Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 511 – 522
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doi:10.3906/mat-1809-119

Categorical structures of Lie–Rinehart crossed module

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Received: 27.09.2018 • **Accepted/Published Online:** 03.01.2019 • **Final Version:** 18.01.2019

Abstract: In this paper we give constructions of pullback, finite product, finite limit, coproduct, colimit, pushout, etc. in a special full subcategory $\mathfrak{XMod}/\mathcal{L}$ of the category of Lie–Rinehart crossed modules.

Key words: Lie–Rinehart algebra, pullback, pushout, crossed module

1. Introduction

Lie–Rinehart algebras are the section spaces of Lie algebroids. In other words, they are the algebraic analogues of Lie algebroids. The theory has been improved from the early 1950s and today it has a large application area in differential geometry, physics, and algebra. For a comprehensive investigation, see [6] and [7].

The notion of crossed modules was introduced by Whitehead in [9], as an algebraic model for homotopy 2-types. After that, crossed modules have been one of the fundamental concepts in several areas of mathematics, namely homotopy theory, (co)homology of groups, algebraic K-theory, and combinatorial group theory.

The Lie–Rinehart algebra version of the crossed module was introduced in [3] and it was shown that the third-dimensional cohomology of Lie–Rinehart algebras classifies Lie–Rinehart crossed modules. Some extra results can be found in [2, 4].

The aim of this paper is to investigate the categorical structure of the category of Lie–Rinehart crossed modules of the same base such as equalizers, products, pullbacks, limits, and dual objects.

Similar works about crossed modules over algebras can be found in the literature [8]. Our case is quite different, because the category of Lie–Rinehart algebras has no zero objects.

2. Preliminaries

2.1. Categorical background

In this subsection we will give definitions of some well-known categorical notions needed in the sequel. See [1] for details.

Definition 1 *Let $\mathfrak{C}$ be a category and $A, B \in \mathfrak{C}$ two objects of $\mathfrak{C}$. A (Cartesian) product of A and B is, by definition, a triple (P, p_A, p_B) where*

(i) $P \in \mathfrak{C}$ is an object,

(ii) $p_A : P \longrightarrow A$ and $p_B : P \longrightarrow B$ are morphisms,

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2010 *AMS Mathematics Subject Classification:* 17B55, 18A30, 18G55

and this triple is such that for every other triple (Q, q_A, q_B) *where*

(i) $Q \in \mathfrak{C}$ *is an object,*

(ii) $q_A : Q \longrightarrow A$ *and* $q_B : Q \longrightarrow B$ *are morphisms,*

there exists a unique morphism $r : Q \longrightarrow P$ *such that* $q_A = p_A \circ r$ *and* $q_B = p_B \circ r$.

Definition 2 *Let* I *be a set and* $(C_i)_{i\in I}$ *a family of objects in a given category* $\mathfrak{C}$. *A coproduct of that family is a pair* $(P, (s_i)_{i\in I})$ *where*

(i) P *is an object of* $\mathfrak{C}$,

(ii) for every $i \in I$, $s_i : C_i \longrightarrow P$ *is a morphism of* $\mathfrak{C}$,

and this pair is such that for every other pair $(Q, (t_i)_{i\in I})$ *where*

(i) Q *is an object of* $\mathfrak{C}$,

(ii) for every $i \in I$, $t_i : C_i \longrightarrow Q$ *is a morphism of* $\mathfrak{C}$,

there exists a unique morphism $r : P \longrightarrow Q$ *such that for every index* i, $t_i = r \circ s_i$.

Definition 3 *Consider two arrows* $f, g : A \longrightarrow B$ *in a category* $\mathfrak{C}$. *An equalizer of* f, g *is a pair* (K, k) *where*

(i) K *is an object of* $\mathfrak{C}$,

(ii) $k : K \longrightarrow A$ *is an arrow of* $\mathfrak{C}$ *such that* $f \circ k = g \circ k$,

and such that for every pair (M, m) *where*

(i) M *is an object of* $\mathfrak{C}$,

(ii) $m : M \longrightarrow A$ *is an arrow of* $\mathfrak{C}$ *such that* $f \circ m = g \circ m$,

there exists a unique morphism $n : M \longrightarrow K$ *such that* $m = k \circ n$.

By duality, one defines the coequalizer of two morphisms when it exists.

Definition 4 *Consider two morphisms* $f : A \longrightarrow C$, $g : B \longrightarrow C$ *in a category* $\mathfrak{C}$. *A pullback of* (f, g) *is a triple* (P, f', g') *where*

(i) P *is an object of* $\mathfrak{C}$,

(ii) $f' : P \longrightarrow B$, $g' : P \longrightarrow A$ *are morphisms of* $\mathfrak{C}$ *such that* $f \circ g' = g \circ f'$,

and for every other triple (Q, f'', g'') *where*

(i) Q *is an object of* $\mathfrak{C}$,

(ii) $f'' : Q \longrightarrow B$, $g'' : Q \longrightarrow A$ *are morphisms of* $\mathfrak{C}$ *such that* $f \circ g'' = g \circ f''$,

there exists a unique morphism $q : Q \longrightarrow P$ *such that* $f'' = f' \circ q$ *and* $g'' = g' \circ q$.

Additionally, one defines the pushout object by duality whenever it exists.

Definition 5 *A category* $\mathfrak{C}$ *is complete when every functor*

$$\mathbf{F} : \mathfrak{D} \longrightarrow \mathfrak{C},$$

with $\mathfrak{D}$ *a small category, has a limit. The category* $\mathfrak{C}$ *is finitely complete when every functor*

$$\mathbf{F} : \mathfrak{D} \longrightarrow \mathfrak{C},$$

with $\mathfrak{D}$ *a finite category, has a limit.*

Proposition 6 *For a category $\mathfrak{C}$, the following conditions are equivalent:*

(1) $\mathfrak{C}$ *is finitely complete;*

(2) $\mathfrak{C}$ *has a terminal object, binary products, and equalizers;*

(3) $\mathfrak{C}$ *has a terminal object and pullbacks.*

By duality we get the notion of a cocomplete category.

2.2. Lie–Rinehart algebras

From now on, we assume that $\mathbf{k}$ is a field, A is a commutative algebra over $\mathbf{k}$, and $Der(A)$ is the set of all $\mathbf{k}$-derivations. Recall that $Der(A)$ is a Lie $\mathbf{k}$-algebra with the bracket

$$[D, D'] = D \circ D' - D' \circ D$$

and is an A-module with

$$a(D(x)) = D(ax),$$

for all $D, D' \in Der(A)$, $a, x \in A$.

Lie–Rinehart algebras were introduced by Herz in [5], named as "pseudo-algebra de Lie".

Definition 7 *Let $\mathcal{L}$ be a Lie $\mathbf{k}$-algebra and an A-module and $\alpha : \mathcal{L} \longrightarrow Der(A)$ is an A-module and a Lie $\mathbf{k}$-algebra homomorphism, which is called the anchor. Then the pair $(\mathcal{L}, \alpha)$ is called a Lie–Rinehart A-algebra over A if*

$$[l, al'] = a[l, l'] + l(a)l',$$

for all $l, l' \in \mathcal{L}$, $a \in A$ where $l(a) = \alpha(l)(a)$. In general this pair is denoted by $\mathcal{L}$ if there is no confusion.

In the rest of this paper we accept that all Lie algebras will be over a fixed field $\mathbf{k}$ and all Lie–Rinehart algebras will be over A.

Definition 8 *Let $(\mathcal{L}, \alpha)$ and $(\mathcal{L}', \alpha')$ be Lie–Rinehart algebras. Let $f : \mathcal{L} \longrightarrow \mathcal{L}'$ be a Lie $\mathbf{k}$-algebra homomorphism and an A-module homomorphism. If $\alpha' f = \alpha$ then f is called a Lie–Rinehart algebra homomorphism.*

Consequently, we get the category $\mathfrak{LR}(A)$ of Lie–Rinehart algebras. The category $\mathfrak{L}(A)$ of Lie A-algebras is a full subcategory of $\mathfrak{LR}(A)$.

Examples a) If $\alpha = 0$ for a Lie–Rinehart algebra $\mathcal{L}$ then obviously $\mathcal{L}$ is a Lie A-algebra. Also, for a $\mathbf{k}$-algebra A, $Der(A)$ is a Lie–Rinehart algebra.

b) If $\mathcal{L}$ is a Lie–Rinehart algebra over A, then $\mathcal{L} \rtimes A$ with Lie bracket $[(l, a), (l', b)] = ([l, l'], l(b) - l'(a))$ and anchor map $\widetilde{\alpha} : \mathcal{L} \rtimes A \longrightarrow Der(A)$, $\widetilde{\alpha}(l, a) = \alpha(l)$, is a Lie–Rinehart algebra, where $\alpha : \mathcal{L} \longrightarrow Der(A)$ is the anchor of $\mathcal{L}$.

Definition 9 *Let* $(\mathcal{L},\alpha)$ *be a Lie–Rinehart algebra. A Lie–Rinehart subalgebra* $\mathcal{N}$ *of* $\mathcal{L}$ *consists of a Lie* $\mathbf{k}$*-subalgebra* $\mathcal{N}$*, which is an* A*-module, and* A *acts on* $\mathcal{N}$ *via the composition*

$$\mathcal{N} \hookrightarrow \mathcal{L} \xrightarrow{\alpha} Der(A).$$

It is said that a Lie–Rinehart subalgebra $\mathcal{N}$ *of* $\mathcal{L}$ *is an ideal if* $\mathcal{N}$ *is an ideal of* $\mathcal{L}$ *as Lie* $\mathbf{k}$*-algebra and the composition*

$$\mathcal{N} \hookrightarrow \mathcal{L} \xrightarrow{\alpha} Der(A)$$

is trivial.

Now we will recall the below definitions from [3].

Definition 10 *Let* $(\mathcal{L},\alpha)$ *be a Lie–Rinehart algebra and* $\mathcal{R}$ *be a Lie* A*-algebra. The action of* $\mathcal{L}$ *on* $\mathcal{R}$ *is a* $\mathbf{k}$*-linear map*

$$\begin{array}{ccc} \mathcal{L}\times\mathcal{R} & \longrightarrow & \mathcal{R} \\ (l,r) & \longmapsto & {}^{l}r \end{array}$$

that satisfies the following axioms:

Act 1) ${}^{[l,l']}r = {}^{l}({}^{l'}r) - {}^{l'}({}^{l}r)$,

Act 2) ${}^{l}[r_1,r_2] = [{}^{l}r_1,r_2] + [r_1,{}^{l}r_2]$,

Act 3) ${}^{al}r = a({}^{l}r)$,

Act 4) ${}^{l}(ar) = a({}^{l}r) + (\alpha(l)(a))r$,

for all $l,l' \in \mathcal{L}$*,* $r,r_1,r_2 \in \mathcal{R}$ *and* $a \in A$*.*

Let $\mathcal{L}$ be a Lie–Rinehart algebra, $\mathcal{R}$ be a Lie A-algebra, and $\mathcal{L}$ act on $\mathcal{R}$. Then $\mathcal{R}\rtimes\mathcal{L}$ is a Lie $\mathbf{k}$-algebra with the Lie bracket

$$[(r,l),(r',l')] = ([r,r'] + {}^{l}r' - {}^{l'}r, [l,l']).$$

Define $\widetilde{\alpha} : \mathcal{R}\rtimes\mathcal{L} \to Der(A)$ by $\widetilde{\alpha}(r,l) = \alpha(l)$, and then the pair $(\mathcal{R}\rtimes\mathcal{L},\widetilde{\alpha})$ is a Lie–Rinehart algebra, where the underlying set of $\mathcal{R}\rtimes\mathcal{L}$ is $\mathcal{R}\times\mathcal{L}$.

An abelian Lie A-algebra $\mathcal{R}$ with an action of $\mathcal{L}$ on it is called an $(\mathcal{L},A)$-module.

2.3. Lie–Rinehart crossed modules

In this subsection, we will recall the notion of the Lie–Rinehart crossed module, which was introduced by Casas et al. in [3].

Definition 11 *A Lie–Rinehart crossed module* $\partial : \mathcal{R} \longrightarrow \mathcal{L}$ *consists of a Lie–Rinehart algebra* $\mathcal{L}$ *and a Lie* A*-algebra* $\mathcal{R}$ *together with the action of* $\mathcal{L}$ *on* $\mathcal{R}$ *and the Lie* $\mathbf{k}$*-algebra homomorphism* ∂ *such that the following conditions hold:*

CM 1) $\partial({}^{l}r) = [l,\partial(r)]$,

CM 2) ${}^{\partial(r')}r = [r',r]$,

CM 3) $\partial(ar) = a\partial(r)$,

CM 4) $\partial(r)(a) = 0$,

for all $r, r' \in \mathcal{R}$, $l \in \mathcal{L}$ *and* $a \in A$.

Such a crossed module will be denoted by $(\mathcal{R}, \mathcal{L}, \partial)$.

Examples a) Let $\mathcal{L}$ be a Lie–Rinehart algebra and I an ideal of $\mathcal{L}$.

$$\begin{array}{rcl} i : I & \longrightarrow & \mathcal{L} \\ l & \longmapsto & l \end{array}$$

is a crossed module with the action of $\mathcal{L}$ on I defined by

$$\begin{array}{rcl} \mathcal{L} \times I & \longrightarrow & I \\ (l, r) & \longmapsto & [l, r]. \end{array}$$

b) Let $\mathcal{R}$ be an $(\mathcal{L}, A)$**-**module. Then the zero morphism $0 : \mathcal{R} \longrightarrow \mathcal{L}$ is a crossed module.

c) For any Lie–Rinehart morphism $f : \mathcal{L} \longrightarrow \mathcal{L}'$, $\ker f \hookrightarrow \mathcal{L}$ is a crossed module.

d) Let $\theta : \mathcal{R} \longrightarrow \mathcal{R}'$ be a homomorphism of $(\mathcal{L}, A)$-modules and $\mathcal{R}' \rtimes \mathcal{L}$ be the semidirect product of the Lie-Rinehart algebra $\mathcal{L}$ and the Lie A-algebra $\mathcal{R}'$. As given before, $\mathcal{R}' \rtimes \mathcal{L}$ is a Lie–Rinehart algebra. We have an action of $\mathcal{R}' \rtimes \mathcal{L}$ on $\mathcal{R}$ defined by $(r', l) \cdot r = lr$ for all $l \in \mathcal{L}$, $r \in \mathcal{R}$, and $r' \in \mathcal{R}'$. Define

$$\begin{array}{rcl} \partial : \mathcal{R} & \longrightarrow & \mathcal{R}' \rtimes \mathcal{L} \\ r & \longmapsto & (\theta(r), 0), \end{array}$$

and then $(\mathcal{R}, \mathcal{R}' \rtimes \mathcal{L}, \partial)$ is a Lie–Rinehart crossed module.

Proposition 12 *Let* $\partial : \mathcal{R} \longrightarrow \mathcal{L}$ *be a crossed module and* $I = \partial(\mathcal{R})$. *Then*

i) $Im(\partial) \trianglelefteq \mathcal{L}$,

ii) $\ker(\partial) \trianglelefteq \mathcal{R}$,

iii) $\ker(\partial)$ *is a* $\mathcal{L}/I$*-module,*

iv) $\mathcal{R}/\mathcal{R}^2$ *and* I/I^2 *are* $\mathcal{L}/I$*-modules.*

Proof Can be checked by direct calculation. □

In light of this information, we can think of Lie–Rinehart crossed modules as the generalizations of Lie–Rinehart algebras and ideals.

2.4. The category of Lie–Rinehart crossed modules

Definition 13 *Let $(\mathcal{R},\mathcal{L},\partial)$ and $(\mathcal{R}',\mathcal{L}',\partial')$ be Lie–Rinehart crossed modules. A homomorphism of crossed modules from $(\mathcal{R},\mathcal{L},\partial)$ to $(\mathcal{R}',\mathcal{L}',\partial')$ is a pair (f,ϕ) of a Lie $\mathbf{k}$-algebra homomorphism $f:\mathcal{R}\longrightarrow\mathcal{R}'$ and a Lie–Rinehart algebra homomorphism $\phi:\mathcal{L}\longrightarrow\mathcal{L}'$ such that*

$$f(l_r)=\phi(l)_{f(r)},\ \ \partial' f(r)=\phi\partial(r),$$

for all $l\in\mathcal{L},\ r\in\mathcal{R}$.

Consequently, we have the category of Lie–Rinehart crossed modules, which we will denote by $\mathfrak{Xmod}(\mathfrak{LR})$.

Now we will give some basic functorial properties of this category. Obviously, we can easily define some forgetful functors as follows:

$$\begin{array}{rcl} U_1:\mathfrak{Xmod}(\mathfrak{LR}) & \longrightarrow & \mathfrak{LR}(A) \\ (\mathcal{R},\mathcal{L},\partial) & \longmapsto & \mathcal{L} \end{array}$$

$$\begin{array}{rcl} U_2:\mathfrak{Xmod}(\mathfrak{LR}) & \longrightarrow & \mathfrak{L}(A) \\ (\mathcal{R},\mathcal{L},\partial) & \longmapsto & \mathcal{R} \end{array}$$

If we denote the category of Lie $\mathbf{k}$-algebras by $\mathfrak{Xmod}(\mathfrak{Lie})$, then we have another forgetful functor:

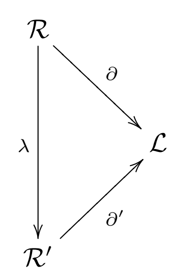
$$U_3:\mathfrak{Xmod}(\mathfrak{LR}) \longrightarrow \mathfrak{Xmod}(\mathfrak{Lie}),$$

which forgets the A-module structure.

3. Categorical structure of $\mathfrak{Xmod}(\mathfrak{LR})$

In this section we will define the category $\mathfrak{Xmod}/\mathcal{L}$ of Lie–Rinehart crossed modules with fixed base $\mathcal{L}$. Many of the following notions do not exist in the category of Lie–Rinehart algebras, since the zero object does not exist in this category.

Let $\mathcal{L}$ be a fixed Lie–Rinehart algebra. We define a full subcategory of $\mathfrak{Xmod}(\mathfrak{LR})$ whose objects are Lie–Rinehart crossed modules with base $\mathcal{L}$. We will denote this category by $\mathfrak{Xmod}/\mathcal{L}$. An object $(\mathcal{R},\mathcal{L},\partial)$ of $\mathfrak{Xmod}/\mathcal{L}$ will be called a crossed $\mathcal{L}$-module and denoted by $(\mathcal{R},\partial)$ for short. A morphism between crossed $\mathcal{L}$-modules $(\mathcal{R},\partial)$ and $(\mathcal{R}',\partial')$ is a Lie $\mathbf{k}$-algebra homomorphism $\lambda:\mathcal{R}\to\mathcal{R}'$ such that the diagram



commutes.

Proposition 14 *In $\mathfrak{Xmod}/\mathcal{L}$, two morphisms between the same crossed modules have an equalizer.*

Proof Let $f, g : (B, \beta) \longrightarrow (C, \varphi)$ be two morphisms of a crossed $\mathcal{L}$-module. Define

$$D = \{b \in B : f(b) = g(b)\}$$

and define ∂ by

$$\beta|_D = \partial : D \longrightarrow \mathcal{L}.$$

For any $a \in A$, $b, b' \in D$, we have

$$\begin{array}{rcl} f([b,b']) & = & [f(b), f(b')] \\ & = & [g(b), g(b')] \end{array}$$

and $f(ab) = af(b) = ag(b) = g(ab)$, so D is a Lie A-algebra.

Since

$$\begin{array}{rcl} f([b,d]) & = & f({}^{\beta(b)}d) \\ & = & {}^{\beta(b)}f(d) \\ & = & {}^{\beta(b)}g(d) \\ & = & g({}^{\beta(b)}d) \\ & = & g([b,d]), \end{array}$$

for all $b \in B$, $d \in D$, the induced bracket of B on D is well-defined. Since

$$\begin{array}{rcl} f({}^l d) & = & {}^l f(d) \\ & = & {}^l g(d) \\ & = & g({}^l d), \end{array}$$

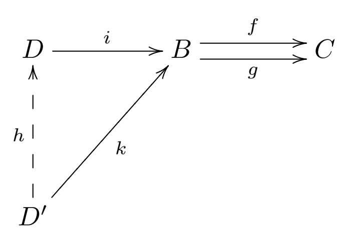
for all $l \in \mathcal{L}$ and $d \in D$, we have ${}^l d \in D$. Thus, (D, ∂) is a crossed $\mathcal{L}$-module, and inclusion

$$i : (D, \alpha) \longrightarrow (C, \partial)$$

is a morphism in $\mathfrak{XMod}/\mathcal{L}$. Suppose that there exist a crossed $\mathcal{L}$-module (D', ∂') and a morphism

$$k : (D', \partial') \longrightarrow (B, \beta)$$

of a crossed $\mathcal{L}$-module such that $fk = gk$. For $x \in D'$, $k(x) \in D$ and hence $k(D') \subseteq D$. Thus, there exists a morphism $h : D' \longrightarrow D$. It is clear that h is unique and the diagram



commutes, as required. □

Theorem 15 *The category $\mathfrak{XMod}/\mathcal{L}$ has pullbacks.*

Proof Let $f:(B,\beta)\longrightarrow(D,\delta)$ and $g:(C,\partial)\longrightarrow(D,\delta)$ be morphisms of a crossed $\mathcal{L}$-module, and

$$X = B \underset{D}{\times} C = \{(b,c) : f(b) = g(c)\}.$$

Define the bracket on X by

$$[(x_1,x_2),(x_3,x_4)] = ([x_1,x_3],[x_2,x_4]),$$

for all $(x_1,x_2),(x_3,x_4)\in X$, which makes X a Lie A-algebra. Define

$$\begin{array}{rccc} \lambda: & X & \longrightarrow & \mathcal{L} \\ & (b,c) & \longmapsto & \lambda(b,c)=\beta(b)=\partial(c). \end{array}$$

A direct calculation shows that (X,λ) is a Lie–Rinehart crossed module where $\lambda(b,c)=\beta(b)=\partial(c)$, for all $(b,c)\in X$ with the action defined by ${}^{l}(b,c)=({}^{l}b,{}^{l}c)$, for all $l\in\mathcal{L}$ and $(b,c)\in X$. We get the commutative diagram

$$\begin{array}{ccc} (X,\lambda) & \xrightarrow{\pi_1} & (B,\beta) \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle f} \\ (C,\partial) & \xrightarrow[g]{} & (D,\delta) \end{array}$$

of Lie–Rinehart crossed $\mathcal{L}$-modules. Suppose that the diagram

$$\begin{array}{ccc} (X',\varphi') & \xrightarrow{\pi_1'} & (B,\beta) \\ {\scriptstyle \pi_2'}\downarrow & & \downarrow{\scriptstyle f} \\ (C,\partial) & \xrightarrow[g]{} & (D,\delta) \end{array}$$

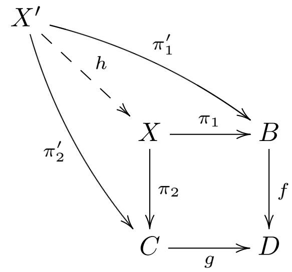
commutes. In this case, we have $(\pi_1'(x),\pi_2'(x))\in X$, since

$$f(\pi_1'(x)) = g(\pi_2'(x)),$$

for all $x\in X'$. Define

$$\begin{array}{rccc} h: & X' & \longrightarrow & X \\ & x & \longmapsto & (\pi_1'(x),\pi_2'(x)). \end{array}$$

Since $\pi_1'=\pi_1 h$ and $\pi_2'=\pi_2 h$, h is unique, from which we have the commutativity of the diagram



as required. □

Theorem 16 $(\ker\alpha, i)$ *is a terminal object in* $\mathfrak{XMod}/\mathcal{L}$*, where* α *is the anchor of* $\mathcal{L}$*.*

Proof Since $\ker\alpha$ is an ideal of $\mathcal{L}$ and a Lie A-algebra (see [3] for details), $(\ker\alpha, \mathcal{L}, i)$ is a crossed module thanks to the Example (a) in Section 1. Let (C, ∂) be a crossed $\mathcal{L}$-module and $f, g : C \longrightarrow \ker\alpha$ be crossed $\mathcal{L}$-module morphisms. Since $if = ig$, we have $f = g$, as required. In other words, $\partial : C \longrightarrow \ker\alpha$ is the unique crossed $\mathcal{L}$-module morphism. □

Theorem 17 $\mathfrak{XMod}/\mathcal{L}$ *has finite products.*

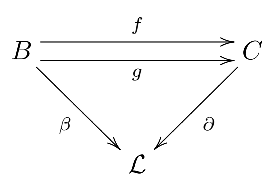
Proof Let (B, β) and (C, ∂) be two objects in $\mathfrak{XMod}/\mathcal{L}$. The product of these objects is pullback of (B, β) and (C, ∂) on terminal object $(\ker\alpha, i)$, as required. □

Corollary 18 $\mathfrak{XMod}/\mathcal{L}$ *has all finite limits. In other words, it is finitely complete.*

Proof Follows from Theorem 15 and 17. □

Theorem 19 *In the category* $\mathfrak{XMod}/\mathcal{L}$*, two morphisms between the same crossed modules have a coequalizer.*

Proof Let $f, g : (B, \beta) \longrightarrow (C, \partial)$ be two morphisms of a crossed $\mathcal{L}$-module and I be the ideal generated by the set $\{f(b) - g(b) : b \in B\}$. Since the diagram



commutes and

$$\partial(f(b) - g(b)) = 0,$$

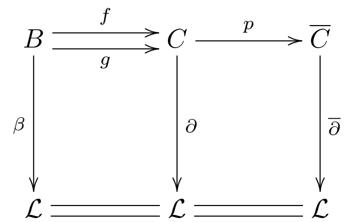
for all $b \in B$, we have $I \subseteq \ker\partial$. Consider the Lie–Rinehart algebra $\overline{C} := C/I$. It is obvious that C/I is a Lie–Rinehart algebra since I is an ideal of C and C/I is a quotient object in the category of Lie–Rinehart algebras. Define an action of $\mathcal{L}$ on $\overline{C}$ by

$${}^{l}(c + I) = {}^{l}c + I,$$

and the morphism $\overline{\partial} : \overline{C} \longrightarrow \mathcal{L}$ by

$$\overline{\partial}(c + I) = \partial(c),$$

for all $c + I \in \overline{C}$ and $l \in \mathcal{L}$. Obviously, $(\overline{C}, \overline{\partial})$ is a crossed $\mathcal{L}$-module. Consequently, we have the following commutative diagram.



Since

$$\begin{array}{rcl} p(f(b)) - p(g(b)) & = & p(f(b) - g(b)) \\ & = & f(b) - g(b) + I \\ & = & I, \end{array}$$

for all $b \in B$, we have $pf = pg$. Suppose that the crossed $\mathcal{L}$-module morphism

$$p' : C \longrightarrow \overline{C}'$$

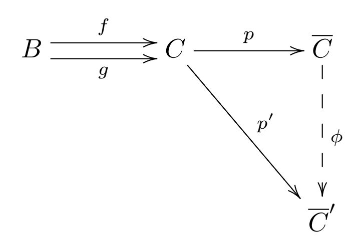
satisfies $p'f = p'g$. Define the morphism $\phi : \overline{C} \longrightarrow \overline{C}'$ by

$$\phi(c + I) = p'(c).$$

Since

$$\begin{array}{rcl} \phi p(c) & = & \phi(c + I) \\ & = & p'(c), \end{array}$$

for all $c \in C$, we have the commutativity of the diagram



and the uniqueness of ϕ, as required. □

Let (C, ∂) and (B, β) be two crossed $\mathcal{L}$-modules. Consider the action of B on C via ∂ defined as

$$c \cdot b =^{\partial(c)} b,$$

for all $b \in B$ and $c \in C$, which makes $B \rtimes C$ a Lie A-algebra. Also, we have the Lie–Rinehart action of $\mathcal{L}$ on $B \rtimes C$ defined by

$$^{l}(b, c) = (^{l}b, ^{l}c),$$

for all $l \in \mathcal{L}$, $(b,c) \in B \rtimes C$. Consider the map δ' defined by

$$\delta'(b,c) = \partial(c) + \beta(b),$$

for all $(b,c) \in B \rtimes C$, and the ideal I of $B \rtimes C$ generated by the elements of the form $({}^{\partial(c')}b, {}^{\beta(b)}c')$. Define

$$\begin{array}{rccl} \delta : & (B \rtimes C)/I & \longrightarrow & \mathcal{L} \\ & (b,c)+I & \longmapsto & \delta'(b,c) = \partial(c) + \beta(b). \end{array}$$

Then $((B \rtimes C)/I, \delta)$ is a crossed $\mathcal{L}$-module. That is,

$$\begin{array}{rcl} \delta({}^{l}(b,c)) & = & \delta({}^{l}b, {}^{l}c)) \\ & = & \partial({}^{l}c) + \beta({}^{l}b) \\ & = & {}^{l}(\partial(c)) + {}^{l}(\beta(b)) \\ & = & {}^{l}(\partial(c) + \beta(b)) \\ & = & {}^{l}(\delta(b,c)) \end{array}$$

$$\begin{array}{rcl} {}^{\delta\overline{(b',c')}}\overline{(b,c)} & = & \overline{{}^{\partial(c')+\beta(b')}(b,c)} \\ & = & \overline{({}^{\partial(c')+\beta(b')}b, {}^{\partial(c')+\beta(b')}c)} \\ & = & \overline{({}^{\partial(c')}b + {}^{\beta(b')}b, {}^{\partial(c')}c + {}^{\beta(b')}c)} \\ & = & \overline{({}^{\partial(c')}b + [b',b], [c',c] + {}^{\beta(b')}c)} \\ & = & \overline{([b',b],[c',c])} + \overline{({}^{\partial(c')}b + {}^{\beta(b')}c)} \\ & = & ([b',b],[c',c]) + ({}^{\partial(c')}b + {}^{\beta(b')}c) + I \\ & = & ([b',b],[c',c]) + I \\ & = & [(b',c'),(b,c)] + I \\ & = & [\overline{(b',c')}, \overline{(b,c)}], \end{array}$$

for all $\overline{(b,c)}, \overline{(b',c')} \in (B \rtimes C)/I$, $l \in \mathcal{L}$. Also, by direct checking we have $\delta(a(b,c)) = a\delta(b,c)$ and $\delta(b,c)(a) = 0$.

Theorem 20 *$((B \rtimes C)/I, i, j)$ is coproduct of (C,∂) and (B,β), where*

$$\begin{array}{rcccccccc} i: & C & \longrightarrow & (B \rtimes C)/I & \text{and} & j: & B & \longrightarrow & (B \rtimes C)/I \\ & c & \longmapsto & (0,c)+I & & & b & \longmapsto & (b,0)+I. \end{array}$$

Proof Can be checked by a direct calculation. □

Theorem 21 *The category $\mathfrak{XMod}/\mathcal{L}$ has pushouts.*

Proof Let $f:(E,\varepsilon) \longrightarrow (C,\partial)$ and $g:(E,\varepsilon) \longrightarrow (B,\beta)$ be two morphisms of crossed $\mathcal{L}$-modules, and let N be an ideal generated by all elements of the forms $({}^{\partial(c)}b, {}^{\beta(b)}(c))$ and $(g(e), -f(e))$ with the morphism

$$\begin{array}{rccl} \overline{\delta}: & \dfrac{B \rtimes C}{N} & \longrightarrow & \mathcal{L} \\ & (b,c)+N & \longmapsto & \partial(c) + \beta(b). \end{array}$$

$(\dfrac{B \rtimes C}{N}, \overline{\delta})$ is a crossed $\mathcal{L}$-module. Also, the following functions can be defined:

$$\begin{array}{rccl} i: & B & \longrightarrow & \dfrac{B \rtimes C}{N} \\ & b & \longmapsto & (b,0)+N \end{array}$$

and

$$j: \begin{array}{ccc} C & \longrightarrow & \dfrac{B \rtimes C}{N} \\ c & \longmapsto & (0,c)+N. \end{array}$$

In this case, $(g(e),-f(e)) = -(0,f(e)) + (g(e),0) \in N$, for all $e \in E$. Consequently, $jf = ig$ since $f(e)+N = g(e)+N$. On the other hand, it can be checked easily that $\overline{\delta}$ satisfies the universal property. □

Corollary 22 *The category $\mathfrak{XMod}/\mathcal{L}$ is cocomplete. In other words, it has finite colimits.*

Proof Since $\mathfrak{XMod}/\mathcal{L}$ has a coequalizer and coproducts, it has finite colimits. □

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