $C \leq x$.*

Lemma 11 *Assume that S is a proper numerical set with the Young sequence $\{u_1, \dots, u_n\}$. If there exists $j \in \{1, \dots, n\}$ such that $u_j = 0$, then the numerical set S' with the Young sequence $\{u_1, \dots, u_{j-1}\}$ and S have the same Arf closure.*

Proof Since $u_j = 0$, we have $s_{j+1} = s_j + 1$. This implies $S \subseteq S' \subseteq \text{Arf}(S)$, and therefore $\text{Arf}(S) = \text{Arf}(S')$. \square

Lemma 12 *Assume that S is a proper numerical set with the Young sequence $\{u_1, u_2, \dots, u_n\}$. If there is $j \in \{1, \dots, n - 1\}$ such that $u_j > u_{j+1}$ and $u_j + 1 \notin -s_j + S_j$, let $k \in \{1, \dots, n - j\}$ such that*

$$-s_j + s_{j+k-1} + 1 < u_j + 1 < -s_j + s_{j+k};$$

define

$$v_i = \begin{cases} u_i & \text{if } i \leq j + k - 1, \\ u_j + s_j - s_{j+k-1} & \text{if } i = j + k, \\ -s_j + s_{j+k} - u_j - 2 & \text{if } i = j + k + 1, \\ u_{i-1} & \text{if } j + k + 2 \leq i \leq n + 1. \end{cases}$$

If $S' = \{s'_0, s'_1, \dots, s'_n, s'_{n+1} = C', \rightarrow\}$ is the numerical set with the Young sequence $\{v_1, \dots, v_n, v_{n+1}\}$, then S and S' have the same Arf closure.

Proof It is easy to see that

$$s'_i = \begin{cases} s_i & \text{if } i \leq j + k - 1, \\ s_j + u_j + 1 & \text{if } i = j + k, \\ s_{j+k} & \text{if } i = j + k + 1, \\ s_{i-1} & \text{if } i \geq j + k + 2. \end{cases}$$

It follows that $S' = S \cup \{s_j + u_j + 1\} = S \cup \{s_j + s_j - s_{j-1}\}$, and $S \subseteq S' \subseteq \text{Arf}(S)$, since $s_j + s_j - s_{j-1} \in \text{Arf}(S)$. Thus, $\text{Arf}(S) = \text{Arf}(S')$. □

Lemma 13 Assume that S is a proper numerical set with the Young sequence $\{u_1, u_2, \dots, u_n\}$. If there is $j \in \{1, \dots, n\}$ such that $u_{j+1} > u_j$ (thus, in that case, $u_j + 1 \notin -s_j + S_j$), define

$$z_i = \begin{cases} u_i & \text{if } i \leq j, \\ u_j & \text{if } i = j + 1, \\ u_{j+1} - u_j - 1 & \text{if } i = j + 2, \\ u_{i-1} & \text{if } j + 3 \leq i \leq n + 1. \end{cases}$$

If $S'' = \{s''_0, s''_1, \dots, s''_n, s''_{n+1} = C'', \rightarrow\}$ is the numerical set with the Young sequence $\{z_1, \dots, z_n, z_{n+1}\}$, then S and S'' have the same Arf closure.

Proof It is easy to see that

$$S' = S \cup \{s_j + u_j + 1\} = S \cup \{s_j + s_j - s_{j-1}\} \subseteq \text{Arf}(S).$$

Hence, $\text{Arf}(S) = \text{Arf}(S')$. □

Given a numerical set S with the Young sequence $\{u_1, u_2, \dots, u_n\}$, which is not an Arf semigroup, if $u_j = 0$ for some $j \in \{1, \dots, n\}$, then we let the smallest element in $\{1, \dots, n\}$ with the property $u_j = 0$ be n_1 . Then the numerical set S' with the Young sequence $\{u_1, u_2, \dots, u_{n_1}\}$ has the same Arf closure as S by Lemma 11. If S' is not an Arf semigroup, we apply Lemma 12 and/or Lemma 13 to S' . The proofs of Lemma 12 and Lemma 13 show that a finite number of applications of these lemmas will yield $\text{Arf}(S)$.

Example 14 The Young sequence of

$$S = \{0, 23, 29, 33, 37, 40, 46, 48, 49, 52, 56, \rightarrow\}$$

is given by

$$u_1 = 22, u_2 = 5, u_3 = 3, u_4 = 3, u_5 = 2, u_6 = 5, u_7 = 1, u_8 = 0, u_9 = 2, u_{10} = 3.$$

Since $u_8 = 0$, if S' is the numerical set with the Young sequence given by $u_1 = 22, u_2 = 5, u_3 = 3, u_4 = 3, u_5 = 2, u_6 = 5, u_7 = 1$, then $\text{Arf}(S) = \text{Arf}(S')$ by Lemma 11. Here $u_2 > u_3$ and $u_2 + 1 = 6 \notin -s'_2 + S'_2 = \{0, 4, 8, 11, 17, 19, \rightarrow\}$. Applying Lemma 12, the semigroup S'' with the Young sequence $v_1 = 22, v_2 = 5, v_3 = 3, v_4 = 1, v_5 = 1, v_6 = 2, v_7 = 5, v_8 = 1$ has the same Arf closure as S' , and thus as S . Here $v_5 < v_6$ and $v_5 + 1 = 2 \notin -s''_5 + S''_5 = \{0, 3, 9, 11, \rightarrow\}$. Applying Lemma 13, the numerical set S''' with the Young sequence given by $z_1 = 22, z_2 = 5, z_3 = 3, z_4 = 1, z_5 = 1, z_6 = 1, z_7 = 0, z_8 = 5, z_9 = 1$ has the same Arf closure as S . Finally we apply Lemma 11 once more and see that the numerical set with the Young sequence $\{22, 5, 3, 1, 1, 1\}$ has the same Arf closure as S . Theorem 8(vi) implies that the numerical set with the Young sequence $\{22, 5, 3, 1, 1, 1\}$ is Arf. Hence,

$$\text{Arf}(S) = \{0, 23, 29, 33, 35, 37, 39, \rightarrow\}.$$

□

There are other procedures and algorithms in the literature for the computation of the Arf closure. The next two examples appear in [9] and [12]. The reader can compare the computations given below with the computations given in the articles cited.

Example 15 ([9] Example 4.6) The Young sequence of the numerical semigroup $S = \langle 4, 10, 25 \rangle$ is $u_1 = 3, u_2 = 3, u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = u_9 = u_{10} = 1, u_{11} = u_{12} = 0, u_{13} = 1, u_{14} = 0, u_{15} = 0, u_{16} = 1$. Hence, Arf closure of S is the same as the Arf closure of the numerical set S' with the Young sequence

$$u_1 = 3, u_2 = 3, u_3 = 1, u_4 = 1, u_5 = 1, u_6 = 1, u_7 = 1, u_8 = 1, u_9 = 1, u_{10} = 1.$$

Theorem 8(vi) implies that S' is Arf. Thus,

$$\text{Arf}(S) = \{0, 4, 8, 10, 12, 14, 16, 18, 20, 22, 24, \rightarrow\}.$$

□

Example 16 ([12] Example 19) The Young sequence of the numerical semigroup $S = \langle 7, 24, 33 \rangle$ is $u_1 = 6, u_2 = 6, u_3 = 6, u_4 = 2, u_5 = 3, u_6 = 2, u_7 = 1, u_8 = 1, u_9 = 2, u_{10} = 1, u_{11} = 1, u_{12} = 2, u_{13} = 1, u_{14} = 0, u_{15} = 0, u_{16} = 2, u_{17} = 1, u_{18} = 0, u_{19} = 0, u_{20} = 0, u_{21} = 1, u_{22} = 1, u_{23} = 0, u_{24} = 0, u_{25} = 0, u_{26} = 1, u_{27} = 1, u_{28} = 0, u_{29} = 0, u_{30} = 0, u_{31} = 0, u_{32} = 0, u_{33} = 1$. Hence, Arf closure of S is the same as the Arf closure of the numerical set S' with the Young sequence $u_1 = 6, u_2 = 6, u_3 = 6, u_4 = 2, u_5 = 3, u_6 = 2, u_7 = 1, u_8 = 1, u_9 = 2, u_{10} = 1, u_{11} = 1, u_{12} = 2, u_{13} = 1$. Since $u_4 + 1 = 3 \notin -s_4 + S_4$, we apply Lemma 13 and see that Arf closure of S is the same as the Arf closure of the numerical set S'' with the Young sequence

$$v_1 = 6, v_2 = 6, v_3 = 6, v_4 = 2, v_5 = 2, v_6 = 0, \dots$$

It follows that

$$\text{Arf}(S) = \{0, 7, 14, 21, 24, 27, \rightarrow\}.$$

□

If $\lambda = [\lambda_1, \dots, \lambda_n] \vdash N$ is a partition, then the Young tableau corresponding to λ is $Y_\lambda = 1^{u_1} \dots n^{u_n}$, where $u_n = \lambda_n$ and $u_i = \lambda_i - \lambda_{i+1}$ for each $i = 1, \dots, n - 1$. The numerical set corresponding to λ is

$$S_\lambda = \{s_0 = 0, s_1 = u_1 + 1, u_1 + u_2 + 2, \dots, u_1 + u_2 + \dots + u_n + n, \rightarrow\}.$$

Note that S_λ is a proper numerical set with conductor $C = \lambda_1 + n$. The correspondence $\lambda \rightarrow S_\lambda$ is a bijection between the set of partitions of positive integers and the set of proper numerical sets. A partition λ is called an *Arf partition* if the numerical set S_λ is an Arf semigroup.

Every positive integer N has at least one Arf partition. For example, $\lambda = [N]$ is an Arf partition for any positive integer N . It corresponds to the Arf semigroup $\{0, N + 1, \rightarrow\}$. Determining the Arf partitions of positive integers is equivalent to determining Arf semigroups. Therefore, Arf partitions seem to deserve more investigation.

Proposition 17 $\lambda = [\lambda_1, \dots, \lambda_n]$ is an Arf partition if and only if

$$\lambda_j - \lambda_{j+1} + 1 \in \{\lambda_{j+1} - \lambda_{j+2} + 1, \lambda_{j+1} - \lambda_{j+3} + 2, \dots, \lambda_{j+1} - \lambda_n + n - j - 1, \lambda_{j+1} + n - j, \rightarrow\}$$

for all $j = 1, \dots, n - 1$.

Proof Let $\{u_1, \dots, u_n\}$ be the Young sequence of S_λ . Then λ is an Arf partition if and only if

$$u_j + 1 \in \{u_{j+1} + 1, u_{j+1} + u_{j+2} + 2, \dots, u_{j+1} + \dots + u_n + n - j, \rightarrow\}$$

for all $j = 1, \dots, n - 1$. The assertion is obvious after observing that $\lambda_j - \lambda_{j+1} = u_j$ and

$$\lambda_{j+1} - \lambda_{j+k} = u_{j+1} + u_{j+2} + \dots + u_{j+k-1}$$

for all $j = 1, \dots, n - 1$. □

Corollary 18 Let $\lambda = [\lambda_1, \dots, \lambda_n]$ be an Arf partition. Then:

- (i) $\lambda^{(j)} = [\lambda_{j+1}, \dots, \lambda_n]$ is an Arf partition for each $j = 1, \dots, n - 1$,
- (ii) For each $i = 2, \dots, n$ and $j = 1, 2, \dots, \lambda_i - 1$, the partition $\lambda^{(j,i)} = [\lambda_1 - j, \dots, \lambda_i - j]$ is an Arf partition.

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