Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 473 – 485
© TÜBİTAK
doi:10.3906/mat-1809-16

Slant helices: a new approximation

Pascual LUCAS*, **José Antonio ORTEGA-YAGÜES**
Department of Mathematics, University of Murcia, Espinardo Campus, Murcia, Spain

Received: 05.09.2018 • **Accepted/Published Online:** 24.12.2018 • **Final Version:** 18.01.2019

Abstract: In this paper, we study a weaker version of classic slant helices in Euclidean space $\mathbb{R}^3$ or Minkowski space $\mathbb{R}^3_1$, which will be called general slant helices. We show that any classic slant helix is a general slant helix but the converse is not true. We also obtain equations involving the curvature and torsion that characterize this family of curves.

Key words: Slant helix, Killing vector field, pseudo-null curve, null curve, Frenet curve

1. Introduction

It is usual practice to extend the concepts defined in Euclidean space to other more general spaces, either allowing the metric to be indefinite or allowing the curvature to be nonzero. Most of the time, when the extended concept is applied over the Euclidean space, the original definition is recovered, but other times this is not the case. In these notes we will discuss one of these cases.

The term "slant helix" was introduced by Izumiya and Takeuchi [7]: slant helices are defined by the property that their principal normals make a constant angle with a fixed direction. However, examples of this kind of curves were studied in the past (see, e.g., Salkowski curves [13]). The characterization of slant helices given in [7] was extended to the Minkowski space $\mathbb{R}^3_1$ by Ali and López [1]. In this case, a regular curve in $\mathbb{R}^3_1$ is said to be a slant helix if there exists a nonzero constant vector $v \in \mathbb{R}^3_1$ such that the function $\langle N, v\rangle$ is constant along the curve, N being the principal normal vector of the curve.

More recently, and following an idea used in [3] to define general helices in 3D-Lorentzian backgrounds (based on a previous work of Barros [2]), the authors defined in [10, 11] slant helices in the three-dimensional sphere $\mathbb{S}^3$ and anti-De Sitter space $\mathbb{H}^3_1$. In [2, 3], the constant vector is replaced by a Killing vector field along the curve with constant length. Since every Killing vector field along a curve can be uniquely extended to a Killing vector field in the ambient space [8], one may think that a plausible hypothesis is that this extension will also be of constant length, which implies (in the case, e.g., of the Euclidean space $\mathbb{R}^3$) that it must be a constant vector field. However, it is easy to find Killing vector fields in $\mathbb{R}^3$ of nonconstant length that are of constant length along a curve. In the study of general helices made in [2, 3], the families of general helices are the same whether or not the extension is of constant length. However, as we will see throughout this work, in the case of slant helices the sets of solutions are not the same with both hypotheses.

As an intermediate step towards our main results, in Section 3 we study the set $M(X,\varepsilon)$ of points $p \in \mathbb{R}^3_\nu$ such that $\langle X_p, X_p\rangle = \varepsilon$, for a certain constant ε, where X is a nonconstant Killing vector field in $\mathbb{R}^3_\nu$. We

*Correspondence: plucas@um.es
2010 *AMS Mathematics Subject Classification:* 53B20, 53B30

show (see Proposition 2) that if $M(X, \varepsilon)$ is a nonempty set, then it must be one of the following submanifolds:

1) A straight line.
2) A null plane.
3) A circular cylinder.
4) A hyperbolic cylinder.
5) A parabolic null cylinder.

Moreover, if P is one of the submanifolds of the above list we show that there exists a Killing vector field $X \in \mathcal{K}(\mathbb{R}^3_\nu)$ such that $P \subseteq M(X, \varepsilon)$ for some $\varepsilon \in \mathbb{R}$.

In Section 4 we recover the notion of slant helix introduced in [10, 11] for a nonflat space, but without the condition that the extension of the Killing vector field to the ambient space be of constant length, and we apply it to the case of $\mathbb{R}^3$ or $\mathbb{R}^3_1$. The new curves are called general slant helices because the Killing vector field is of constant length only along the curve. Obviously every classic slant helix is a general slant helix, but the converse is not true. It is important to point out that in the Minkowskian ambient space $\mathbb{R}^3_1$ the situation is much richer than in the Euclidean case $\mathbb{R}^3$ since a curve γ can be of three types, according to the causal characters of γ' and γ'': (1) γ' and γ'' are nonnull (we say γ is a Frenet curve); (2) γ' is spacelike and γ'' is null (we say γ is a pseudo-null curve); (3) γ' is a null vector (we say γ is a null curve). The following characterizations of general slant helices in $\mathbb{R}^3_\nu$ are given (see Theorems 3, 4, and 5):

(a) A Frenet curve γ is a general slant helix if and only if the function

$$\frac{\kappa^2}{\sqrt{|\varepsilon_3\kappa^2 + \varepsilon_1(\tau-\lambda)^2|^3}}\left(\frac{\tau-\lambda}{\kappa}\right)'$$

is constant (in the open set where $\varepsilon_3\kappa^2 + \varepsilon_1(\tau-\lambda)^2$ does not vanish) and the differentiable function λ is a solution of the ODE (4.19). In this case, an axis of the general slant helix is given by

$$V = \frac{\mu(\tau-\lambda)}{\sqrt{|\varepsilon_3\kappa^2 + \varepsilon_1(\tau-\lambda)^2|}}\, T + b\, N + \frac{\mu\kappa}{\sqrt{|\varepsilon_3\kappa^2 + \varepsilon_1(\tau-\lambda)^2|}}\, B,$$

where constants b and μ are given by (4.24) and (4.25).

(b) Any pseudo-null curve is a general slant helix.

(c) A null curve γ is a general slant helix if and only if its pseudo-torsion is given by

$$\tau(s) = \frac{-n}{(-bs+m)^2}, \quad n, b, m \in \mathbb{R}.$$

In this case, an axis of the general slant helix is given by

$$V(s) = \frac{n}{-bs+m}\, T(s) + b\, N(s) + (-bs+m)\, B(s).$$

2. Preliminaries

The semi-Euclidean space $\mathbb{R}^3_\nu$, $\nu \in \{0,1\}$, is defined as the vector space $\mathbb{R}^3$ endowed with the semi-Riemannian metric

$$\langle \cdot,\cdot \rangle = \delta \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2, \quad \delta = (-1)^\nu,$$

where (x,y,z) denotes the usual rectangular coordinates in $\mathbb{R}^3$.

The local geometry of a regular curve γ in $\mathbb{R}^3_\nu$ can be described by using the Frenet apparatus of the curve. Since there are three families of regular curves in $\mathbb{R}^3_1$, we briefly recall the Frenet apparatus in each one of the three families [9, 14].

2.1. The Frenet apparatus of a regular curve

Let $\gamma = \gamma(s) : I \subset \mathbb{R} \to \mathbb{R}^3_\nu$ be an immersed regular curve. If ∇^0 denotes the Levi-Civita connection in $\mathbb{R}^3_\nu$, we will write $V'(s) \equiv \nabla^0_{\gamma'(s)} V$ for any differentiable vector field $V \in \mathfrak{X}(\gamma)$ along the curve γ. The parameter s is the arc length parameter in the case when γ is a nonnull curve, or the pseudo-arc length parameter in the case when γ is a null curve.

2.1.1. $\boldsymbol{\gamma}$ is a nonnull curve with nonnull acceleration vector

In this case, there exists an orthonormal frame along γ, $\{T = \gamma', N, B\}$, satisfying the following equations:

$$\begin{aligned} T' &= \varepsilon_2 \kappa N, \\ N' &= -\varepsilon_1 \kappa T + \varepsilon_3 \tau B, \\ B' &= -\varepsilon_2 \tau N, \end{aligned} \tag{2.1}$$

where $\varepsilon_1 = \langle T,T \rangle$, $\varepsilon_2 = \langle N,N \rangle$, and $\varepsilon_3 = \langle B,B \rangle$. The differentiable functions κ and τ denote the curvature and torsion of γ. A curve of this family will be called a Frenet curve.

2.1.2. $\boldsymbol{\gamma}$ is a spacelike curve with null acceleration vector

In this case, there exists a pseudo-orthonormal frame along γ, $\{T = \gamma', N, B\}$, satisfying the following equations:

$$\begin{aligned} T' &= N, \\ N' &= \tau N, \\ B' &= T - \tau B, \end{aligned} \tag{2.2}$$

where τ stands for the torsion (also called pseudo-torsion) of γ. The matrix of the metric in that pseudo-orthonormal reference is given by $\langle T,T \rangle = 1$, $\langle T,N \rangle = \langle T,B \rangle = 0$, $\langle N,N \rangle = \langle B,B \rangle = 0$, and $\langle N,B \rangle = -1$. A curve of this family is called a pseudo-null curve [14] (or 2-degenerate curve [4]).

2.1.3. γ is a null curve

Finally, for a curve of this family there exists a pseudo-orthonormal frame along γ, $\{T=\gamma', N, B\}$, satisfying the following equations:

$$\begin{aligned} T' &= N, \\ N' &= \tau T + B, \\ B' &= \tau N, \end{aligned} \tag{2.3}$$

where τ stands for the torsion (also called pseudo-torsion) of γ. The metric is given by $\langle T, T\rangle = \langle B, B\rangle = 0$, $\langle T, B\rangle = -1$, $\langle N, N\rangle = 1$, and $\langle T, N\rangle = \langle N, B\rangle = 0$.

2.2. Killing vector fields along regular curves

Given a regular curve $\gamma = \gamma(t) : I \subset \mathbb{R} \to \mathbb{R}^3_\nu$, we will consider variations $\Gamma = \Gamma(t,z) : I \times (-\varepsilon, \varepsilon) \subset \mathbb{R}^2 \to \mathbb{R}^3_\nu$, with $\Gamma(t,0) = \gamma(t)$, such that all the t-curves $\gamma_z(t) = \Gamma(t,z)$ belong to the same family as the curve γ. Let $V(t,z) = \frac{\partial \Gamma}{\partial z}(t,z)$ and $\overline{\Gamma}(t,z) = \frac{\partial \Gamma}{\partial t}(t,z)$ be the tangent vector fields to the z-curves and t-curves, respectively. In particular, the vector field along γ defined by $V(t) = \frac{\partial \Gamma}{\partial z}(t,0)$ is called the variational vector field of Γ. We will write $T(s,z)$, $N(s,z)$, $B(s,z)$, $\kappa(s,z)$, $\tau(s,z)$, $V(s,z)$, etc. for the corresponding elements when s is the arc length (or pseudo-arc length) parameter of the curve γ.

2.2.1. Killing vector fields along Frenet curves

The vector field $V(s)$ is said to be a Killing vector field along γ if the following conditions are satisfied:

$$\left.\frac{\partial v}{\partial z}\right|_{z=0} = \left.\frac{\partial \kappa}{\partial z}\right|_{z=0} = \left.\frac{\partial \tau}{\partial z}\right|_{z=0} = 0, \tag{2.4}$$

where $v(s,z)$ stands for the velocity of the s-curves. These conditions are well defined in the sense that they do not depend on the V-variation of γ one chooses to compute the derivatives involved in (2.4). In fact, by using [3, Lemma 3.1] and (2.4) it is easy to see that V is a Killing vector field along γ if and only if it satisfies the following conditions:

$$\begin{aligned} &a) \quad \langle V', T\rangle = 0, \\ &b) \quad \langle V'', N\rangle \, \kappa = 0, \\ &c) \quad \Big\langle \frac{1}{\kappa} V''' - \frac{\kappa'}{\kappa^2} V'' + \varepsilon_1 \varepsilon_2 \kappa V', B \Big\rangle \tau = 0. \end{aligned} \tag{2.5}$$

2.2.2. Killing vector fields along pseudo-null curves

The tangent vector to s-curves is given by $\overline{T}(s,z) = v(s,z) T(s,z)$, and the acceleration $a(s,z)$ of s-curves is defined by

$$a(s,z)^2 = \left\langle \nabla^0_{\overline{T}} \overline{T}, \nabla^0_{\overline{T}} \overline{T} \right\rangle.$$

Note that s is the arc length parameter of the curve γ, but this does not imply that it is also the arc length parameter of the variational curves. In this family of curves the three functions that characterize a curve are the velocity, the acceleration, and the torsion. Hence, it is said that the vector field $V(s)$ is a Killing vector field along γ if the following conditions are satisfied:

$$\left.\frac{\partial v}{\partial z}\right|_{z=0} = \left.\frac{\partial a}{\partial z}\right|_{z=0} = \left.\frac{\partial \tau}{\partial z}\right|_{z=0} = 0.$$

A straightforward computation yields that these equations are equivalent to the following conditions:

$$\begin{aligned} &a) \quad \langle V', T\rangle = 0, \\ &b) \quad \langle V'', N\rangle = 0, \\ &c) \quad \langle V''' - \tau V'', B\rangle = 0. \end{aligned} \tag{2.6}$$

2.2.3. Killing vector fields along null curves

Finally, in the case of null curves, and in a similar way as before, we will say that the vector field $V(s)$ is a Killing vector field along γ if the following conditions are satisfied [5, 6]:

$$\begin{aligned} &a) \quad \langle V', T\rangle = 0, \\ &b) \quad \langle V'', N\rangle = 0, \\ &c) \quad \langle V''' - \tau V', B\rangle = 0. \end{aligned} \tag{2.7}$$

Bearing in mind that the solutions of (2.5) or (2.6) or (2.7) constitute a six-dimensional linear space, and that the restriction to γ of a Killing vector field of $\mathbb{R}^3_\nu$ is a Killing vector field along γ, we can state the following result [8].

Proposition 1 *Let γ be a regular curve immersed in $\mathbb{R}^3_\nu$. A vector field $V \in \mathfrak{X}(\gamma)$ is a Killing vector field along γ if and only if it is the restriction to γ of a Killing vector field of $\mathbb{R}^3_\nu$.*

3. Killing vector fields of $\mathbb{R}^3_\nu$ of constant length on a submanifold

Let $X \in \mathfrak{X}(\mathbb{R}^3_\nu)$ be a Killing vector field. Then there exist two constant vectors v and w such that $X_p = v + w \times p$, for any point $p \in \mathbb{R}^3_\nu$, where the cross product $\times$ is defined as usual: $u \times v$ is the only vector in $\mathbb{R}^3_\nu$ such that $\langle u \times v, w\rangle = \det(u, v, w)$ for any $w \in \mathbb{R}^3_\nu$. Then a basis of the vector space $\mathfrak{K}(\mathbb{R}^3_\nu)$ of Killing vector fields is given by

$$\{\partial_x, \partial_y, \partial_z, -\delta y\partial_x + x\partial_y, -\delta z\partial_x + x\partial_z, -z\partial_y + y\partial_z\},$$

and so the vector field X can be expressed as a linear combination:

$$X = (a - \delta dy - \delta ez)\,\partial_x + (b + dx - fz)\,\partial_y + (c + ex + fy)\,\partial_z,$$

for certain constants a, b, c, d, e, f. Hence, we can write

$$X_p = (a, b, c) + (f, -e, d) \times p.$$

Given a constant $\varepsilon \in \mathbb{R}$, how is the geometry of the set $M(X, \varepsilon)$ of points $p \in \mathbb{R}^3_\nu$ such that $\langle X_p, X_p \rangle = \varepsilon$? If $p = (x, y, z)$ then $\langle X_p, X_p \rangle = \varepsilon$ if and only if

$$\begin{aligned} (d^2+e^2)x^2 + (\delta d^2 + f^2)y^2 + (\delta e^2 + f^2)z^2 + \\ 2efxy - 2dfxz + 2\delta deyz + \\ 2(bd+ec)x + 2(-ad+cf)y + 2(-ae-bf)z + \\ \delta a^2 + b^2 + c^2 - \varepsilon = 0. \end{aligned} \tag{3.1}$$

In other words,

$$\langle Ap, p \rangle + 2\langle p, u \rangle + u_0 = 0, \tag{3.2}$$

where A is a self-adjoint matrix, $u \in \mathbb{R}^3_\nu$ is a constant vector, and $u_0 \in \mathbb{R}$. That equation can be rewritten as

$$p^t(A^tG)p + 2p^tGu + u_0 = 0, \tag{3.3}$$

where $()^t$ denotes the transpose and G stands for the matrix of metric, which can be solved in a standard way. It is easy to show that the eigenvalues of A^tG are given by

$$\tilde{\lambda}_1 = 0, \quad \text{associated to the eigenvector } u_1 = (f, -e, d), \tag{3.4}$$

$$\tilde{\lambda}_2 = d^2 + e^2 + f^2, \tag{3.5}$$

$$\tilde{\lambda}_3 = \delta(d^2 + e^2 + \delta f^2). \tag{3.6}$$

As a consequence, the eigenvalues of self-adjoint matrix A are given by

$$\lambda_1 = 0, \quad \text{associated to the eigenvector } u_1 = (f, -e, d), \tag{3.7}$$

$$\lambda_2 = \lambda_3 = \delta(d^2 + e^2 + \delta f^2). \tag{3.8}$$

Now we distinguish three cases.

Case 1: $d^2 + e^2 = 0$. Then $M(X, \varepsilon)$ is characterized by the equation

$$(fy + c)^2 + (fz - b)^2 + \delta a^2 - \varepsilon = 0.$$

Then depending on whether $\delta a^2 - \varepsilon$ is positive, zero, or negative, $M(X, \varepsilon)$ will be the empty set, a straight line, or a circular cylinder with axis $(1, 0, 0)$ and radius $\sqrt{\varepsilon - \delta a^2}$.

Case 2: $d^2 + e^2 \neq 0$ and $d^2 + e^2 + \delta f^2 \neq 0$. With an appropriate change of coordinates $(x, y, z) \to (\overline{x}, \overline{y}, \overline{z})$, Eq. (3.1) can be rewritten as

$$(d^2 + e^2 + \delta f^2)\overline{x}^2 + \delta(d^2 + e^2 + \delta f^2)\overline{z}^2 = R, \quad \text{if } \delta\varepsilon_1 = -1, \tag{3.9}$$

$$\delta(d^2 + e^2 + \delta f^2)\overline{y}^2 + \delta(d^2 + e^2 + \delta f^2)\overline{z}^2 = R, \quad \text{if } \delta\varepsilon_1 = 1, \tag{3.10}$$

where

$$R = \varepsilon - \frac{(cd - be + \delta af)^2}{d^2 + e^2 + \delta f^2} = \langle X, X \rangle - \frac{\langle u_1, (a, b, c) \rangle^2}{\langle u_1, u_1 \rangle}.$$

Therefore, if $M \neq \varnothing$ then M must be a straight line, a plane, a circular cylinder, or a hyperbolic cylinder.

Case 3: $d^2+e^2+\delta f^2=0$. Then $\delta=-1$ and the axis $u_1=(f,-e,d)$ is a null vector. When $\langle u_1,(a,b,c)\rangle \neq 0$, we can find a change of coordinates $(x,y,z)\to(\overline{x},\overline{y},\overline{z})$, $\overline{x}$ being a null coordinate, such that equation (3.1) yields

$$\overline{x}^2-2R\,\overline{y}=0, \quad R=af+be-cd=-\langle u_1,(a,b,c)\rangle\,,$$

which represents a parabolic null cylinder (see [12]). Otherwise, in the case $\langle u_1,(a,b,c)\rangle=0$, there exists a constant λ_0 such that

$$(\overline{x}+\lambda_0)^2=\varepsilon,$$

which represents a pair of parallel null planes.

In conclusion, we have proved the following characterization of $M(X,\varepsilon)$.

Proposition 2 *Let $X\in\mathcal{K}(\mathbb{R}^3_\nu)$ be a nonconstant Killing vector field and consider $\varepsilon\in\mathbb{R}$. If $M(X,\varepsilon)$ is a nonempty set, then it must be one of the following submanifolds:*

1) *A straight line.*

2) *A null plane.*

3) *A circular cylinder.*

4) *A hyperbolic cylinder.*

5) *A parabolic null cylinder.*

Let P be one of the submanifolds listed in this proposition. We are going to show that there exists a Killing vector field $X\in\mathcal{K}(\mathbb{R}^3_\nu)$ such that $P\subseteq M(X,\varepsilon)$ for some $\varepsilon\in\mathbb{R}$.

Case 1: P is a straight line. If the line is spanned by the vector (v_1,v_2,v_3) then we can consider the vector field

$$X=a\partial_x+b\partial_y+c\partial_z+mv_3(-\delta y\partial_x+x\partial_y)-mv_2(-\delta z\partial_x+x\partial_z)+mv_1(-z\partial_y+y\partial_z),$$

for any constants a,b,c,m. It is easy to show that $X|_P$ is constant and so $\langle X,X\rangle$ is constant along P.

Case 2: P is a null plane. Let us suppose that P is characterized by the equation

$$d_1x+d_2y+d_3z+d_4=0, \quad d_1^2=d_2^2+d_3^2.$$

Let us consider the vector field

$$\begin{aligned}X=&a\partial_x+b\partial_y+c\partial_z+\\ &\lambda[-d_3(y\partial_x+x\partial_y)+d_2(z\partial_x+x\partial_z)+d_1(-z\partial_y+y\partial_z)]+\\ &\mu[d_2(y\partial_x+x\partial_y)+d_3(z\partial_x+x\partial_z)],\end{aligned}$$

with $ad_1+bd_2+cd_3=d_1d_4\mu$. Then $\langle X,X\rangle$ is constant along the plane; in fact, we have

$$\langle X,X\rangle=-a^2+\Big[b-\frac{d_4}{d_1}\Big(\mu d_2-d_3\lambda\Big)\Big]^2+\Big[c-\frac{d_4}{d_1}\Big(\mu d_3+d_2\lambda\Big)\Big]^2.$$

Cases 3 and 4: P is either a circular cylinder or a hyperbolic cylinder with axis spanned by a nonnull vector w_1. Then there exists an orthonormal basis $\{w_1, w_2, w_3\}$ of $\mathbb{R}^3_\nu$, w_3 being a spacelike vector, such that P can be parametrized as follows:

$$\varphi(t,s) = p_0 + R\big(f(t)w_2 + g(t)w_3\big) + sw_1, \tag{3.11}$$

where f and g are differentiable functions satisfying

$$f^2 + \varepsilon_2 g^2 = 1, \quad f' = -\varepsilon_2 g, \quad g' = f.$$

It is not difficult to see that a Killing vector field with constant length along N can be written as

$$X_p = \mu w_1 + \lambda w_1 \times (p - p_0),$$

for certain constants μ, λ. Note that equation (3.11) implies the following characterization of cylinders:

$$\varepsilon_2 \langle p - p_0, w_3\rangle^2 + \langle p - p_0, w_2\rangle^2 = R^2.$$

Case 5: Finally, let us consider P as a parabolic cylinder with axis spanned by a null vector w_1. Then there exists a pseudo-orthonormal basis $\{w_1, w_2, w_3\}$ of $\mathbb{R}^3_\nu$, w_3 being a spacelike vector, such that P can be parametrized as follows:

$$\varphi(t,s) = p_0 + R\Big(tw_2 + \frac{t^2}{2}w_3\Big) + sw_1,$$

which leads to the following equation:

$$\langle w_1 \times (p - p_0), w_3\rangle^2 - 2R\,\langle w_1 \times (p - p_0), w_2\rangle = 0.$$

As before, it is easy to see that a Killing vector field with constant length along P can be written as

$$X_p = \mu w_1 + \lambda R w_2 + \lambda w_1 \times (p - p_0),$$

for certain constants μ, λ.

4. General slant helices

A regular curve $\gamma = \gamma(s)$ immersed in $\mathbb{R}^3_\nu$ is said to be a general slant helix if there is a Killing vector field V along γ with constant length such that the product $\langle V, N\rangle$ is a constant function along γ. V is called an axis of the general slant helix γ.

Note that every plane curve is a general slant helix, so in the following reasoning we will consider that our curves do not live on a plane.

The usual concept of slant helix in $\mathbb{R}^3_\nu$ (see [1, 7]) is more restrictive than the above definition, because we can find a Killing vector field V along γ with constant length such that its extension to $\mathbb{R}^3_\nu$ (which must be a Killing vector field $X \in \mathfrak{K}(\mathbb{R}^3_\nu)$) is not necessarily of constant length. For example, in the Euclidean space $\mathbb{R}^3$ we can consider the curve $\gamma(s) = (\cos s, \sin s, s)$ and the vector field $V(s) = (-\sin s, \cos s, 0)$ whose extension to $\mathbb{R}^3$ is $X = -y\partial_x + x\partial_y$. In other words, every classic slant helix is a general slant helix, but the converse is not true in general, as we shall see below. The family of general slant helices consists of the classic slant helices (when the axis V is extended to a constant vector) and the *local slant helices* (when the axis V is extended to a nonconstant vector field). In view of Proposition 2, every local slant helix lives in a circular, hyperbolic, or parabolic cylinder.

4.1. Frenet general slant helices

Let $\gamma(s)$ be a regular Frenet curve and let us suppose it is a general slant helix with axis $V(s)$. If $\{T, N, B\}$ is the orthonormal Frenet frame of γ in $\mathbb{R}^3_\nu$, write

$$V(s) = x(s)T(s) + bN(s) + z(s)B(s), \tag{4.1}$$

where x, z are differentiable functions and b is a constant. Without loss of generality, we can assume $b \neq 0$; otherwise, it is not difficult to see that $\gamma(s)$ would be a general helix. From the condition $\langle V, V\rangle = \varepsilon$ it is easy to get

$$\varepsilon_1 x^2 + \varepsilon_2 b^2 + \varepsilon_3 z^2 = \varepsilon \Longrightarrow \varepsilon_1 x x' + \varepsilon_3 z z' = 0. \tag{4.2}$$

From (4.1) and the Frenet equations (2.1) we get

$$V' = (x' - \varepsilon_1 b\kappa)T + (\varepsilon_2 x\kappa - \varepsilon_2 z\tau)N + (z' + \varepsilon_3 b\tau)B, \tag{4.3}$$

and by using the Killing equation (2.5a) we obtain

$$x' = \varepsilon_1 b\kappa, \tag{4.4}$$

$$V' = \lambda\,(T \times V), \qquad \lambda = \frac{\varepsilon_3}{b}(z' + \varepsilon_3 b\tau). \tag{4.5}$$

From these equations we find

$$\frac{\tau - \lambda}{\kappa} = -\varepsilon_1\varepsilon_3\frac{z'}{x'} = \frac{x}{z}, \tag{4.6}$$

which jointly with (4.2) yields

$$\left(\frac{\tau - \lambda}{\kappa}\right)' = (\varepsilon - \varepsilon_2 b^2)\frac{\varepsilon_3 x'}{z^3}. \tag{4.7}$$

A straightforward computation shows that

$$\frac{\kappa^2}{\sqrt{|\varepsilon_3\kappa^2 + \varepsilon_1(\tau - \lambda)^2|^3}}\left(\frac{\tau - \lambda}{\kappa}\right)' = m, \tag{4.8}$$

for a certain real constant $m = \varepsilon_1\varepsilon_3 b(\varepsilon - \varepsilon_2 b^2)^{-1/2}$, everywhere $\varepsilon_3\kappa^2 + \varepsilon_1(\tau - \lambda)^2$ does not vanish. Note that $m \neq 0$; otherwise, γ is a general helix. If $\varepsilon_3\kappa^2 + \varepsilon_1(\tau - \lambda)^2$ vanishes everywhere then γ is also a general helix.

What functions λ can appear in Eq. (4.8)? From (4.5) we get

$$V' = \lambda(-\varepsilon_2 z N + \varepsilon_3 b B), \tag{4.9}$$

and then bearing (2.5b) in mind we obtain

$$\varepsilon_3 z\lambda' + b\lambda^2 = 0, \tag{4.10}$$

and therefore

$$V'' = (\varepsilon_1\varepsilon_2 z\kappa\lambda)T + (\varepsilon_3 b\lambda' - \varepsilon_2\varepsilon_3 z\tau\lambda)B. \tag{4.11}$$

Since γ is not a plane curve, Eq. (2.5c) is equivalent to

$$\left(\frac{1}{\kappa}\left\langle V'', B\right\rangle\right)' + \varepsilon_1\varepsilon_2\kappa\left\langle V', B\right\rangle = 0, \tag{4.12}$$

which, bearing (4.9)–(4.10) in mind, leads to

$$\lambda''(\varepsilon - \varepsilon_1 x^2)\kappa - \lambda'(3bx\kappa^2 + (\varepsilon - \varepsilon_1 x^2)\kappa') = 0. \tag{4.13}$$

We can assume $\varepsilon - \varepsilon_1 x^2 \neq 0$; otherwise, γ would be a general helix. A trivial solution of (4.13) is obtained when λ is constant; in this case, from (4.10) we get $\lambda = 0$. This is the solution of classic slant helices. In fact, if γ is not a straight line, from $\lambda = 0$ and by using (4.5) we obtain that V would be a parallel vector field along γ. If $X_p = v + w \times p$ is its extension to $\mathbb{R}^3_\nu$, then $V' = \nabla^0_T X = w \times T$, and then w must vanish (recall that γ is not a line), so that X is a constant vector.

In the case when $\lambda' \neq 0$, from (4.13) we get

$$\frac{\lambda''}{\lambda'} = \frac{3bx\kappa}{\varepsilon - \varepsilon_1 x^2} + \frac{\kappa'}{\kappa} \tag{4.14}$$

and a first integration of this equation is given by

$$\lambda' = \frac{c_1\kappa}{(\varepsilon - \varepsilon_1 x^2)^{3/2}}, \tag{4.15}$$

for a certain constant c_1. By using (4.4) we can integrate (4.15) to obtain

$$\lambda = \frac{c_1\varepsilon\varepsilon_1}{b}\frac{x}{\sqrt{\varepsilon - \varepsilon_1 x^2}} + c_2, \tag{4.16}$$

for a certain constant c_2. From here we get

$$x^2 = \frac{\varepsilon b^2(\lambda - c_2)^2}{\varepsilon_1 b^2(\lambda - c_2)^2 + c_1^2} \tag{4.17}$$

and from (4.2) we obtain

$$\varepsilon_3 z^2 = \varepsilon - \varepsilon_2 b^2 - \frac{\varepsilon\varepsilon_1 b^2(\lambda - c_2)^2}{\varepsilon_1 b^2(\lambda - c_2)^2 + c_1^2}. \tag{4.18}$$

This equation, jointly with (4.10), yields

$$b^2\lambda^4(\varepsilon_1 b^2(\lambda - c_2)^2 + c_1^2) = (\lambda')^2[\varepsilon\varepsilon_3 c_1^2 - \varepsilon_2\varepsilon_3 b^2(\varepsilon_1 b^2(\lambda - c_2)^2 + c_1^2)]. \tag{4.19}$$

This differential equation characterizes the functions λ that can appear in (4.8). If λ is a solution of (4.19), then from (4.17) and (4.4) we obtain

$$\kappa^2 = \frac{\varepsilon b^2 c_1^4 \lambda^4}{F(\lambda)^2(\varepsilon\varepsilon_3 c_1^2 - \varepsilon_2\varepsilon_3 b^2 F(\lambda))}, \tag{4.20}$$

where $F(\lambda) = \varepsilon_1 b^2(\lambda - c_2)^2 + c_1^2$. Now from (4.18), (4.10), and (4.19) we get

$$z^2 = \frac{\varepsilon\varepsilon_3 c_1^2 - \varepsilon_2\varepsilon_3 b^2 F(\lambda)}{F(\lambda)}, \tag{4.21}$$

and bearing (4.6) and (4.17) in mind we obtain

$$\tau = \lambda \pm \frac{b^2 c_1^2 \lambda^2(\lambda - c_2)}{F(\lambda)(\varepsilon\varepsilon_3 c_1^2 - \varepsilon_2\varepsilon_3 b^2 F(\lambda))}. \tag{4.22}$$

Now we will prove that a regular Frenet curve $\gamma(s)$ satisfying (4.8), when m is constant and λ is a solution of (4.19), is a general slant helix. Let

$$\eta = \mathrm{sgn}(\varepsilon_3\kappa^2 + \varepsilon_1(\tau - \lambda)^2) = \pm 1, \quad \varepsilon = \mathrm{sgn}(\eta + \varepsilon_2 m^2) \in \{-1, 0, +1\}, \tag{4.23}$$

where sgn denotes the signum function. Consider $V(s)$ the vector field along γ given by (4.1) where x, b, z are defined by

$$x = \frac{\mu(\tau - \lambda)}{\sqrt{|\varepsilon_3\kappa^2 + \varepsilon_1(\tau - \lambda)^2|}}, \quad b = \varepsilon_2\delta\eta\mu m, \quad z = \frac{\mu\kappa}{\sqrt{|\varepsilon_3\kappa^2 + \varepsilon_1(\tau - \lambda)^2|}}, \tag{4.24}$$

with

$$\begin{cases} \mu = \dfrac{\varepsilon_2\delta\eta}{\sqrt{\varepsilon(\eta + \varepsilon_2 m^2)}}, & \text{if } \varepsilon \neq 0, \\ \mu \neq 0 \text{ is an arbitrary constant}, & \text{if } \varepsilon = 0. \end{cases} \tag{4.25}$$

It is easy to see that $\langle V, V\rangle$ and $\langle V, N\rangle$ are constant. Let us prove that V is a Killing vector field along γ. A long but straightforward computation yields the following equations:

$$V' = \lambda(-\varepsilon_2 z N + \varepsilon_3 b B), \tag{4.26}$$

$$V'' = \delta\lambda(\varepsilon_3 z\kappa T - \varepsilon_1 b\lambda N - \varepsilon_1 z\tau B), \tag{4.27}$$

$$\begin{aligned} V''' = {} & \delta\lambda\big(-b\kappa(\tau - 2\lambda) + \varepsilon_3 z\kappa'\big)T + \lambda(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2)zN + \\ & \lambda\big(\varepsilon_2 b\tau(\tau - 2\lambda) - \delta\varepsilon_1 z\tau'\big)B. \end{aligned} \tag{4.28}$$

From these equations, it is not difficult to see that V satisfies Killing equations (2.5).

Therefore, we have shown the following result.

Theorem 3 *Let γ be an arc length parametrized immersed Frenet curve in $\mathbb{R}^3_\nu$ with nonzero curvature κ and torsion τ. The curve γ is a general slant helix if and only if the function*

$$\frac{\kappa^2}{\sqrt{|\varepsilon_3\kappa^2 + \varepsilon_1(\tau - \lambda)^2|^3}}\left(\frac{\tau - \lambda}{\kappa}\right)' \tag{4.29}$$

is a constant m (in the open set where $\varepsilon_3\kappa^2 + \varepsilon_1(\tau - \lambda)^2$ does not vanish) and the differentiable function λ is a solution of the ODE (4.19). In this case, an axis of the general slant helix is given by

$$V = \frac{\mu(\tau - \lambda)}{\sqrt{|\varepsilon_3\kappa^2 + \varepsilon_1(\tau - \lambda)^2|}}\, T + b\, N + \frac{\mu\kappa}{\sqrt{|\varepsilon_3\kappa^2 + \varepsilon_1(\tau - \lambda)^2|}}\, B,$$

where constants b, μ are given by (4.24) and (4.25).

4.2. Pseudo-null general slant helices

We will prove that any pseudo-null curve γ immersed in $\mathbb{R}^3_1$ is a general slant helix. The proof is similar to that given in [1] for classic slant helices.

Take the vector field along γ given by $V = yN$, where N is the principal normal of γ and y is any nontrivial solution of the ODE $y' + \tau y = 0$. Since $\langle V, V\rangle = \langle V, N\rangle = 0$, we only need to show that V is a Killing vector field along γ. An easy calculation from (2.2) gives $V' = V'' = V''' = 0$, and then equations (2.6) are trivially fulfilled.

Hence, we have shown the following result.

Theorem 4 *Let γ be an immersed pseudo-null curve in $\mathbb{R}^3_1$ with pseudo-torsion τ. Then γ is a general slant helix with axis $V = yN$, where y is any nontrivial solution of the ODE $y' + \tau y = 0$.*

Bearing [1, Theorem 1.3 (c)] in mind, the last result yields that every pseudo-null general slant helix is also a pseudo-null slant helix.

4.3. Null general slant helices

Let $\gamma(s)$ be a null general slant helix with axis $V(s)$ and Frenet frame $\{T, N, B\}$. Write

$$V = xT + bN + zB, \tag{4.30}$$

where x, z are differentiable functions and $b \in \mathbb{R}$ is constant. Let us assume that γ is not a general helix, so that τ is not constant and $b \neq 0$. From (4.30) and Frenet equations (2.3) we have

$$V' = (x' + b\tau)T + (x + z\tau)N + (z' + b)B. \tag{4.31}$$

From the Killing equation (2.7a) we obtain

$$z(s) = -bs + m, \quad m \in \mathbb{R}. \tag{4.32}$$

Putting $\langle V, V\rangle = \varepsilon$, we can write $xz = \frac{1}{2}(b^2 - \varepsilon) \equiv n$ constant, and then

$$x(s) = \frac{n}{-bs + m}. \tag{4.33}$$

By taking the covariant derivative in (4.31) and using (2.7b) we get $\tau' = -2x'/z$, and then from (2.7c) we deduce

$$\tau(s) = \frac{-n}{(-bs + m)^2}. \tag{4.34}$$

Let us see now that the above equation characterizes null general slant helices. Let γ be a null curve with torsion satisfying (4.34) for certain real constants n, b, m. Consider the following vector field along γ:

$$V(s) = \frac{n}{-bs + m}T(s) + bN(s) + (-bs + m)B(s). \tag{4.35}$$

It is an easy task to check equations (2.7), so that V is a Killing vector field along γ, with constant length $\langle V, V\rangle = b^2 - 2n$ and constant function $\langle V, N\rangle$. Then we have shown the following result.

Theorem 5 *Let γ be a null curve, pseudo-arc length parametrized, with nonconstant torsion $\tau \neq 0$. The curve γ is a null general slant helix if and only if*

$$\tau(s) = \frac{-n}{(-bs+m)^2}, \quad n, b, m \in \mathbb{R}.$$

In this case, an axis of the null slant helix is given by

$$V(s) = \frac{n}{-bs+m} T(s) + bN(s) + (-bs+m)B(s).$$

Bearing [1, Theorem 1.4] in mind, the previous result yields that every null general slant helix is also a null slant helix.

Acknowledgments

This work was partially supported by MINECO (Ministerio de Economía Competitividad, Spain) and FEDER (Fondo Europeo de Desarrollo Regional) project MTM2015-65430-P, and by Fundación Séneca (Región de Murcia, Spain) project 19901/GERM/15.

References

[1] Ali AT, López R. Slant helices in Minkowski space $\mathbf{E}_1^3$. J Korean Math Soc 2011; 48: 159-167.

[2] Barros M. General helices and a theorem of Lancret. P Am Math Soc 1997; 125: 1503-1509.

[3] Barros M, Ferrández A, Lucas P, Meroño MA. General helices in the 3-dimensional Lorentzian space forms. Rocky Mt J Math 2001; 31: 373-388.

[4] Ferrández A, Giménez A, Lucas P. s-Degenerate curves in Lorentzian space forms. J Geom Phys 2003; 45: 116-129.

[5] Giménez A. Relativistic particles along null curves in 3D Lorentzian space forms. Int J Bifurcat Chaos 2010; 20: 2851-2859.

[6] Huang R, Liao C. Geometrical particle models on 3D lightlike curves. Mod Phys Lett A: 2006; 21: 3039-3048.

[7] Izumiya S, Takeuchi N. New special curves and developable surfaces. Turk J Math 2004; 28: 153-163.

[8] Langer J, Singer DA. The total squared curvature of closed curves. J Differ Geom 1984; 20: 1-22.

[9] López R. Differential geometry of curves and surfaces in Lorentz-Minkowski space. Int Electron J Geom 2014; 7: 44-107.

[10] Lucas P, Ortega-Yagües JA. Slant helices in the three-dimensional sphere. J Korean Math Soc 2017; 54: 1331-1343.

[11] Lucas P, Ortega-Yagües JA. Helix surfaces and slant helices in the three-dimensional anti-De Sitter space. RACSAM Rev R Acad A 2017; 11: 1201-1222.

[12] Mira P, Pastor JA. Helicoidal maximal surfaces in Lorentz-Minkowski space. Monatsh Math 2003; 140: 315-334.

[13] Salkowski E. Zur Transformation von Raumkurven. Math Ann 1909; 66: 517-557.

[14] Walrave J. Curves and surfaces in Minkowski space. PhD, Katholieke Universiteit Leuven, Leuven, Belgium, 1995.