Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 486 – 497
© TÜBİTAK
doi:10.3906/mat-1811-56

A new class of generalized polynomials associated with Laguerre and Bernoulli polynomials

Nabiullah KHAN[1], **Talha USMAN**[1], **Junesang CHOI**[2,*]
[1]Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh, India
[2]Department of Mathematics, Dongguk University, Gyeongju, Republic of Korea

Received: 14.11.2018 • **Accepted/Published Online:** 25.12.2018 • **Final Version:** 18.01.2019

Abstract: Motivated by their importance and potential for applications in certain problems in number theory, combinatorics, classical and numerical analysis, and other fields of applied mathematics, a variety of polynomials and numbers with their variants and extensions have recently been introduced and investigated. In this paper, we aim to introduce generalized Laguerre–Bernoulli polynomials and investigate some of their properties such as explicit summation formulas, addition formulas, implicit formulas, and symmetry identities. Relevant connections of the results presented here with those relatively simple numbers and polynomials are considered.

Key words: Laguerre polynomials, Hermite polynomials, Bernoulli polynomials, generalized Laguerre–Bernoulli polynomials, summation formulae, symmetric identities

1. Introduction and preliminaries

The two variable Laguerre polynomials $L_n(x,y)$ are defined by the following generating function (see, e.g., [13, 24]):

$$\frac{1}{1-yt}\exp\left(\frac{-xt}{1-yt}\right)=\sum_{n=0}^{\infty}L_n(x,y)\,t^n \quad (|\,yt\,|<1). \tag{1}$$

The polynomials $L_n(x,y)$ are also given by the following generating function (see [14]):

$$\mathrm{e}^{yt}\,C_0(xt)=\sum_{n=0}^{\infty}L_n(x,y)\frac{t^n}{n!}, \tag{2}$$

where $C_0(x)$ denotes the 0th order Tricomi function and the nth order Tricomi functions $C_n(x)$ are given by the following generating function:

$$\exp\left(t-\frac{x}{t}\right)=\sum_{n=-\infty}^{\infty}C_n(x)\,t^n \quad (t\in\mathbb{C}\setminus\{0\};\ x\in\mathbb{C}). \tag{3}$$

Here and in the following, let $\mathbb{C}$, $\mathbb{R}^+$, $\mathbb{Z}$, and $\mathbb{N}$ be the sets of complex numbers, positive real numbers, integers, and positive integers, respectively, and let $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. The nth Tricomi functions $C_n(x)$ are explicitly given

*Correspondence: junesang@mail.dongguk.ac.kr
2010 *AMS Mathematics Subject Classification:* 05A10, 05A15, 11B68, 11C45, 11C90

by

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!} \quad (n \in \mathbb{N}_0)\,. \tag{4}$$

The Tricomi functions $C_n(x)$ are associated with the Bessel function of the first kind $J_n(x)$ (see, e.g., [12, 24]):

$$C_n(x) = x^{-\frac{n}{2}} J_n(2\sqrt{x}). \tag{5}$$

It follows from (2) and (4) that

$$L_n(x,y) = n! \sum_{s=0}^{n} \frac{(-1)^s x^s y^{n-s}}{(s!)^2 (n-s)!} = y^n L_n(x/y), \tag{6}$$

where $L_n(x)$ are the ordinary Laguerre polynomials (see, e.g., [3, 33]). We have

$$L_n(x,0) = \frac{(-1)^n x^n}{n!}, \quad L_n(0,y) = y^n, \quad L_n(x,1) = L_n(x). \tag{7}$$

Derre and Simsek [16] modified the Milne–Thomson polynomials $\Phi_n^{(\alpha)}(x)$ (see [28]) slightly to give polynomials $\Phi_n^{(\alpha)}(x,\nu)$ of degree n and order α by means of the following generating function (see also [24]):

$$f(t,\alpha)\, \mathrm{e}^{xt+h(t,\nu)} = \sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x,\nu) \frac{t^n}{n!}, \tag{8}$$

where $f(t,\alpha)$ and $h(t,\nu)$ are functions of t and $\alpha \in \mathbb{Z}$ and t and $\nu \in \mathbb{N}_0$, respectively, which are analytic in a neighborhood of $t=0$. Observe that $\Phi_n^{(\alpha)}(x,0) = \Phi_n^{(\alpha)}(x)$ (see, for details, [28]).

Here, by setting $f(t,\alpha) = \left(\frac{t}{\mathrm{e}^t - 1}\right)^{\alpha}$ in (8), we introduce the polynomials $B_n^{(\alpha)}(x,\nu)$ defined by

$$\left(\frac{t}{\mathrm{e}^t - 1}\right)^{\alpha} \mathrm{e}^{xt+h(t,\nu)} := \sum_{n=0}^{\infty} B_n^{(\alpha)}(x,\nu) \frac{t^n}{n!}. \tag{9}$$

We find that the polynomials $B_n^{(\alpha)}(x,\nu)$ in (9) are related to Bernoulli polynomials and Hermite polynomials. For example, setting $h(t,0) = 0$ in (9), we obtain

$$\left(\frac{t}{\mathrm{e}^t - 1}\right)^{\alpha} \mathrm{e}^{xt} := \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}, \tag{10}$$

where $B_n^{(\alpha)}(x)$ are generalized Bernoulli polynomials (see, e.g., [35, Section 1.7]). Further, taking $x=0$ in (10), we get

$$\left(\frac{t}{\mathrm{e}^t - 1}\right)^{\alpha} := \sum_{n=0}^{\infty} B_n^{(\alpha)} \frac{t^n}{n!}, \tag{11}$$

where $B_n^{(\alpha)}(x)$ are generalized Bernoulli numbers (see, e.g., [35, Section 1.7]). For more information about Bernoulli numbers and Bernoulli polynomials, we refer, for example, to [9, 10, 15, 16, 25, 29, 32, 35].

Setting $h(t,\nu) = h(t,y) = yt^2$ in (8), we have the generalized Hermite–Bernoulli polynomials of two variables ${}_HB_n^{(\alpha)}(x,y)$, which were introduced and investigated by Pathan [29], given by

$$\left(\frac{t}{\mathsf{e}^t - 1}\right)^{\alpha} \mathsf{e}^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(\alpha)}(x,y)\frac{t^n}{n!}. \tag{12}$$

Obviously, the polynomials ${}_HB_n^{(\alpha)}(x,y)$ in (12) are generalizations of several known polynomials and numbers, e.g., Bernoulli numbers, Bernoulli polynomials, Hermite polynomials, and Hermite–Bernoulli polynomials ${}_HB_n^{(\alpha)}(x,y)$ defined by (see [11, Eq. (1.6)])

$$\frac{t}{\mathsf{e}^t - 1}\,\mathsf{e}^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n(x,y)\frac{t^n}{n!}. \tag{13}$$

We also note that, by setting $f(t,\alpha) = \left(\frac{2}{\mathsf{e}^t+1}\right)^{\alpha}$ in (8), Khan et al. [24] introduced the polynomials $E_n^{(\alpha)}(x,\nu)$.

The sum of integer power (or simply power sum)

$$S_k(n) := \sum_{r=0}^{n} r^k \quad (k \in \mathbb{N}_0,\, n \in \mathbb{N}) \tag{14}$$

is generated by

$$\sum_{k=0}^{\infty} S_n(k)\frac{t^n}{n!} = 1 + \mathsf{e}^t + \mathsf{e}^{2t} + \cdots + \mathsf{e}^{nt} = \frac{\mathsf{e}^{(n+1)t} - 1}{\mathsf{e}^t - 1}. \tag{15}$$

Guo and Qi [17] (see also [32]) introduced the following generalized Bernoulli numbers $B_n(a,b)$ defined by

$$\frac{t}{a^t - b^t} := \sum_{n=0}^{\infty} B_n(a,b)\frac{t^n}{n!} \quad \left(a,\, b \in \mathbb{R}^+,\, a \neq b,\, |t| < \frac{2\pi}{|\ln a - \ln b|}\right). \tag{16}$$

Luo et al. [27] generalized the numbers $B_n(a,b)$ in (16) to introduce and investigate the generalized Bernoulli polynomials $B_n(x;a,b,\mathsf{e})$ defined by

$$\frac{t\,\mathsf{e}^{xt}}{a^t - b^t} := \sum_{n=0}^{\infty} B_n(x;a,b,\mathsf{e})\frac{t^n}{n!} \tag{17}$$

$$\left(x \in \mathbb{C},\, a,\, b \in \mathbb{R}^+,\, a \neq b,\, |t| < \frac{2\pi}{|\ln a - \ln b|}\right).$$

The polynomials $B_n(x;a,b,\mathsf{e})$ in (17) reduce to yield the Bernoulli polynomials $B_n(x)$ and the Bernoulli numbers B_n (see, e.g., [35, Section 1.7]).

The 2-variable Hermite Kampé de Fériet polynomials $H_n(x,y)$ (see [4, 11]) are defined by

$$H_n(x,y) = n!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^r x^{n-2r}}{r!(n-2r)!}, \tag{18}$$

which are generated by

$$\mathsf{e}^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}. \tag{19}$$

We find the ordinary Hermite polynomials $H_n(x) = H_n(2x,-1)$ (see, e.g., [4, 33]).

Motivated by their importance and potential for applications in certain problems in number theory, combinatorics, classical and numerical analysis, and other fields of applied mathematics, a variety of polynomials and numbers with their variants and extensions have recently been investigated (see, e.g., the references). In this paper, we aim to introduce generalized Laguerre–Bernoulli polynomials ${}_LB_n^{(\alpha)}(x,y,z;a,b,\mathsf{e})$ in (20) and investigate some of their properties such as explicit summation formulas, addition formulas, implicit formula, and symmetry identities.

2. Generalized Laguerre–Bernoulli polynomials

In view of (8), we introduce the following so-called generalized Laguerre–Bernoulli polynomials and investigate their properties.

Definition 2.1 *The generalized Laguerre-Bernoulli polynomials* ${}_LB_n^{(\alpha)}(x,y,z;a,b,\mathsf{e})$ *are defined by the following generating function:*

$$\left(\frac{t}{a^t-b^t}\right)^{\alpha}\ \mathsf{e}^{yt+zt^2}\, C_0(xt) := \sum_{n=0}^{\infty} {}_LB_n^{(\alpha)}(x,y,z;a,b,\mathsf{e})\frac{t^n}{n!} \tag{20}$$

$$\left(\alpha,\ x,\ y,\ z\in\mathbb{C},\ a,\ b\in\mathbb{R}^+,\ a\neq b,\ |t|<\frac{2\pi}{|\ln a-\ln b|}\right).$$

We consider some special cases of (20) in the following remark.

Remark 1 (i) The special case of (20) when $x=0$ reduces to the generalized Hermite–Bernoulli polynomials ${}_HB_n^{(\alpha)}(y,z;a,b,\mathsf{e})$ (see [30])

$$\left(\frac{t}{a^t-b^t}\right)^{\alpha}\ \mathsf{e}^{yt+zt^2} := \sum_{n=0}^{\infty} {}_HB_n^{(\alpha)}(y,z;a,b,\mathsf{e})\frac{t^n}{n!} \tag{21}$$

$$\left(\alpha,\ y,\ z\in\mathbb{C},\ a,\ b\in\mathbb{R}^+,\ a\neq b,\ |t|<\frac{2\pi}{|\ln a-\ln b|}\right).$$

(ii) Setting $x = z = 0$ in (20), we obtain the generalized Bernoulli polynomials $B_n^{(\alpha)}(y; a, b, \mathsf{e})$ defined by (see [27])

$$\left(\frac{t}{a^t - b^t}\right)^{\alpha} \mathsf{e}^{yt} := \sum_{n=0}^{\infty} B_n^{(\alpha)}(y; a, b, \mathsf{e}) \frac{t^n}{n!} \tag{22}$$

$$\left(\alpha,\, y \in \mathbb{C},\, a,\, b \in \mathbb{R}^+,\, a \neq b,\, |t| < \frac{2\pi}{|\ln a - \ln b|}\right).$$

(iii) Setting $x = y = z = 0$ in (20), we get the generalized Bernoulli numbers $B_n^{(\alpha)}(a, b)$ defined by (see [17, 32])

$$\left(\frac{t}{a^t - b^t}\right)^{\alpha} := \sum_{n=0}^{\infty} B_n^{(\alpha)}(a, b) \frac{t^n}{n!} \tag{23}$$

$$\left(\alpha,\, a,\, b \in \mathbb{R}^+,\, a \neq b,\, |t| < \frac{2\pi}{|\ln a - \ln b|}\right).$$

For easy reference, we begin by recalling some formal manipulations of double series in the following lemma (see, e.g., [8, 23, 24, 33, 36]).

Lemma 2.2 *The following identities hold:*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A_{k,n} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} A_{k,n-pk} \quad (p \in \mathbb{N}) \tag{24}$$

and

$$\sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(m+n) \frac{x^n}{n!} \frac{y^m}{m!}. \tag{25}$$

Here, $A_{k,n}$ and $f(N)$ $(k,\, n,\, N \in \mathbb{N}_0)$ are real or complex valued functions indexed by k, n, and N, respectively, and x and y are real or complex numbers. Also, for possible rearrangements of the involved double series, all the associated series should be absolutely convergent.

We present several explicit summation formulas of the generalized Laguerre–Bernoulli polynomials ${}_LB_n^{(\alpha)}(x, y, z; a, b, \mathsf{e})$ in (20), which are expressed in terms of some known polynomials as in Theorem 2.3.

Theorem 2.3 *Let $a, b \in \mathbb{R}^+$ with $a \neq b$, $n \in \mathbb{N}_0$, and $\alpha,\, x,\, y,\, z \in \mathbb{C}$. Then each of the following identities holds:*

$${}_LB_n^{(\alpha)}(x, y, z; a, b, \mathsf{e}) = \sum_{m=0}^{n} \sum_{l=0}^{[m/2]} \frac{m! \binom{n}{m}}{(m-2l)!\, l!} B_{n-m}^{(\alpha)}(a, b)\, L_{m-2l}(x, y)\, z^l. \tag{26}$$

$${}_LB_n^{(\alpha)}(x, y, z; a, b, \mathsf{e}) = \sum_{m=0}^{n} \sum_{l=0}^{m} \frac{(-1)^l \binom{n}{m} \binom{m}{l}}{l!} B_{n-m}^{(\alpha)}(a, b)\, H_{m-l}(y, z)\, x^l. \tag{27}$$

$$ {}_L B_n^{(\alpha)}(x,y,z;a,b,\mathsf{e}) = \sum_{m=0}^{n} \sum_{l=0}^{[m/2]} \frac{(-1)^m\, m!\, \binom{n}{m}}{((m-2l)!)^2\, l!}\, B_{n-m}^{(\alpha)}(y;a,b,\mathsf{e})\, x^{m-2l}\, z^l. \tag{28} $$

$$ {}_L B_n^{(\alpha)}(x,y,z;a,b,\mathsf{e}) = \sum_{m=0}^{n} \frac{(-1)^m \binom{n}{m}}{m!}\, {}_H B_{n-m}^{(\alpha)}(y,z;a,b,\mathsf{e})\, x^m. \tag{29} $$

Proof We will prove only (26). A similar argument will establish the other identities. Let $\mathcal{R}$ be the right side of (20). By using (23) and (2), we have

$$ \begin{aligned} \mathcal{R} &= \left(\frac{t}{a^t - b^t}\right)^{\alpha} \cdot \mathsf{e}^{yt}\, C_0(xt) \cdot \mathsf{e}^{zt^2} \\ &= \left(\sum_{n=0}^{\infty} B_n^{(\alpha)}(a,b)\, \frac{t^n}{n!}\right) \left(\sum_{m=0}^{\infty} L_m(x,y)\, \frac{t^m}{m!}\right) \left(\sum_{l=0}^{\infty} \frac{z^l\, t^{2l}}{l!}\right). \end{aligned} \tag{30} $$

Applying (24) with $p=2$ to the last two summations of (30), we obtain

$$ \mathcal{R} = \left(\sum_{n=0}^{\infty} B_n^{(\alpha)}(a,b)\, \frac{t^n}{n!}\right) \left(\sum_{m=0}^{\infty} \sum_{l=0}^{[m/2]} \frac{L_{m-2l}(x,y)\, z^l}{(m-2l)!\, l!}\, t^m\right). \tag{31} $$

Using (24) with $p=1$ in the two summations of (31), we get

$$ \begin{aligned} &\sum_{n=0}^{\infty} {}_L B_n^{(\alpha)}(x,y,z;a,b,\mathsf{e}) \frac{t^n}{n!} \\ &\quad = \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \sum_{l=0}^{[m/2]} \frac{B_{n-m}^{(\alpha)}(a,b) L_{m-2l}(x,y)\, z^l}{(n-m)!(m-2l)!\, l!} \right\} t^n. \end{aligned} \tag{32} $$

Finally, equating the coefficients of t^n in (32), we obtain the desired identity (26). □

We give some addition formulas for the generalized Laguerre–Bernoulli polynomials (20) in the following theorem.

Theorem 2.4 *Let $a, b \in \mathbb{R}^+$ with $a \neq b$, $n \in \mathbb{N}_0$, and α, β, x, y, y_1, y_2, z, z_1, $z_2 \in \mathbb{C}$. Then each of the following identities holds:*

$$ {}_L B_n^{(\alpha+\beta)}(x,y,z;a,b,\mathsf{e}) = \sum_{m=0}^{n} \binom{n}{m} B_m^{(\beta)}(a,b)\, {}_L B_{n-m}^{(\alpha)}(x,y,z;a,b,\mathsf{e}). \tag{33} $$

$$ \begin{aligned} {}_L B_n^{(\alpha)}(x,y_1+y_2,z;a,b,\mathsf{e}) &= \sum_{m=0}^{n} \binom{n}{m} y_1^m\, {}_L B_{n-m}^{(\alpha)}(x,y_2,z;a,b,\mathsf{e}) \\ &= \sum_{m=0}^{n} \binom{n}{m} y_2^m\, {}_L B_{n-m}^{(\alpha)}(x,y_1,z;a,b,\mathsf{e}). \end{aligned} \tag{34} $$

$$
\begin{aligned}
{}_LB_n^{(\alpha)}(x,y,z_1+z_2;a,b,\mathsf{e}) &= \sum_{m=0}^{[n/2]} \frac{n!}{m!\,(n-2m)!}\, z_1^m \, {}_LB_{n-2m}^{(\alpha)}(x,y,z_2;a,b,\mathsf{e}) \\
&= \sum_{m=0}^{[n/2]} \frac{n!}{m!\,(n-2m)!}\, z_2^m \, {}_LB_{n-2m}^{(\alpha)}(x,y,z_1;a,b,\mathsf{e}).
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
&{}_LB_n^{(\alpha+\beta)}(x,y_1+y_2,z;a,b,\mathsf{e}) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m} {}_LB_{n-m}^{(\alpha)}(x,y_1,z;a,b,\mathsf{e})\, B_m^{(\beta)}(y_2;a,b,\mathsf{e}).
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
&{}_LB_n^{(\alpha+\beta)}(x,y_1+y_2,z_1+z_2;a,b,\mathsf{e}) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m} {}_LB_{n-m}^{(\alpha)}(x,y_1,z_1;a,b,\mathsf{e})\, {}_HB_m^{(\beta)}(y_2,z_2;a,b,\mathsf{e}).
\end{aligned} \tag{37}
$$

$$
{}_LB_n^{(\alpha)}(x,y+\alpha,z;a,b,\mathsf{e}) = {}_LB_n^{(\alpha)}\left(x,y,z;\frac{a}{\mathsf{e}},\frac{b}{\mathsf{e}},\mathsf{e}\right). \tag{38}
$$

Proof Similarly as in the proof of Theorem 2.3, we can establish the identities here, so we omit the details. □

3. An implicit summation formula involving the generalized Laguerre–Bernoulli polynomials

We give an implicit summation formula for the generalized Laguerre–Bernoulli polynomials (20) in Theorem 3.1.

Theorem 3.1 *Let $a, b \in \mathbb{R}^+$ with $a \neq b$, $n, m \in \mathbb{N}_0$, and $\alpha, v, x, y, z \in \mathbb{C}$. Then the following implicit formula holds:*

$$
{}_LB_{m+n}^{(\alpha)}(x,v,z;a,b,\mathsf{e}) = \sum_{k=0}^{n}\sum_{s=0}^{m} \binom{n}{k}\binom{m}{s} (v-y)^{k+s}\, {}_LB_{m+n-k-s}^{(\alpha)}(x,y,z;a,b,\mathsf{e}). \tag{39}
$$

Proof Replacing t by $t+u$ in and using (25), we obtain

$$
\begin{aligned}
&\left(\frac{t+u}{a^{t+u}-b^{t+u}}\right)^{\alpha} \mathsf{e}^{z(t+u)^2} C_0(x(t+u)) \\
&\quad = \mathsf{e}^{-y(t+u)} \sum_{m,n=0}^{\infty} {}_LB_{m+n}^{(\alpha)}(x,y,z;a,b,\mathsf{e})\, \frac{t^n}{n!}\, \frac{u^m}{m!}.
\end{aligned} \tag{40}
$$

We find that the left side of (40) is independent of the variable y. Substituting any variable v for the variable

y in the right side of (40) and equating the right sides of (40) and the resulting identity, we get

$$\sum_{m,n=0}^{\infty} {}_LB_{m+n}^{(\alpha)}(x,v,z;a,b,\mathsf{e})\frac{t^n}{n!}\frac{u^m}{m!} \\ = \mathsf{e}^{(v-y)(t+u)} \sum_{m,n=0}^{\infty} {}_LB_{m+n}^{(\alpha)}(x,y,z;a,b,\mathsf{e})\frac{t^n}{n!}\frac{u^m}{m!}. \tag{41}$$

By using (25), we find

$$\mathsf{e}^{(v-y)(t+u)} = \sum_{N=0}^{\infty} \frac{(v-y)^N\,(t+u)^N}{N!} = \sum_{k,s=0}^{\infty} \frac{(v-y)^{k+s}\; t^k u^s}{k!s!}. \tag{42}$$

Setting the double series of (42) in (41) and applying (24) with $p=1$ two times to the resulting quadruple series, we have

$$\sum_{m,n=0}^{\infty} {}_LB_{m+n}^{(\alpha)}(x,v,z;a,b,\mathsf{e})\frac{t^n}{n!}\frac{u^m}{m!} \\ = \sum_{m,n=0}^{\infty} \sum_{k=0}^{n} \sum_{s=0}^{m} {}_LB_{m+n-k-s}^{(\alpha)}(x,y,z;a,b,\mathsf{e})\,\frac{(v-y)^{k+s}}{(n-k)!\,(m-s)!\,k!\,s!}\,t^n\,u^m. \tag{43}$$

Equating the coefficients of $t^n u^m$ on both sides of (43), we obtain the desired identity (39).

□

It is interesting to see that the left side of (39) is *independent of* the variable y.

We consider some special cases of (39) in the following corollary.

Corollary 3.2 *Let $a, b \in \mathbb{R}^+$ with $a \neq b$, $n, m \in \mathbb{N}_0$, and $\alpha, v, x, y, z \in \mathbb{C}$. Then each of the following implicit formulas holds:*

$${}_LB_n^{(\alpha)}(x,v,z;a,b,\mathsf{e}) = \sum_{k=0}^{n} \binom{n}{k}\,(v-y)^k\,{}_LB_{n-k}^{(\alpha)}(x,y,z;a,b,\mathsf{e}). \tag{44}$$

$$B_{m+n}^{(\alpha)}(v+y;a,b,\mathsf{e}) = \sum_{k=0}^{n}\sum_{s=0}^{m} \binom{n}{k}\binom{m}{s}\,v^{k+s}\,B_{m+n-k-s}^{(\alpha)}(y;a,b,\mathsf{e}). \tag{45}$$

$${}_HB_{m+n}^{(\alpha)}(v,z;a,b,\mathsf{e}) = \sum_{k=0}^{n}\sum_{s=0}^{m} \binom{n}{k}\binom{m}{s}\,(v-y)^{k+s}\,{}_HB_{m+n-k-s}^{(\alpha)}(y,z;a,b,\mathsf{e}). \tag{46}$$

$$B_{m+n}^{(\alpha)}(a,b) = \sum_{k=0}^{n}\sum_{s=0}^{m} \binom{n}{k}\binom{m}{s}\,(-y)^{k+s}\,B_{m+n-k-s}^{(\alpha)}(y;a,b,\mathsf{e}). \tag{47}$$

$$H_{m+n}(v,z) = \sum_{k=0}^{n}\sum_{s=0}^{m} \binom{n}{k}\binom{m}{s}\,(v-y)^{k+s}\,H_{m+n-k-s}(y,z). \tag{48}$$

Proof Setting $m=0$ in (39), we obtain (44).

Replacing v by $v+y$ in (39) and setting $x=z=0$ in the resulting identity, in view of (22), we get (45).

Setting $x=0$ in (39), in view of (22), we get (46).

Setting $x=v=z=0$ in (39), in view of (23) and (22), we obtain (47).

Setting $\alpha=x=0$ in (39), in view of (19), we get (48).

□

4. Symmetry identities for the generalized Laguerre–Bernoulli polynomials

Khan et al. [18–24], Yang [40], and Zhang and Yang [41] have established some interesting symmetry identities for various polynomials. Here we present certain symmetry identities for the generalized Laguerre–Bernoulli polynomials (20) in the following theorem.

Theorem 4.1 *Let $a,\, b \in \mathbb{R}^+$ with $a \neq b$, $n \in \mathbb{N}_0$, and $\alpha,\, \beta,\, x_j,\, y_j,\, z_j \in \mathbb{C}$ $(j=1,\,2)$. Then the following symmetry identities hold:*

$$\begin{aligned}
&\sum_{m=0}^{n} \binom{n}{m}\, {}_L B_{n-m}^{(\alpha)}\left(x_1, y_1, z_1; a, b, \mathsf{e}\right)\, {}_L B_{m}^{(\beta)}\left(x_2, y_2, z_2; a, b, \mathsf{e}\right) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m}\, {}_L B_{n-m}^{(\alpha)}\left(x_1, y_1, z_2; a, b, \mathsf{e}\right)\, {}_L B_{m}^{(\beta)}\left(x_2, y_2, z_1; a, b, \mathsf{e}\right) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m}\, {}_L B_{n-m}^{(\alpha)}\left(x_1, y_2, z_1; a, b, \mathsf{e}\right)\, {}_L B_{m}^{(\beta)}\left(x_2, y_1, z_2; a, b, \mathsf{e}\right) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m}\, {}_L B_{n-m}^{(\alpha)}\left(x_1, y_2, z_2; a, b, \mathsf{e}\right)\, {}_L B_{m}^{(\beta)}\left(x_2, y_1, z_1; a, b, \mathsf{e}\right) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m}\, {}_L B_{n-m}^{(\alpha)}\left(x_2, y_1, z_1; a, b, \mathsf{e}\right)\, {}_L B_{m}^{(\beta)}\left(x_1, y_2, z_2; a, b, \mathsf{e}\right) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m}\, {}_L B_{n-m}^{(\alpha)}\left(x_2, y_1, z_2; a, b, \mathsf{e}\right)\, {}_L B_{m}^{(\beta)}\left(x_1, y_2, z_1; a, b, \mathsf{e}\right) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m}\, {}_L B_{n-m}^{(\alpha)}\left(x_2, y_2, z_1; a, b, \mathsf{e}\right)\, {}_L B_{m}^{(\beta)}\left(x_1, y_1, z_2; a, b, \mathsf{e}\right) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m}\, {}_L B_{n-m}^{(\alpha)}\left(x_2, y_2, z_2; a, b, \mathsf{e}\right)\, {}_L B_{m}^{(\beta)}\left(x_1, y_1, z_1; a, b, \mathsf{e}\right).
\end{aligned} \tag{49}$$

Proof Consider the following function:

$$g(t) := \left(\frac{t}{a^t - b^t}\right)^{\alpha+\beta} \mathsf{e}^{(y_1+y_2)t+(z_1+z_2)t^2}\, C_0(x_1 t)\, C_0(x_2 t). \tag{50}$$

Here the involved variable and parameters in (50) are assumed to satisfy the conditions of Theorem 4.1. We find that the function $g(t)$ is symmetric with respect to the parameters in each of the following pairs: (α,β), (x_1,x_2), (y_1,y_2), and (z_1,z_2). In view of the generalized Laguerre–Bernoulli polynomials (20), we have the following eight combinations:

$(\alpha,x_1,y_1,z_1)\leftrightarrow(\beta,x_2,y_2,z_2)$; $(\alpha,x_1,y_1,z_2)\leftrightarrow(\beta,x_2,y_2,z_1)$;
$(\alpha,x_1,y_2,z_1)\leftrightarrow(\beta,x_2,y_1,z_2)$; $(\alpha,x_1,y_2,z_2)\leftrightarrow(\beta,x_2,y_1,z_1)$;
$(\alpha,x_2,y_1,z_1)\leftrightarrow(\beta,x_1,y_2,z_2)$; $(\alpha,x_2,y_1,z_2)\leftrightarrow(\beta,x_1,y_2,z_1)$;
$(\alpha,x_2,y_2,z_1)\leftrightarrow(\beta,x_1,y_1,z_2)$; $(\alpha,x_2,y_2,z_2)\leftrightarrow(\beta,x_1,y_1,z_1)$.

Choosing the first two combinations to make the generalized Laguerre–Bernoulli polynomials (20) and using (24) with $p=1$, we obtain

$$g(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}{}_LB_{n-m}^{(\alpha)}(x_1,y_1,z_1;a,b,\mathtt{e})\;{}_LB_{m}^{(\beta)}(x_2,y_2,z_2;a,b,\mathtt{e})\;\frac{t^n}{(n-m)!m!} \tag{51}$$

and

$$g(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}{}_LB_{n-m}^{(\alpha)}(x_1,y_1,z_2;a,b,\mathtt{e})\;{}_LB_{m}^{(\beta)}(x_2,y_2,z_1;a,b,\mathtt{e})\;\frac{t^n}{(n-m)!m!}. \tag{52}$$

Equating the coefficients of t^n on the right sides of (51) and (52), we obtain the first identity in (49). Similarly, using the remaining combinations in the order, we can establish the third identity to the seventh in (49). We omit the details. □

5. Concluding remarks

In 1929, Lidstone [26] introduced a generalization of Taylors series. It approximates a given function in the neighborhood of two points instead of one. This series includes the polynomials later called Lidstone polynomials. These polynomials have been studied in the works of Boas [6, 7], Poritsky [31], Schoenberg [34], Whittaker [37], Widder [38, 39], and others [1, 2]. Recall that Λ_n is a Lidstone polynomial of degree $(2n+1)$ defined by the relations

$$\begin{aligned}\Lambda_0(t)&=t,\\ \Lambda_n''(t)&=\Lambda_{n-1}(t),\\ \Lambda_n(0)&=\Lambda_n(1)=0, \qquad n\in\mathbb{N}.\end{aligned}$$

Another explicit representation of Lidstone polynomials is given by

$$\begin{aligned}\Lambda_n(t)=&\frac{1}{6}\left[\frac{6t^{2n+1}}{(2n+1)!}-\frac{t^{2n-1}}{(2n-1)!}\right]\\ &-\sum_{k=0}^{n-2}\frac{2(2^{2k+3}-1)}{(2k+4)!}\;B_{2k+4}\;\frac{t^{2n-2k-3}}{(2n-2k-3)!},\quad n\in\mathbb{N},\\ \Lambda_n(t)=&\frac{2^{2n+1}-1}{(2n+1)!}\;B_{2n+1}\;\left(\frac{1+t}{2}\right),\quad n\in\mathbb{N},\end{aligned}$$

where B_{2k+4} is the $(2k+4)$th Bernoulli number and $B_{2n+1}\left(\frac{1+t}{2}\right)$ is Bernoulli polynomial.

Since our result of Laguerre–Bernoulli polynomials given by (20) can be connected to Bernoulli polynomials, Bernoulli numbers, and Bernoulli-Laguerre numbers, we can apply the above connection of Lidstone polynomials and Bernoulli polynomials to obtain new results and connections of Laguerre–Bernoulli polynomials and Lidstone polynomials.

Furthermore, the generalized Laguerre–Bernoulli polynomials ${}_LB_n^{(\alpha)}(x, y, z; a, b, \mathtt{e})$ in (20), being very general, can be specialized to yield various known polynomials and numbers, for example, Bernoulli numbers B_n, Bernoulli polynomials $B_n(x)$, generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ (see, e.g., [35, Section 1.7]), generalized Bernoulli polynomials $B_n(x; a, b, \mathtt{e})$ (see [27]), Hermite–Bernoulli polynomials ${}_HB_n(x, y)$ (see [11]), and ${}_HB_n^{(\alpha)}(x, y)$ (see [29]). In this regard, the results presented here can be specialized to yield or be closely connected with some known identities and formulas (see, e.g., [5,13,18-24,27-29,40,41], and the references cited therein). Therefore, the results presented in this article seem to be potentially useful in arising problems of the aforementioned fields.

References

[1] Agarwal R, Wong P. Error Inequalities in Polynomial Interpolation and Their Applications. London, UK: Kluwer, 1993.

[2] Agarwal R, Wong P. Error bounds for derivatives of Lidstone interpolation and applications. In: Govil NK, editor. Approximate Theory. New York, NY, USA: Dekker, 1998. pp. 1-43.

[3] Andrews LC. Special Functions of Mathematics for Engineers. New York, NY, USA: Macmillan Company, 1985.

[4] Bell ET. Exponential polynomials. Ann Math 1934; 35: 258-277.

[5] Betti G, Ricci PE. Multidimensional extensions of the Bernoulli and Appell polynomials. Taiwanese J Math 2004; 8: 415-428.

[6] Boas RP. A note on functions of exponential type. B Am Math Soc 1941; 47: 750-754.

[7] Boas RP. Representation of functions by Lidstone series. Duke Math J 1943; 10: 239-245.

[8] Choi J. Notes on formal manipulations of double series. Commun Korean Math Soc 2003; 18: 781-789.

[9] Costabile FA. Expansions of real functions in Bernoulli polynomial and applications. In: Conferences Seminars of Mathematics University of Bari; 1999. pp. 1-13.

[10] Costabile FA, Dell Accio F, Gualtieri MI. A new approach to Bernoulli polynomials. Rend Mat VII 2005; 26: 1-12.

[11] Dattoli G, Lorenzutta S, Cesarano C. Finite sums and generalized forms of Bernoulli polynomials. Rend Mat 1999; 19: 385-391.

[12] Dattoli G, Torre A. Theory and Applications of Generalized Bessel Function. Rome, Italy: Aracne, 1996.

[13] Dattoli G, Torre A. Operational methods and two variable Laguerre polynomials. Atti Acad Torino 1998; 132: 1-7.

[14] Dattoli G, Torre A, Mancho AM. The generalized Laguerre polynomials, the associated Bessel functions and applications to propagation problems. Radiat Phys Chem 2000; 59: 229-237.

[15] Deeba E, Rodrigues D. Stirling series and Bernoulli numbers. Am Math Mon 1991; 98: 423-426.

[16] Dere R, Simsek Y. Bernoulli type polynomials on umbral algebra. Russian J Math Phys 2015; 23: 1-6.

[17] Guo BN, Qi F. Generalization of Bernoulli polynomials. J Math Ed Sci Tech 2002; 33: 428-431.

[18] Khan NU, Usman T. A new class of Laguerre-based generalized Apostol polynomials. Fasciculli Math 2016; 57: 67-89.

[19] Khan NU, Usman T. A new class of Laguerre-based poly-Euler and multi poly-Euler polynomials. J Anal Num Theor 2016; 4: 113-120.

[20] Khan NU, Usman T. A new class of Laguerre poly-Bernoulli numbers and polynomials. Adv Stud Contemporary Math 2017; 27: 229-241.

[21] Khan NU, Usman T, Aman A. Generating functions for Legendre-Based poly-Bernoulli numbers and polynomials. Honam Math J 2017; 39: 217-231.

[22] Khan NU, Usman T, Choi J. Certain generating function of Hermite-Bernoulli-Laguerre polynomials. Far East J Math Sci 2017; 101: 893-908.

[23] Khan NU, Usman T, Choi J. A new generalization of Apostol type Laguerre-Genocchi polynomials. C R Acad Sci Paris Ser I 2017; 355: 607-617.

[24] Khan NU, Usman T, Choi J. A new class of generalized polynomials. Turk J Math 2018; 42: 1366-1379.

[25] Kurt B, Simsek Y. Notes on generalization of the Bernoulli type polynomials. Appl Math Comput 2011; 218: 906-911.

[26] Lidstone GL. Note on the extension of Aitken's theorem (for polynomial interpolation) to the Everett types. Proc Edinb Math Soc 1929; 2: 16-19.

[27] Luo QM, Guo BN, Qi F, Debnath L. Generalization of Bernoulli numbers and polynomials. Int J Math Math Sci 2003; 61: 3769-3776.

[28] Milne-Thomsons LM. Two classes of generalized polynomials. Proc London Math Soc 1933; 35: 514-522.

[29] Pathan MA. A new class of generalized Hermite-Bernoulli polynomials. Georgian Math J 2012; 19: 559-573.

[30] Pathan MA, Khan WA. A new class of generalized polynomials associated with Hermite and Bernoulli polynomials. Le Matematiche 2015; 70: 53-70.

[31] Poritsky H. On certain polynomial and other approximation to analytic functions. T Am Math Soc 1932; 34: 274-331.

[32] Qui F, Guo BN. Generalization of Bernoulli polynomials. RGMIA Res Rep Coll 2001; 4: 691-695.

[33] Rainville ED. Special Functions. New York, NY, USA: Macmillan Company, 1960.

[34] Schoenberg LJ. On certain two-point expansion of integral functions of exponential type. B Am Math Soc 1936; 42: 284-288.

[35] Srivastava HM, Choi J. Zeta and q-Zeta Functions and Associated Series and Integrals. Amsterdam, the Netherlands: Elsevier Science Publishers, 2012.

[36] Srivastava HM, Manocha HL. A Treatise on Generating Functions. New York, NY, USA: John Wiley and Sons, 1984.

[37] Whittaker JM. On Lidstone's series and two point expansions of analytic functions. Proc London Math Soc 1934; 36: 451-469.

[38] Widder DV. Functions whose even derivatives have a prescribed sign. Proc Natl Acad Sci 1940; 26: 657-659.

[39] Widder DV. Completely convex functions and Lidstone series. T Am Math Soc 1942; 51: 387-398.

[40] Yang S. An identity of symmetry for the Bernoulli polynomials. Discrete Math 2008; 308: 550-554.

[41] Yang H. Several identities for the generalized Apostol Bernoulli polynomials. Comput Math Appl 2008; 56: 2993-2999.