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A Bernstein-type theorem for ξ-submanifolds with flat normal bundle in the Euclidean spaces

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Abstract: ξ-Submanifolds in the Euclidean spaces are a natural extension of self-shrinkers and a generalization of λ-hypersurfaces. Moreover, ξ-submanifolds are expected to take the place of submanifolds with parallel mean curvature vector. In this paper, we establish a Bernstein-type theorem for ξ-submanifolds in the Euclidean spaces. More precisely, we prove that an n-dimensional smooth graphic ξ-submanifold with flat normal bundle in $\mathbb{R}^{n+p}$ is an affine n-plane.

Key words: ξ-Submanifold, Bernstein-type theorem

1. Introduction

In this paper, we are concerned with ξ-submanifolds in the Euclidean spaces. This concept was introduced by Li and Chang [9] in 2016. Its precise definition is as follows. Let $X : M^n \to \mathbb{R}^{n+p}$ be an n-dimensional smooth immersed hypersurface in the $(n+p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$. Then X is called a ξ-submanifold if there is a parallel normal vector field ξ such that the mean curvature vector field H satisfies

$$H + X^{\perp} = \xi, \tag{1.1}$$

where $X^{\perp}$ is the orthogonal projection of the position vector X to the normal space $T^{\perp}M^n$ of X in $\mathbb{R}^{n+p}$.

ξ-Submanifolds in $\mathbb{R}^{n+p}$ are a natural extension of self-shrinkers to the mean curvature flow in $\mathbb{R}^{n+p}$. In fact, when $\xi \equiv 0$, equation (1.1) becomes

$$H + X^{\perp} = 0, \tag{1.2}$$

which is the equation of a self-shrinker. It is known that self-shrinkers play an important role in the study of the mean curvature flow because they describe all possible blow-ups at a given type I singularity of the mean curvature flow. Moreover, ξ-submanifolds in $\mathbb{R}^{n+p}$ are a generalization of λ-hypersurfaces in $\mathbb{R}^{n+1}$ to arbitrary codimensions. When the codimension $p = 1$, a nonzero normal vector field ξ is parallel in the normal bundle if and only if $|\xi|$ is constant. In that case, there is a constant λ such that $\xi = \lambda N$, where N is unit normal vector of M^n. Then equation (1.1) becomes

$$|H| + \langle X, N\rangle = \lambda, \tag{1.3}$$

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which was called a λ-hypersurface by Cheng and Wei [3] in 2014. There have been some interesting results for self-shrinkers and λ-hypersurfaces. Equation (1.3) was first studied by McGonagle and Ross [12]. Denote by $\mu(\Sigma)$ the weighted area functional defined by $\mu(\Sigma)=\int_\Sigma e^{-\frac{|X|^2}{4}}dA$ for any n-dimensional hypersurface Σ of $\mathbb{R}^{n+1}$. They investigated the Gaussian isoperimetric problem as follows: which one has the least weighted boundary area among all Σ enclosing regions with the same weighted volume? They proved that critical points of the Gaussian isoperimetric problem are λ-hypersurfaces and the only smooth stable ones are hyperplanes. Cheng and Wei [3] proved that λ-hypersurfaces are critical points of the weighted area functional for the weighted volume-preserving variations. Moreover, they extended a result of Colding and Minicozzi[5] by defining a F-functional of λ-hypersurfaces and studying their F-stability.

It is known that self-shrinkers can be characterized as minimal hypersurfaces in Gaussian metric space $(\mathbb{R}^{n+p}, g_{AB})$ where $g_{AB}=e^{-\frac{|X|^2}{n}}\delta_{AB}$. The λ-hypersurfaces can be characterized as having constant weighted mean curvature $\tilde{H}=e^{-\frac{|X|^2}{2n}}\bar{H}$, where $\bar{H}$ is the mean curvature of M^n in Gaussian metric space. As Li and Li[10] pointed out, if self-shrinkers and λ-hypersurfaces take the places of minimal submanifolds and hypersurfaces with constant mean curvature, respectively, then ξ-submanifolds are expected to take the place of submanifolds with parallel mean curvature vector. Therefore, the research of ξ-submanifolds is interesting and significant.

In the last few years, a few results about the rigidity and characterization of ξ-submanifolds have been obtained. In 2016, Li and Chang [9] derived a rigidity theorem for Lagrangian ξ-submanifolds in the complex 2-plane $\mathbb{C}^2$. Li and Li [10] gave some characterizations for ξ-submanifolds. They showed that a submanifold in $\mathbb{R}^{m+p}$ is a ξ-submanifold if and only if its modified mean curvature $\tilde{H}=e^{-\frac{|X|^2}{2n}}\bar{H}$ is parallel when it is viewed as a submanifold in the Gaussian space $(\mathbb{R}^{n+p}, e^{-\frac{|x|^2}{n}}\delta_{AB})$. In [10], the authors also proved that any complete and properly immersed ξ-submanifold with a normal bundle must be an n-plane if it is W-stable.

As we know, the Bernstein theorem for minimal surface in $\mathbb{R}^3$ states that if it is a graph defined on $\mathbb{R}^2$, then it is a plane (cf. [1]). In the theory of minimal surfaces, the Bernstein theorem for entire minimal graphs plays a fundamental role. Some researchers have derived some Bernstein-type results (see [7, 13–16] and the reference therein). According to the counter-example given by Lawson and Osserman [8], we know that minimal submanifolds of higher codimension in the Euclidean spaces are more complicated. In 2006, Smoczyk et al. [15] considered minimal n-submanifolds in $\mathbb{R}^{n+p}$ with flat normal bundle. They obtained some Bernstein-type theorems. In 2011, Wang [16] proved that smooth self-shrinkers in $\mathbb{R}^{n+1}$ that are entire graphs are hyperplanes. In 2012, Luo [11] proved that if M is an n-dimensional graphic self-shrinker in $\mathbb{R}^{n+m}$ with flat normal bundle, then M is a linear subspace.

In this paper, we establish the following Bernstein-type theorem for ξ-submanifolds in Euclidean spaces.

Theorem 1.1 *If M^n is an n-dimensional graphic ξ-submanifold in $\mathbb{R}^{n+p}$ with flat normal bundle, then M^n is an affine n-plane.*

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we first discuss the volume growth for proper ξ-submanifolds and then establish Theorem 3.1. Then we use Theorem 3.1 to give the proof of Theorem 1.1.

2. Preliminaries

In this section, we give some notations and formulas for ξ-submanifolds in the Euclidean space. Let $X : M^n \to \mathbb{R}^{n+p}$ be an isometric immersion. Denote by ∇ and $\bar{\nabla}$ the Levi-Civita connections on M^n and $\mathbb{R}^{n+p}$, respectively. The induced connections on the tangent bundle TM and the normal bundle NM are defined by

$$\nabla_V W = (\bar{\nabla}_V W)^T \quad \text{and} \quad \nabla^{\perp}_V \nu = (\bar{\nabla}_V \nu)^N$$

for $V, W \in \Gamma(TM), \nu \in \Gamma(NM)$, where $(\cdots)^T$ and $(\cdots)^N$ are the projections onto the tangent bundle TM and the normal bundle NM, respectively. The second fundamental form B is defined by

$$B(V,W) = (\bar{\nabla}_V W)^N = \bar{\nabla}_V W - \nabla_V W,$$

and taking the trace of B gives the mean curvature vector H of M^n in $\mathbb{R}^{n+p}$.

Let $\{e_1, \cdots, e_n\}$ be a local tangent orthonormal frame field on M^n with respect to the induced metric, $\{\theta_1, \cdots, \theta_n\}$ be their dual 1-forms, and $\{\nu_1, \cdots, \nu_p\}$ be a local normal orthonormal frame field on M^n. From now on, we use the following convention on the range of indices:

$$i, j, k, l = 1, \cdots, n; \quad \alpha, \beta = 1, \cdots, p.$$

Denote by $B = \sum_{i,j,\alpha} h^{\alpha}_{ij} \theta_i \otimes \theta_j \otimes \nu_\alpha$ the second fundamental form of M^n. The Gauss equations are given by

$$R_{ijkl} = \sum_{\alpha} (h^{\alpha}_{ik} h^{\alpha}_{jl} - h^{\alpha}_{il} h^{\alpha}_{jk}). \tag{2.1}$$

The Codazzi equations are given by

$$h^{\alpha}_{ijk} = h^{\alpha}_{ikj}. \tag{2.2}$$

where h^{α}_{ijk} are the covariant derivatives of h^{α}_{ij}. The Ricci equations are given by

$$R_{\alpha\beta kl} = \sum_{i} (h^{\alpha}_{il} h^{\beta}_{ik} - h^{\beta}_{il} h^{\alpha}_{ik}), \tag{2.3}$$

where $R_{\alpha\beta kl}$ are the components of the normal curvature tensor $R^{\perp}$. If $R^{\perp} = 0$, we say that M^n is a submanifold with flat normal bundle. Then the Ricci equation becomes

$$\sum_{i} (h^{\alpha}_{il} h^{\beta}_{ik} - h^{\beta}_{il} h^{\alpha}_{ik}) = 0. \tag{2.4}$$

Typical examples of ξ-submanifolds include the ξ-curves, the standard spheres centered at the origin, submanifolds in a sphere with parallel mean curvature vector, and so on (cf. [10]). For convenience, here we give the details of the n-planes not necessarily passing through the origin. As subplanes of the Euclidean spaces, they are important examples of ξ-submanifolds. An n-plane $x : P^n \to \mathbb{R}^{n+p}$ $(p \geq 0)$ is the inclusion map of a n-dimensional connected, complete, and totally geodesic submanifold of $\mathbb{R}^{n+p}$. Let p_0 be the orthogonal projection of the origin 0 onto P^n and ξ be the position vector of p_0, which is constant and is parallel along P^n. Then it is clear that P^n is a ξ-submanifold because $H \equiv 0$ and the tangential part $x^{\top}$ of x is precisely $x - \xi$.

Let U, V be two Hausdoff topological spaces. We say that a continuous mapping $f : U \to V$ is proper if $f^{-1}(K)$ is compact in U for any compact subset K in V.

3. The volume growth for proper ξ-submanifolds and the proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. For this goal, we first study the volume growth for complete and noncompact properly immersed ξ-submanifolds.

For n-dimensional complete and noncompact self-shrinkers, Ding and Xin [6] proved that any complete noncompact properly immersed self-shrinker in the Euclidean space has polynomial area growth. Cheng and Zhou [4] showed that any complete immersed self-shrinker with polynomial volume growth in the Euclidean space is proper. Hence, we can know that a complete immersed self-shrinker is proper if and only if it has polynomial area growth. Cheng and Wei[3] proved some similar results for λ-hypersurfaces.

The definition of the volume growth for complete submanifolds can be given as follows: Let $X : M^n \to \mathbb{R}^{n+p}$ be an n-dimensional complete submanifold in $\mathbb{R}^{n+p}$. We say that M^n has polynomial volume growth if there exist two constants C and d such that for all $r \geq 1$, the inequality

$$\mathrm{Vol}\left(B_r(0) \cap X(M^n)\right) \leq C r^d \tag{3.1}$$

holds, where $B_r(0)$ is a ball centered at the origin with radius r.

In the proof of Theorem 3.1, we need the result derived by Cheng and Zhou [4].

Lemma 3.1 *Let M be a complete and noncompact Riemannian manifold. If f is a proper C^∞ function on M satisfying $|\nabla f|^2 \leq f$ on the level set $D_r = \{x \in M : 2\sqrt{f} < r\}$ of $2\sqrt{f}$ for all r, and $\Delta_f f + f \leq k$ for some constant k, then*

$$V_f(M) = \int_M e^{-f} dv < +\infty$$

and

$$V(r) = Vol(D_r) = \int_{D_r} dv \leq C r^{2k}$$

for $r \geq 1$, where C is a constant depending only on $\int_M e^{-f} dv$.

Now we give the following result about the volume growth for proper ξ-submanifolds in Euclidean spaces.

Theorem 3.1 *Let $X : M^n \to \mathbb{R}^{n+p}$ be an n-dimensional complete and noncompact properly immersed ξ-submanifold in the $(n+p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$. Then M^n has polynomial volume growth.*

Proof Set $f = \frac{|X|^2}{4}$ and $\beta = \frac{1}{4}\inf|\xi - H|^2 \geq 0$. Since ξ is parallel in the normal bundle, we know that $|\xi|$ is constant. Because the immersion X is proper, we know that $\widetilde{f} = f - \beta$ is proper. Noticing that

$$\nabla|X|^2 = (\bar{\nabla}|X|^2)^\top = 2X^\top,$$

we have

$$\begin{aligned} f - |\nabla f|^2 &= \frac{|X|^2}{4} - \frac{|X^\top|^2}{4} = \frac{|X^\perp|^2}{4} \\ &= \frac{1}{4}|\xi - H|^2 \geq \beta. \end{aligned} \tag{3.2}$$

It implies

$$|\nabla\widetilde{f}|^2 = |\nabla(f-\beta)|^2 = |\nabla f|^2 \le f-\beta = \widetilde{f}. \tag{3.3}$$

On the other side, we deduce

$$\begin{aligned}\frac{1}{4}|\xi|^2 - \frac{1}{4}|H|^2 - f + |\nabla f|^2 =& \frac{1}{4}|\xi|^2 - \frac{1}{4}|H|^2 - \frac{1}{4}|\xi - H|^2\\ =& \frac{1}{2}\langle H, \xi - H\rangle\\ =& \frac{1}{2}\langle H, X^\perp\rangle.\end{aligned} \tag{3.4}$$

Hence, using (3.4), we get

$$\begin{aligned}\Delta f =& \frac{1}{2}\langle \Delta X, X\rangle + \frac{1}{2}|\nabla X|^2\\ =& \frac{1}{2}\langle H, X^\perp\rangle + \frac{1}{2}n\\ =& \frac{1}{2}n + \frac{1}{4}|\xi|^2 - \frac{1}{4}|H|^2 - f + |\nabla f|^2.\end{aligned} \tag{3.5}$$

Therefore, it is from (3.2) and (3.5) that

$$\begin{aligned}\Delta\widetilde{f} - |\nabla\widetilde{f}|^2 + \widetilde{f} =& \Delta(f-\beta) - |\nabla(f-\beta)|^2 + (f-\beta)\\ \le& \frac{1}{2}n + \frac{1}{4}|\xi|^2 - \beta - \frac{1}{4}\inf|H|^2.\end{aligned} \tag{3.6}$$

Denote by $\widetilde{D}_r$ the level set of $2\sqrt{\widetilde{f}}$ as follows:

$$\widetilde{D}_r = \{x \in M^n : 2\sqrt{\widetilde{f}} < r\}.$$

That is to say,

$$\widetilde{D}_r = \{x \in M^n : |x| < \sqrt{r^2 + 4\beta}\} = B_{\sqrt{r^2+4\beta}}(0) \cap M^n.$$

Applying Lemma 3.1 to $\widetilde{f} = f - \beta$ with the constant $\kappa = \frac{1}{2}n + \frac{1}{4}|\xi|^2 - \beta - \frac{1}{4}\inf|H|^2$, we get

$$\mathrm{Vol}(B_{\sqrt{r^2+4\beta}}(0) \cap M^n) < Cr^{2\kappa}.$$

Hence, we have

$$\mathrm{Vol}(B_r(0) \cap M^n) < Cr^{2\kappa},$$

where C is a constant. In other words, M^n has polynomial volume growth. This completes the proof of Theorem 3.1. □

Now we can give the proof of Theorem 1.1.

Proof of theorem 1.1 Let M^n be an n-dimensional complete submanifold in $\mathbb{R}^{n+p}$. Denote by ∇ and $\bar{\nabla}$ Levi-Civita connections on M^n and $\mathbb{R}^{n+p}$, respectively. Let $\{e_1, \cdots, e_n\}$ be a local tangent orthonormal frame field on M^n with respect to the induced metric and $\{\nu_1, \cdots, \nu_p\}$ be a local normal orthonormal frame field on M^n. Let

$$\zeta = a_1 \wedge \cdots \wedge a_n$$

be any fixed n-vector on M^n. Set

$$\omega = \langle e_1 \wedge \cdots \wedge e_n, \zeta\rangle = \det(\langle e_i, a_j\rangle).$$

Now we want to calculate the Laplacian of ω. One can find similar computations in some references (see [2] by Chen and Piccinni). For the convenience of the reader, we give these computations here. We have

$$\begin{aligned} e_i(\omega) &= \sum_k \langle e_1 \wedge \cdots \wedge \bar{\nabla}_{e_i} e_k \wedge \cdots \wedge e_n, \zeta\rangle \\ &= \sum_{k,\alpha} h^\alpha_{ik} \langle e_1 \wedge \cdots \wedge e_{k-1} \wedge \nu_\alpha \wedge e_{k+1} \wedge \cdots \wedge e_n, \zeta\rangle. \end{aligned} \tag{3.7}$$

Using (3.7), and noticing that

$$\begin{aligned} e_1 \wedge \cdots \wedge e_{k-1} \wedge \nabla_{e_j} \nu_\alpha \wedge e_{k+1} \wedge \cdots \wedge e_n &= -\sum_{l,\alpha} e_1 \wedge \cdots \wedge e_{k-1} \wedge h^\alpha_{jl} e_l \wedge e_{k+1} \wedge \cdots \wedge e_n \\ &= -h^\alpha_{jk} e_1 \wedge \cdots \wedge e_n, \end{aligned}$$

we obtain

$$\begin{aligned} \nabla_{e_j}\nabla_{e_i}\omega =& \sum_{k,\alpha} (\nabla_{e_j} h^\alpha_{ik}) \langle e_1 \wedge \cdots \wedge e_{k-1} \wedge \nu_\alpha \wedge e_{k+1} \wedge \cdots \wedge e_n, \zeta\rangle \\ &+ \sum_{k,\alpha} h^\alpha_{ik} \langle e_1 \wedge \cdots \wedge e_{k-1} \wedge \nabla_{e_j}\nu_\alpha \wedge e_{k+1} \wedge \cdots \wedge e_n, \zeta\rangle \\ &+ \sum_{\substack{k,\alpha \\ l\neq k}} h^\alpha_{ik} \langle e_1 \wedge \cdots \wedge \nabla_{e_j} e_l \wedge \cdots \wedge e_{k-1} \wedge \nu_\alpha \wedge e_{k+1} \wedge \cdots \wedge e_n, \zeta\rangle \\ =& \sum_{k,\alpha} h^\alpha_{ijk} \langle e_1 \wedge \cdots \wedge e_{k-1} \wedge \nu_\alpha \wedge e_{k+1} \wedge \cdots \wedge e_n, \zeta\rangle - \sum_{k,\alpha} h^\alpha_{ik} h^\alpha_{jk} \langle e_1 \wedge \cdots \wedge e_n, \zeta\rangle \\ &+ \sum_{\substack{k,l\neq k \\ \alpha,\beta\neq\alpha}} h^\alpha_{ik} h^\beta_{jl} \langle e_1 \wedge \cdots \wedge e_{l-1} \wedge \nu_\beta \wedge e_{l+1} \wedge \cdots \wedge e_{k-1} \wedge \nu_\alpha \wedge e_{k+1} \wedge \cdots \wedge e_n, \zeta\rangle. \end{aligned} \tag{3.8}$$

It implies

$$\begin{aligned} \Delta\omega =& -|B|^2\omega + \sum_{i,k,\alpha} h^\alpha_{iik} \langle e_1 \wedge \cdots \wedge e_{k-1} \wedge \nu_\alpha \wedge e_{k+1} \wedge \cdots \wedge e_n, \zeta\rangle \\ &+ \sum_{\substack{i,k,l \\ \beta<\alpha}} (h^\alpha_{ik} h^\beta_{il} - h^\beta_{ik} h^\alpha_{il}) \langle e_1 \wedge \cdots \wedge e_{l-1} \wedge \nu_\beta \wedge e_{l+1} \wedge \cdots \wedge e_{k-1} \wedge \nu_\alpha \wedge e_{k+1} \wedge \cdots \wedge e_n, \zeta\rangle. \end{aligned} \tag{3.9}$$

According to the definition of the ξ-submanifold, normal vector field ξ is parallel. Hence, it holds that $\nabla^\perp_{e_k}\xi = 0$. Thus, we have

$$\sum_{i,\alpha} h^\alpha_{iik}\nu_\alpha = \nabla^\perp_{e_k} H = \nabla^\perp_{e_k}\xi - \nabla^\perp_{e_k} X^\perp = \sum_{i,\alpha} h^\alpha_{ik} \langle X, e_i\rangle \nu_\alpha, \tag{3.10}$$

where we use $\nabla^{\perp}\xi = 0$. Hence, it follows from (3.10) that

$$\begin{aligned}
&\sum_{i,k,\alpha} h^{\alpha}_{iik}\langle e_1 \wedge \cdots \wedge e_{k-1} \wedge \nu_\alpha \wedge e_{k+1} \wedge \cdots \wedge e_n, \zeta\rangle \\
=&\sum_{i,k,\alpha} h^{\alpha}_{ik}\langle X, e_i\rangle\langle e_1 \wedge \cdots \wedge e_{k-1} \wedge \nu_\alpha \wedge e_{k+1} \wedge \cdots \wedge e_n, \zeta\rangle \\
=&\sum_{i}\langle X, e_i\rangle \nabla_{e_i}\omega \\
=&\langle X, \nabla\omega\rangle.
\end{aligned} \tag{3.11}$$

Moreover, since M^n is a submanifold in $\mathbb{R}^{n+p}$ with flat normal bundle, (2.4) holds. Therefore, substituting (2.4) and (3.11) into (3.9), we know that for ξ-submanifold with flat normal bundle, ω satisfies the following equation:

$$\Delta\omega - \langle X, \nabla\omega\rangle + |B|^2\omega = 0. \tag{3.12}$$

Note that M^n is an entire graphic submanifold of $\mathbb{R}^{n+p}$. Without loss of generality, we can find an n-vector ζ such that ω is positive on M^n. Set $\rho = e^{-\frac{|X|^2}{2}}$. Let η be a smooth function with a compact support on M^n. Multiplying equation (3.12) by $\omega^{-1}\eta^2\rho$, and integrating on M^n, we derive

$$\begin{aligned}
0 =& \int_{M^n} \frac{\eta^2}{\omega}\left(\Delta\omega - \langle X, \nabla\omega\rangle\right)\rho dv + \int_{M^n} \eta^2|B|^2\rho dv \\
=& \int_{M^n} \frac{\eta^2}{\omega}\mathrm{div}(\rho\nabla\omega) dv + \int_{M^n} \eta^2|B|^2\rho dv \\
=& -\int_{M^n} \langle \nabla(\frac{\eta^2}{\omega}), \nabla\omega\rangle\rho dv + \int_{M^n} \eta^2|B|^2\rho dv \\
=& -2\int_{M^n} \frac{\eta}{\omega}\langle\nabla\eta, \nabla\omega\rangle\rho dv + \int_{M^n} \frac{\eta^2}{\omega^2}|\nabla\omega|^2\rho dv + \int_{M^n} \eta^2|B|^2\rho dv.
\end{aligned} \tag{3.13}$$

Using the Cauchy–Schwarz inequality, we deduce

$$\begin{aligned}
0 \geq& -\int_{M^n} |\nabla\eta|^2\rho - \int_{M^n} \frac{\eta^2}{\omega^2}|\nabla\omega|^2\rho + \int_{M^n} \frac{\eta^2}{\omega^2}|\nabla\omega|^2\rho + \int_{M^n} \eta^2|B|^2\rho \\
=& -\int_{M^n} |\nabla\eta|^2\rho + \int_{M^n} \eta^2|B|^2\rho.
\end{aligned} \tag{3.14}$$

Namely, we get the following stability inequality:

$$\int_{M^n} \eta^2|B|^2\rho \leq \int_{M^n} |\nabla\eta|^2\rho, \qquad \text{for } \eta \in C_0^\infty(M^n). \tag{3.15}$$

Now we can choose $0 \leq \eta \leq 1$ to be a function defined on M^n, which equals 1 on $B_r(0) \cap M^n$ and equals 0 outside $B_{2r}(0) \cap M^n$, with first derivatives bounded by $2/r$. Then we have

$$\begin{aligned}
\int_{M^n} \eta^2|B|^2\rho dv =& \int_{B_r(0)\cap M^n} \eta^2|B|^2\rho dv + \int_{M^n \setminus (B_r(0)\cap M^n)} \eta^2|B|^2\rho dv \\
\geq& \int_{M^n \cap B_r(0)} |B|^2\rho dv.
\end{aligned} \tag{3.16}$$

Combining (3.15) with (3.16), we obtain

$$\begin{aligned}\int_{B_r(0)\cap M^n}|B|^2\rho &\le \int_{M^n}|\nabla\eta|^2\rho dv \le \frac{4}{r^2}\int_{\left(B_{2r}(0)\setminus B_r(0)\right)\cap M^n}\rho dv \\ &\le \frac{4}{r^2}e^{-\frac{r^2}{2}}\mathrm{Vol}\left(B_{2r}(0)\cap M^n\right).\end{aligned} \tag{3.17}$$

Noticing that M^n is a graph in $\mathbb{R}^{n+p}$, it is proper. According to Theorem 3.1, we know that M^n has polynomial volume growth. Therefore, the following holds:

$$r^{-2}e^{-r^2/2}\mathrm{Vol}\left(B_{2r}(0)\cap M^n\right)\to 0,$$

when $r\to\infty$. It implies $|B|\equiv 0$. Then we know that M^n is an affine n-plane. This concludes the proof of Theorem 1.1. □

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