An asymptotic-numerical hybrid method for singularly perturbed system of two-point reaction-diffusion boundary-value problems

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Abstract: This article focuses on the numerical approximate solution of singularly perturbed systems of second-order reaction-diffusion two-point boundary-value problems for ordinary differential equations. To handle these types of problems, a numerical-asymptotic hybrid method has been used. In this hybrid approach, an efficient asymptotic method, the so-called successive complementary expansion method (SCEM) is employed first, and then a numerical method based on finite differences is applied to approximate the solution of corresponding singularly perturbed reaction-diffusion systems. Two illustrative examples are provided to demonstrate the efficiency, robustness, and easy applicability of the present method with convergence properties.

Key words: Singular perturbation problems, reaction-diffusion equations, asymptotic approximations, boundary layers, finite difference method

1. Introduction

Systems of differential equations arise in many branches of science and engineering, such as modeling of electrical networks, mechanical systems, marketing problems, earthquake/tsunami problems, chemotaxis processes, and semiconductor physics. Because of their physical importance, many studies have been devoted to this area. In [11], systems of differential equations are studied in detail, and in [1, 3] some numerical treatments are examined. On the other hand, singularly perturbed differential equations that involve positive small perturbation parameter(s) $0 < \varepsilon \ll 1$ multiplied with the highest order derivative term(s) are also important concepts of applied sciences and it is a well-known fact that the standard numerical techniques are very often insufficient to handle them. Control theory, fluid mechanics, quantum mechanics, combustion theory, signal and image processing, and pharmacokinetics are some areas where singularly perturbed problems arise. One can find theoretical considerations regarding singular perturbation problems in [6, 9, 14, 23, 26, 29] and various approximation methods in [4, 5, 7, 12, 13, 16, 17, 30].

The present paper deals with obtaining accurate approximations to the solution of singularly perturbed systems of reaction-diffusion boundary-value problems (BVPs) that frequently arise in electroanalytical chemistry and population dynamics problems. In recent years, various methods were employed to obtain approximations to the solution of this kind of problem. In [2, 20], finite difference methods (FDMs) were used to obtain

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the approximate numerical solution of singularly perturbed convection-diffusion systems, while in [18, 19] finite element methods (FEMs) are used. In [10, 21, 25, 27] FDMs are examined for singularly perturbed reaction-diffusion systems. For singularly perturbed systems of reaction-diffusion BVPs, the authors provided a robust computational technique in [24] and obtained optimal order error estimates on equidistributed grids in [8].

In this paper, we apply an asymptotic-numerical hybrid method for approximating the solution of singularly perturbed systems of two-point boundary-value problems of reaction-diffusion type. At the first step, an efficient asymptotic method, known as the successive complementary expansion method (SCEM) and introduced in [22], is applied, and later the FDM given in [15] is applied to solve the system of ODEs obtained through the SCEM.

In order to examine the convergence properties of approximations to the solution of singularly perturbed systems of differential equations, we need to define the norm that we will use in the remaining parts of this study. In [9], one can find that the appropriate norm for this purpose is the maximum norm, which is given by:

$$\| r \| = \max_{\overline{\Omega}} |r(x)|, \quad \| \overrightarrow{r} \| = \max_{\overline{\Omega}} \{ \| r_i \| \},$$

where $\overrightarrow{r} = (r_1, r_2, \cdots, r_n)$.

The rest of the article is organized in the following manner: In Section 2, the continuous problem and its general properties are explained. In Section 3, the numerical-asymptotic hybrid method that we apply to solve the system of singularly perturbed two-point reaction-diffusion equations is described. In Section 4, two illustrative examples are provided to show the efficiency of the method. The paper ends with some conclusions in the last section.

2. The continuous problem

In general, a coupled system of singularly perturbed reaction-diffusion ODEs is given by

$$\begin{cases} -\varepsilon y_1''(x) + a_{11}(x) y_1(x) + a_{12}(x) y_2(x) = f_1(x), \\ -\eta y_2''(x) + a_{21}(x) y_1(x) + a_{22}(x) y_2(x) = f_2(x), \end{cases} \tag{2.1}$$

with certain suitable boundary conditions. The presence of the positive small parameters $0 < \varepsilon \ll 1$ and $0 < \eta \ll 1$ causes rapid and sharp changes (boundary layers) near the end-points of the domain $\overline{\Omega}$. The layer behaviors can be examined in three different cases:

Case A: $0 < \varepsilon < \eta \ll 1 \Rightarrow$ In this case, both components of the solution have boundary layers of width $O(\eta \ln \eta)$ and $y_1(x)$ has an additional sublayer of width $O(\varepsilon \ln \varepsilon)$.

Case B: $0 < \varepsilon \ll 1$ and $\eta = 1 \Rightarrow$ In this case, only the first component of the solution has a boundary layer of width $O(\varepsilon \ln \varepsilon)$.

Case C: $0 < \varepsilon = \eta \ll 1 \Rightarrow$ In this case, both components of the solution have boundary layers of width $O(\varepsilon \ln \varepsilon)$ (or equivalently $O(\eta \ln \eta)$).

We are interested only in the Case C and in finding an approximation to the solution $\overrightarrow{y}(x) \in C^2(\overline{\Omega})$ such that for all $x \in \Omega = (0, 1)$ for the problem

$$\begin{cases} -\varepsilon y_1''(x) + a_{11}(x) y_1(x) + a_{12}(x) y_2(x) = f_1(x), \\ -\varepsilon y_2''(x) + a_{21}(x) y_1(x) + a_{22}(x) y_2(x) = f_2(x), \\ y_1(0) = y_1(1) = 0, \quad y_2(0) = y_2(1) = 0. \end{cases} \tag{2.2}$$

The system of BVPs given by (2.2) can be rewritten in matrix form as

$$
\begin{cases}
L_\varepsilon \overrightarrow{y}(x) = \begin{bmatrix} -\varepsilon \frac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon \frac{d^2}{dx^2} \end{bmatrix} \overrightarrow{y}(x) + A(x)\overrightarrow{y}(x) = \overrightarrow{f}(x), \\
\overrightarrow{y}(0) = [y_1(0), y_2(0)]^T, \quad \overrightarrow{y}(1) = [y_1(1), y_2(1)]^T,
\end{cases}
\tag{2.3}
$$

where $A(x) = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix}$ and $\overrightarrow{f}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$. The functions $a_{ij}(x)$, $f_i(x) \in C^2(\Omega)$ for $i, j = 1, 2$.

Now we should impose two assumptions on problem (2.3). The first one is for strictly the diagonally dominance of matrix A and the second one is required in order to establish the maximum principle theorem:

Assumption 1. $a_{11}(x) > |a_{12}(x)|$ and $a_{22}(x) > |a_{21}(x)|$, for all $x \in \overline{\Omega}$,

Assumption 2. $a_{12}(x) \leq 0$ and $a_{21}(x) \leq 0$, for all $x \in \overline{\Omega}$.

The proof of the following lemma and the corollary can be seen in [21].

Lemma 2.1 *(Maximum Principle) Consider the system of singularly perturbed BVPs (2.3). If $\overrightarrow{y}(0) \geqslant \overrightarrow{0}$, $\overrightarrow{y}(1) \geqslant \overrightarrow{0}$, and $L_\varepsilon \overrightarrow{y}(x) \geqslant \overrightarrow{0}$ for all $x \in \Omega$, then $\overrightarrow{y}(x) \geqslant \overrightarrow{0}$, $\forall x \in \overline{\Omega}$.*

Corollary 2.2 *(Stability) If $\overrightarrow{y}(x)$ is the solution of (2.3), then the stability bound inequality*

$$
\|\overrightarrow{y}(x)\| \leq \frac{1}{\delta}\|\overrightarrow{f}\| + \|\overrightarrow{y}(0)\| + \|\overrightarrow{y}(1)\|
$$

holds, where $\delta = \min_{\overline{\Omega}} \{a_{11}(x) + a_{12}(x), a_{21}(x) + a_{22}(x)\}$.

Under the above-mentioned assumptions and conditions, the hybrid method is explained in the following section.

3. The hybrid method

In this section, we give a brief overview of asymptotic expansions and approximations first, then explain the hybrid method by which we obtain highly accurate approximations to the solution of systems of singularly perturbed reaction-diffusion equations.

Let E be the set of all real-valued and strictly positive functions that are continuous in an interval $(0, \varepsilon_0]$ and depend on ε. Moreover, let the limit $\lim_{\varepsilon \to 0} \delta(\varepsilon)$ exist, and for each $\delta_1(\varepsilon), \delta_2(\varepsilon) \in E$, let the relation $\delta_1(\varepsilon)\delta_2(\varepsilon) \in E$ hold. A function $\delta_i(\varepsilon)$ that satisfies these conditions is called an order function. Given that two functions $\phi(x, \varepsilon)$ and $\phi_a(x, \varepsilon)$ defined in a domain $\overline{\Omega}$ are asymptotically identical to order $\delta(\varepsilon)$ if their difference is asymptotically smaller than $\delta(\varepsilon)$, where $\delta(\varepsilon)$ is an order function, in mathematical terms, this fact is expressed by

$$
\phi(x, \varepsilon) - \phi_a(x, \varepsilon) = o(\delta(\varepsilon)),
\tag{3.1}
$$

where ε is a small parameter arising from the physical problem under consideration. The function $\phi_a(x, \varepsilon)$ is referred to as an asymptotic approximation of the function $\phi(x, \varepsilon)$.

Asymptotic approximations, in their general form, are defined by

$$\phi_a(x, \varepsilon) = \sum_{i=1}^{n} \delta_i(\varepsilon)\varphi_i(x, \varepsilon), \tag{3.2}$$

where the asymptotic sequences of order functions $\delta_i(\varepsilon)$ satisfy the condition $\delta_{i+1}(\varepsilon) = o(\delta_i(\varepsilon))$, as $\varepsilon \to 0$. Under these conditions, approximation (3.2) is called a generalized asymptotic expansion. If expansion (3.2) is given in the form of

$$\phi_a(x, \varepsilon) = E_0\phi = \sum_{i=1}^{n} \delta_i^{(0)}(\varepsilon)\varphi_i^{(0)}(x), \tag{3.3}$$

then it is called a regular asymptotic expansion where the special operator E_0 is an outer expansion operator at a given order $\delta(\varepsilon)$, with $\phi - E_0\phi = o(\delta(\varepsilon))$. Interesting cases occur when the function ϕ is not sufficiently regular in $\overline{\Omega}$. Therefore, (3.2) or/and (3.3) is/are valid only in a restricted region $\overline{\Omega}_0 \in \overline{\Omega}$, called the outer region. This is a singular perturbation problem and we shall introduce boundary layer domains. We introduce an inner domain, which can be formally denoted by $\overline{\Omega}_1 = \overline{\Omega} \setminus \overline{\Omega}_0$, and a corresponding inner layer variable located near the point $x = x_0$, as $\overline{x} = \frac{x-x_0}{\xi(\varepsilon)}$, where $\xi(\varepsilon)$ is the order of thickness of this boundary layer. If a regular expansion can be constructed in $\overline{\Omega}_1$, it can be written as

$$\phi_a(x, \varepsilon) = E_1\phi = \sum_{i=1}^{n} \delta_i^{(1)}(\varepsilon)\varphi_i^{(1)}(\overline{x}), \tag{3.4}$$

where the inner expansion operator E_1 is defined in $\overline{\Omega}_1$ at the same order of $\delta(\varepsilon)$ as the outer expansion operator E_0. Therefore, $\phi - E_1\phi = o(\delta(\varepsilon))$ holds and it is clear that

$$\phi_a = E_0\phi + E_1\phi - E_1E_0\phi$$

is a uniformly valid approximation (UVA) [4–6]. For only one singularly perturbed differential equation, the uniformly valid SCEM approximation is in the regular form given by

$$y_n^{scem}(x, \overline{x}, \varepsilon) = \sum_{i=1}^{n} \delta_i(\varepsilon)\left[y_i(x) + \Psi_i(\overline{x})\right], \tag{3.5}$$

where $\{\delta_i(\varepsilon)\}$ is an asymptotic sequence and functions $\Psi_i(\overline{x})$ are the complementary functions that depend on $\overline{x}$. If the functions $y_i(x)$ and $\Psi_i(\overline{x})$ depend also on ε, the uniformly valid SCEM approximation is called generalized SCEM approximation and given by

$$y_{ng}^{scem}(x, \overline{x}, \varepsilon) = \sum_{i=1}^{n} \delta_i(\varepsilon)\left[y_i(x, \varepsilon) + \Psi_i(\overline{x}, \varepsilon)\right]. \tag{3.6}$$

Since the solution of problem (2.3) exhibits boundary layers at both end points of the interval $\Omega = (0, 1)$, the generalized SCEM approximation is adopted as follows:

For the outer region, i.e. far from the end points, the outer approximation will be in the following form:

$$\overrightarrow{y}^{out}(x, \delta) = \begin{bmatrix} y_1^{out}(x, \delta(\varepsilon)) \\ y_2^{out}(x, \delta(\varepsilon)) \end{bmatrix} = \begin{bmatrix} y_1^{out(1)}(x) + \delta(\varepsilon)\, y_1^{out(2)}(x) + \delta^2(\varepsilon)\, y_1^{out(3)}(x) + \cdots \\ y_2^{out(2)}(x) + \delta(\varepsilon)\, y_2^{out(2)}(x) + \delta^2(\varepsilon)\, y_2^{out(3)}(x) + \cdots \end{bmatrix}. \tag{3.7}$$

If equation (3.7) is substituted into (2.3) and the powers of ε are balanced, one gets the asymptotic approximation for the outer region. In order to cope with the approximation difficulties at the end points, stretching variables will be introduced as $\overline{x}_L = \frac{x}{\sqrt{\varepsilon}}$ for the left end and $\overline{x}_R = \frac{x-1}{\sqrt{\varepsilon}}$ for the right end. Using these transformations with the help of the chain rule and substituting them into equation (2.3), one obtains the complementary functions as the solution to following subproblems:

$$
\begin{cases}
-\left(\Psi_1^{Lcomp}\right)''(\overline{x}_L, \varepsilon) + a_{11}(\overline{x}_L)\Psi_1^{Lcomp}(\overline{x}_L, \varepsilon) + a_{12}(\overline{x}_L)\Psi_2^{Lcomp}(\overline{x}_L, \varepsilon) = f_1(\overline{x}_L), \\
-\left(\Psi_2^{Lcomp}\right)''(\overline{x}_L, \varepsilon) + a_{21}(\overline{x}_L)\Psi_1^{Lcomp}(\overline{x}_L, \varepsilon) + a_{22}(\overline{x}_L)\Psi_2^{Lcomp}(\overline{x}_L, \varepsilon) = f_2(\overline{x}_L), \\
\Psi_1^{Lcomp}(0, \varepsilon) = -y_1^{out(}(0), \quad \Psi_1^{Lcomp}\left(\frac{1}{\sqrt{\varepsilon}}, \varepsilon\right) = -y_1^{out}(1), \\
\Psi_2^{Lcomp}(0, \varepsilon) = -y_2^{out(}(0), \quad \Psi_2^{Lcomp}\left(\frac{1}{\sqrt{\varepsilon}}, \varepsilon\right) = -y_2^{out}(1),
\end{cases}
\tag{3.8}
$$

and

$$
\begin{cases}
-\left(\Psi_1^{Rcomp}\right)''(\overline{x}_R, \varepsilon) + a_{11}(\overline{x}_R)\Psi_1^{Rcomp}(\overline{x}_R, \varepsilon) + a_{12}(\overline{x}_R)\Psi_2^{Rcomp}(\overline{x}_R, \varepsilon) = f_1(\overline{x}_R), \\
-\left(\Psi_2^{Rcomp}\right)''(\overline{x}_R, \varepsilon) + a_{21}(\overline{x}_R)\Psi_1^{Rcomp}(\overline{x}_R, \varepsilon) + a_{22}(\overline{x}_R)\Psi_2^{Rcomp}(\overline{x}_R, \varepsilon) = f_2(\overline{x}_R), \\
\Psi_1^{Rcomp}\left(\frac{-1}{\sqrt{\varepsilon}}, \varepsilon\right) = -y_1^{out}(0), \quad \Psi_1^{Rcomp}(0, \varepsilon) = -y_1^{out}(1), \\
\Psi_2^{Rcomp}\left(\frac{-1}{\sqrt{\varepsilon}}, \varepsilon\right) = -y_2^{out}(0), \quad \Psi_2^{Rcomp}(0, \varepsilon) = -y_2^{out}(1),
\end{cases}
\tag{3.9}
$$

where the subscripts $1, 2$ denote the first and second components of the approximations. If an asymptotic approximation for the left complementary function (3.8) and the right complementary function (3.9) is adopted in the following form,

$$
\Psi(\overline{x}, \varepsilon) = \begin{bmatrix} \Psi_1(\overline{x}, \delta(\varepsilon)) \\ \Psi_2(\overline{x}, \delta(\varepsilon)) \end{bmatrix} = \begin{bmatrix} \Psi_1^{(1)}(\overline{x}, \delta(\varepsilon)) + \delta(\varepsilon)\Psi_1^{(2)}(\overline{x}, \delta(\varepsilon)) + \delta^2(\varepsilon)\Psi_1^{(3)}(\overline{x}, \delta(\varepsilon)) + \cdots \\ \Psi_2^{(1)}(\overline{x}, \delta(\varepsilon)) + \delta(\varepsilon)\Psi_2^{(2)}(\overline{x}, \delta(\varepsilon)) + \delta^2(\varepsilon)\Psi_2^{(3)}(\overline{x}, \delta(\varepsilon)) + \cdots \end{bmatrix},
$$

where the superscript (i) denotes the ith complementary approximation, then substituting this approximation into problem (2.3), one gets the asymptotic approximations for complementary functions, again balancing them with respect to the powers of parameter ε. To this end, the first iteration of the hybrid method is of the following form:

$$
\overrightarrow{y}^{(1)}(x, \overline{x}_L, \overline{x}_R, \varepsilon) = \overrightarrow{y}^{out(1)}(x, \varepsilon) + \left[\frac{\overrightarrow{\Psi}^{Lcomp(1)}(\overline{x}_L, \varepsilon) + \overrightarrow{\Psi}^{Rcomp(1)}(\overline{x}_R, \varepsilon)}{2}\right],
\tag{3.10}
$$

where the superscripts $Lcomp(1)$ and $Rcomp(1)$ are the first approximation terms of complementary functions (3.8) and (3.9), respectively, that are numerically solved by the numerical method given in [15] based on finite differences and that uses the three-stage Lobatto IIIa formula.

4. Illustrative examples

In this section, two numerical examples are studied. In the first one, all the processes are explained in detail, and for the second one, only results are presented. All the computations are performed in the MATLAB2015B environment using double precision.

Example 4.1 *[28] Consider the singularly perturbed system of coupled reaction-diffusion two-point BVPs:*

$$\begin{cases} -\varepsilon y_1''(x) + 4y_1(x) - 2y_2(x) = 1, & x \in \Omega = (0,1) \\ -\varepsilon y_2''(x) - y_1(x) + 3y_2(x) = 2, \\ y_1(0) = y_1(1) = 0, & y_2(0) = y_2(1) = 0. \end{cases} \tag{4.1}$$

As shown in Figure 1, the solution of Example 4.1 exhibits boundary layer behavior at both end points of the interval when $\varepsilon \to 0^+$. Therefore, for both end points, the stretching variables should be introduced as $\overline{x}_L = \frac{x}{\sqrt{\varepsilon}}$ for the left-end and $\overline{x}_R = \frac{x-1}{\sqrt{\varepsilon}}$ for the right-end, respectively. First, however, the reduced problem should be obtained by taking $\varepsilon = 0$:

$$\begin{cases} 4y_1^{out(1)}(x) - 2y_2^{out(1)}(x) = 1, \\ -y_1^{out(1)}(x) + 3y_2^{out(1)}(x) = 2, \end{cases} \tag{4.2}$$

and it is obvious that the solution of the reduced system (4.2) is $y_1^{out(1)}(x) = 0.7$ and $y_2^{out(1)}(x) = 0.9$, where the superscript $out(i)$ denotes the ith approximation to the outer layer problem. For the left and right inner layer problems, adopting the stretching variables $\overline{x}_L$ and $\overline{x}_R$ respectively, one gets the systems

$$\begin{cases} -\left(\Psi_1^{Lcomp(1)}\right)''(\overline{x}_L, \varepsilon) + 4\Psi_1^{Lcomp(1)}(\overline{x}_L, \varepsilon) - 2\Psi_2^{Lcomp(1)}(\overline{x}_L, \varepsilon) = 1, \\ -\left(\Psi_2^{Lcomp(1)}\right)''(\overline{x}_L, \varepsilon) - \Psi_1^{Lcomp(1)}(\overline{x}_L, \varepsilon) + 3\Psi_2^{Lcomp(1)}(\overline{x}_L, \varepsilon) = 2, \\ \Psi_1^{Lcomp(1)}(0, \varepsilon) = -0.7, \quad \Psi_1^{Lcomp(1)}\left(\frac{1}{\sqrt{\varepsilon}}, \varepsilon\right) = -0.7, \\ \Psi_2^{Lcomp(1)}(0, \varepsilon) = -0.9, \quad \Psi_2^{Lcomp(1)}\left(\frac{1}{\sqrt{\varepsilon}}, \varepsilon\right) = -0.9, \end{cases} \tag{4.3}$$

and

$$\begin{cases} -\left(\Psi_1^{Rcomp(1)}\right)''(\overline{x}_R, \varepsilon) + 4\Psi_1^{Rcomp(1)}(\overline{x}_R, \varepsilon) - 2\Psi_2^{Rcomp(1)}(\overline{x}_R, \varepsilon) = 1, \\ -\left(\Psi_2^{Rcomp(1)}\right)''(\overline{x}_R, \varepsilon) - \Psi_1^{Rcomp(1)}(\overline{x}_R, \varepsilon) + 3\Psi_2^{Rcomp(1)}(\overline{x}_R, \varepsilon) = 2, \\ \Psi_1^{Rcomp(1)}\left(\frac{-1}{\sqrt{\varepsilon}}, \varepsilon\right) = -0.7, \quad \Psi_1^{Rcomp(1)}(0, \varepsilon) = -0.7, \\ \Psi_2^{Rcomp(1)}\left(\frac{-1}{\sqrt{\varepsilon}}, \varepsilon\right) = -0.9, \quad \Psi_2^{Rcomp(1)}(0, \varepsilon) = -0.9. \end{cases} \tag{4.4}$$

In order to apply the numerical method [15] and code the problem in MATLAB, we should transform these new systems into first-order systems. Transformed systems (4.3)–(4.4) corresponding to problem (4.1) can be coded using MATLAB BVP4C as given by Listing 1-3:

The system of equations 4.4 is transformed into systems of first-order equations and these corresponding systems are stored in a TWOODE function as shown in Listing 1. Later, the boundary conditions are imposed by the function TWOBC as given in Listing 2. In the final step, numerical approximations and corresponding plots are obtained running the code that is given in Listing 3.

The double mesh principle is used for estimating the maximum point-wise errors and for computing the rate of convergence in the computed approximations:

$$D_{\varepsilon,i}^N = \max_{x_j \in \overline{\Omega}_\varepsilon^N} |Y_i^{2N}(x_j) - Y_i^N(x_j)|$$

```matlab
function dydx = twoode(x,y)
    dydx = [y(2)
        -1+4*y(1)-2*y(3)
        y(4)
        -2-y(1)+3*y(3)];
```

Listing 1. TWOODE component of MATLAB BVP4C

```matlab
function res = twobc(ya,yb)
    res = [ ya(1)+7/10
        yb(1)+7/10
        ya(3)+9/10
        yb(3)+9/10];
```

Listing 2. TWOBC component of MATLAB BVP4C

```matlab
ep=1;
solinit1 = bvpinit(linspace(0,1,1024),[1 0 1 0]);
sol1 = bvp4c(@twoode1,@twobc1,solinit1);
x1=0:0.001:1;
double(x1);
y1= deval(sol1,x1);
double(y1);
yg=y1(1,:);
plot(x1,yg);
hold on
y13=y1(3,:);
plot(x1,y13)
grid on
hold on
```

Listing 3. The solver that runs the components TWOBC and TWOODE of MATLAB BVP4C

for $i = 1, 2$ and $D_i^N = \max_\varepsilon D_{\varepsilon,i}^N$, where $Y_i^N(x_j)$ and $Y_i^{2N}(x_j)$ denote the computed approximations at the point $x = x_j$, on N and $2N$ mesh sizes, respectively. The order of convergence is calculated by the formula $p_i = \log_2\left(\frac{D_i^N}{D_i^{2N}}\right)$.
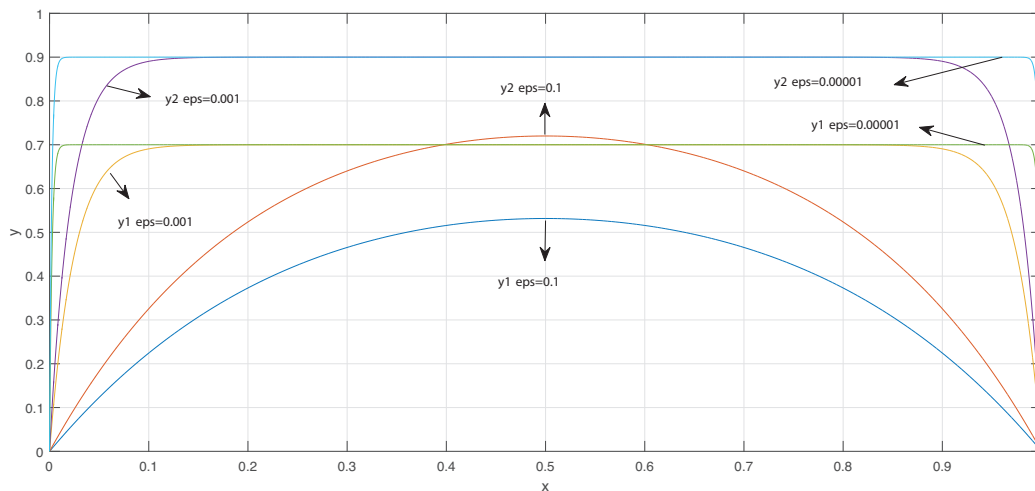

Figure 1. SCEM approximations to Example 4.1 for certain values of ε.

Numerical approximations to y_1 and y_2 of Example 4.1 that are generated by the hybrid method are presented for various ε values in Table 1 and Table 2, respectively. In Figure 2 and Table 3, the finer mesh errors that are based the on double-mesh principle are illustrated.

Table 1. Approximations to y_1 of Example 4.1 for various values of ε, and $N = 1024$.

x	y_1^{hybrid} for $\varepsilon = 1$	y_1^{hybrid} for $\varepsilon = 0.01$	y_1^{hybrid} for $\varepsilon = 0.0001$
0.000	0.000000000000000	0.000000000000000	0.000000000000000
0.001	0.000476481355558	0.008753818527279	0.083181084554497
0.003	0.001426444995724	0.025964925472142	0.222962651683962
0.070	0.030951927479662	0.418207225230131	0.699958187708815
0.090	0.038906391001189	0.484445481559528	0.699997527634950
0.100	0.042736593719825	0.511650798815475	0.699999398893554
0.300	0.099085092371664	0.688146304878580	0.700000000000000
0.500	0.117696173594857	0.698588175505725	0.700000000000000
0.700	0.099085092371662	0.688146304878580	0.700000000000000
0.900	0.042736593719823	0.511650798815475	0.699999398893554
0.910	0.038906391001187	0.484445481559528	0.699997527634950
0.930	0.030951927479661	0.418207225230131	0.699958187708815
0.997	0.001426444995724	0.025964925472143	0.222962651683962
0.999	0.000476481355559	0.008753818527279	0.083181084554497
1.000	0.000000000000000	0.000000000000000	0.000000000000000

Table 2. Approximations to y_2 of Example 4.1 for various values of ε and $N = 1024$.

x	y_2^{hybrid} for $\varepsilon = 1$	y_2^{hybrid} for $\varepsilon = 0.01$	y_2^{hybrid} for $\varepsilon = 0.0001$
0.000	0.000000000000000	0.000000000000000	0.000000000000000
0.001	0.000836831145262	0.013176322987849	0.123255082602135
0.003	0.002504501565593	0.038941217642193	0.320704685367825
0.070	0.053859654789803	0.576399973393426	0.899958155847379
0.090	0.067538818218910	0.657713587952894	0.899997527270993
0.100	0.074102135126685	0.690275199692044	0.899999398854655
0.300	0.168930685338889	0.887902110952330	0.900000000000000
0.500	0.199554186019598	0.898582598753880	0.900000000000000
0.700	0.168930685338888	0.887902110952330	0.900000000000000
0.900	0.074102135126684	0.690275199692044	0.899999398854655
0.910	0.067538818218910	0.657713587952894	0.899997527270993
0.930	0.053859654789803	0.576399973393426	0.899958155847379
0.997	0.002504501565593	0.038941217642193	0.320704685367825
0.999	0.000836831145262	0.013176322987849	0.123255082602135
1.000	0.000000000000000	0.000000000000000	0.000000000000000

Table 3. D_1^N and p_1^N for Example 4.1 for various values of ε and N.

| ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	$6.12679e-10$	$3.70626e-11$	$2.27984e-12$	$1.40748e-13$	$6.27276e-15$
2^{-2}	$1.81544e-09$	$1.09791e-10$	$6.76309e-12$	$4.16611e-13$	$3.53050e-14$
2^{-3}	$6.22012e-09$	$3.76598e-10$	$2.31717e-11$	$1.43662e-12$	$8.91509e-14$
2^{-4}	$2.47671e-08$	$1.50369e-09$	$9.25755e-11$	$5.73113e-12$	$3.58879e-13$
2^{-5}	$1.01353e-07$	$6.14983e-09$	$3.78493e-10$	$2.34682e-11$	$1.45650e-12$
2^{-6}	$4.03660e-07$	$2.46994e-08$	$1.52268e-09$	$9.43695e-11$	$5.87935e-12$
2^{-7}	$1.61007e-06$	$9.89155e-08$	$6.09315e-09$	$3.77304e-10$	$2.34998e-11$
2^{-8}	$6.44894e-06$	$3.92055e-07$	$2.43357e-08$	$1.51185e-09$	$9.41339e-11$
2^{-9}	$2.43061e-05$	$1.56232e-06$	$9.74256e-08$	$6.04717e-09$	$3.76837e-10$
2^{-10}	$4.88247e-07$	$6.24635e-06$	$3.86149e-07$	$2.41664e-08$	$1.50718e-09$
2^{-11}	$1.42363e-06$	$2.36149e-05$	$1.53878e-06$	$9.66679e-08$	$6.02468e-09$
2^{-12}	$4.88247e-07$	$2.39459e-07$	$6.14740e-06$	$3.83279e-07$	$2.40598e-08$
2^{-13}	$2.20197e-05$	$7.04572e-07$	$2.32768e-05$	$1.52714e-06$	$9.62998e-08$
2^{-14}	$1.92995e-05$	$2.39028e-07$	$1.18093e-07$	$6.09883e-06$	$3.81771e-07$
2^{-15}	$4.12438e-07$	$2.16196e-05$	$3.43788e-07$	$2.31095e-05$	$1.52132e-06$
D_1^N	$2.43061e-05$	$2.36149e-05$	$2.32768e-05$	$2.31095e-05$	$1.52132e-06$
p_1^N	0.0416210001	0.0208046902	0.0312113804	3.9250904939	3.9889941035
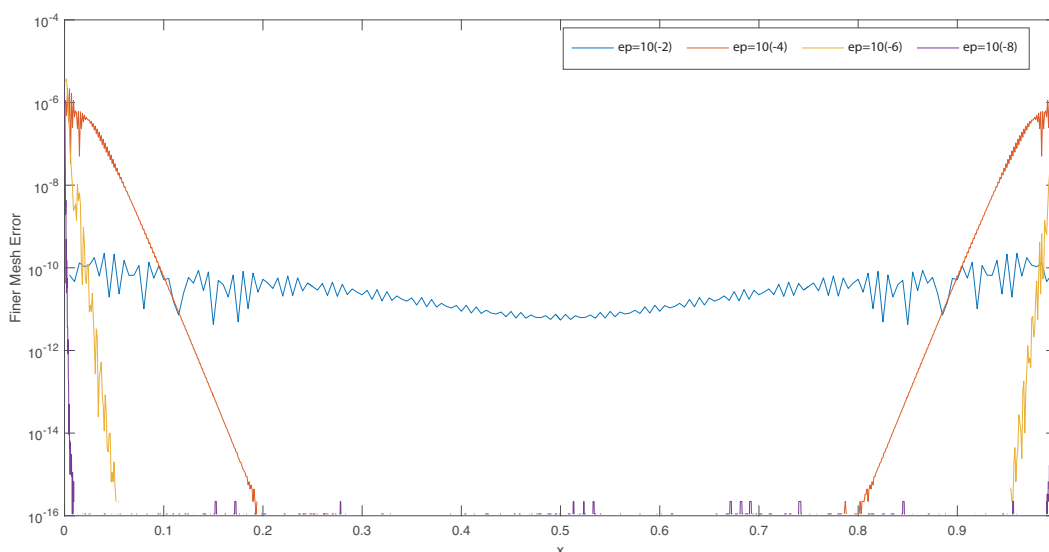
Figure 2. Errors in y_1 approximations of Example 4.1 for various values of ε.

Example 4.2 [18] *Consider the singularly perturbed system of coupled reaction-diffusion two-point BVPs:*

$$
\begin{cases}
-\varepsilon y_1''(x) + 3y_1(x) - y_2(x) - y_3(x) = 0, \\
-\varepsilon y_2''(x) - y_1(x) + 3y_2(x) - y_3(x) = 1, \\
-\varepsilon y_3''(x) - y_2(x) + 3y_3(x) = x, \\
y_1(0) = y_1(1) = 0, \quad y_2(0) = y_2(1) = 0, \quad y_3(0) = y_3(1) = 0.
\end{cases}
\tag{4.5}
$$

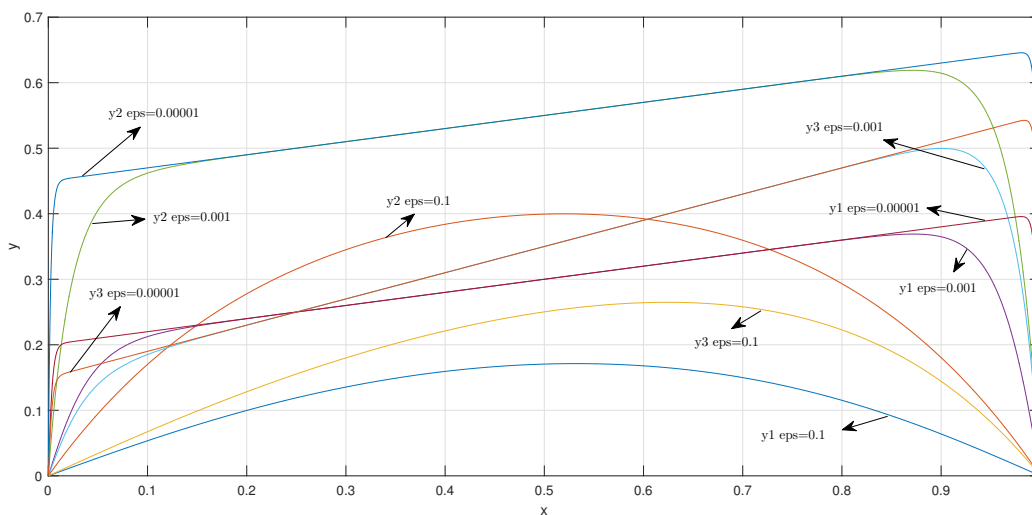The solution of this problem exhibits boundary layer behavior in all the components y_1, y_2, and y_3.

Figure 3. SCEM approximations to Example 4.2 for certain values of ε.

Table 4. Approximations to y_3 of Example 4.2 for various values of ε, and $N = 1024$.

x	y_3^{hybrid} for $\varepsilon = 1$	y_3^{hybrid} for $\varepsilon = 0.01$	y_3^{hybrid} for $\varepsilon = 0.0001$
0.000	0.000000000000000	0.000000000000000	0.000000000000000
0.001	0.000296931914224	0.001555243868093	0.011907872130351
0.003	0.000445395437314	0.004664568760707	0.034807883220829
0.070	0.010337895722462	0.098994889589599	0.177946516870041
0.090	0.013246127387130	0.121783382519711	0.185994913258379
0.100	0.014688076281396	0.132245460929406	0.189998430525371
0.300	0.040416458359500	0.263967353662464	0.270000000000000
0.500	0.055600510502286	0.348297797967831	0.350000000000000
0.700	0.053365373735929	0.417475745785387	0.430000000000000
0.900	0.025982747779700	0.360670915745580	0.509996860535433
0.910	0.023793671058585	0.344253981906367	0.513989822709124
0.930	0.019152362750862	0.302349198379968	0.521892825852710
0.997	0.000917731428012	0.020288336492293	0.178812725352010
0.999	0.000306908478447	0.006860930176790	0.067932980344570
1.000	0.000000000000000	0.000000000000000	0.000000000000000

Numerical approximations to y_1, y_2 and y_3 of Example 4.2 that are generated by the hybrid method are presented for various ε values in Figure 3. In Table 4, approximations to component y_3 are considered only and in Table 5 and Figure 4 corresponding finer mesh errors for y_3 are given.

5. Conclusions

In this paper, singularly perturbed systems of two-point boundary-value problems of reaction-diffusion type are examined. In order to obtain better approximations to the solution of this kind of problem, an asymptotic-numerical hybrid method that consists of an asymptotic method, known as SCEM, and a numerical method based on finite differences given in [15] is applied. In Section 4, the implementation of the present method is given in detail in the illustrative Example 4.1. In Table 1, Table 2, and Table 4, numerical results obtained by the hybrid method and in Table 3 and Table 5 maximum point-wise errors and convergence based on finer

Table 5. D_3^N and p_3^N for Example 4.2, for various values of ε and N.

| ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-1}	$3.97376e - 08$	$2.40516e - 09$	$1.48046e - 10$	$9.04537e - 12$	$5.14129e - 13$
2^{-2}	$1.60003e - 07$	$9.68587e - 09$	$5.95688e - 10$	$3.69476e - 11$	$2.45948e - 12$
2^{-3}	$4.55184e - 07$	$2.75605e - 08$	$1.69540e - 09$	$1.05172e - 10$	$6.50464e - 12$
2^{-4}	$1.86005e - 06$	$1.12657e - 07$	$6.93077e - 09$	$4.29730e - 10$	$2.67455e - 11$
2^{-5}	$5.30766e - 06$	$3.21668e - 07$	$1.97922e - 08$	$1.22731e - 09$	$7.63886e - 11$
2^{-6}	$7.19988e - 05$	$1.28455e - 06$	$7.90643e - 08$	$4.90311e - 09$	$3.05273e - 10$
2^{-7}	$8.50572e - 05$	$3.62682e - 06$	$2.23376e - 07$	$1.38547e - 08$	$8.62547e - 10$
2^{-8}	$1.48198e - 04$	$4.15936e - 05$	$8.93032e - 07$	$5.54076e - 08$	$3.44981e - 09$
2^{-9}	$1.20331e - 04$	$4.88494e - 05$	$2.52367e - 06$	$1.56680e - 07$	$9.75681e - 09$
2^{-10}	$1.12466e - 05$	$1.02047e - 04$	$2.89999e - 05$	$6.26449e - 07$	$3.90227e - 08$
2^{-11}	$1.42610e - 05$	$8.24795e - 05$	$3.41249e - 05$	$1.77034e - 06$	$1.10348e - 07$
2^{-12}	$1.25522e - 04$	$3.92817e - 06$	$7.24787e - 05$	$1.73510e - 05$	$4.41207e - 07$
2^{-13}	$7.95760e - 05$	$4.97760e - 06$	$7.11641e - 05$	$3.54793e - 05$	$1.24685e - 06$
2^{-14}	$5.17968e - 05$	$8.69193e - 05$	$1.41263e - 06$	$5.00248e - 05$	$1.43538e - 05$
2^{-15}	$5.35070e - 06$	$5.53248e - 05$	$1.76076e - 06$	$5.00871e - 05$	$2.50118e - 05$
D_3^N	$1.48198e - 04$	$1.02047e - 04$	$7.24787e - 05$	$5.00871e - 05$	$2.50118e - 05$
p_3^N	0.5382881157	0.4936047862	0.5331179957	1.0018301969	4.7990910350
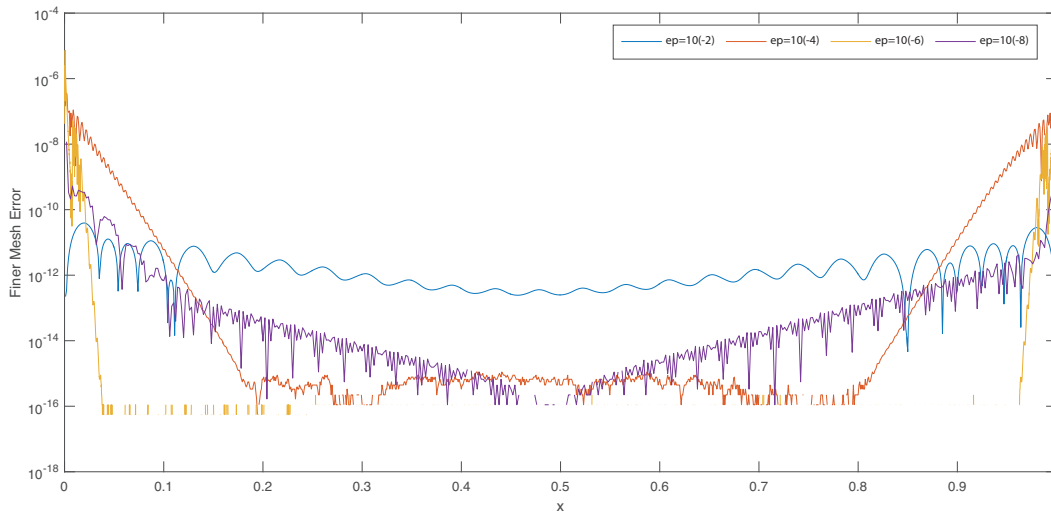
Figure 4. Errors in y_3 approximations of Example 4.2 for various values of ε.

mesh strategy are presented. In addition, approximations for certain values of ε are given in Figure 1 and Figure 3 to illustrate the layer behavior. Maximum point-wise errors are given in Figure 2 and Figure 4 for certain values of ε. From the numerical values given in the tables and the figures, one can easily conclude that the proposed hybrid method gives highly accurate results and is well-suited for singularly perturbed systems of reaction-diffusion equations.

Acknowledgment

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