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Some operator inequalities associated with Kantorovich and Hölder–McCarthy inequalities and their applications

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Abstract: We prove analogs of certain operator inequalities, including Hölder–McCarthy inequality, Kantorovich inequality, and Heinz–Kato inequality, for positive operators on the Hilbert space in terms of the Berezin symbols and the Berezin number of operators on the reproducing kernel Hilbert space.

Key words: Reproducing kernel Hilbert space, Berezin symbol, Berezin number, positive operator, Hölder–McCarthy-type inequality, Kantorovich-type inequality, Heinz–Kato inequality

1. Introduction

We study operator inequalities associated mainly with Kantorovich inequality and Hölder–McCarthy inequality. In this paper, we give analogies of these inequalities for the Berezin symbols of operators and give their applications in proving some direct and inverse power inequalities for Berezin number of some concrete classes of operators on the reproducing kernel Hilbert space. Recall that an operator on a Hilbert space is called positive, if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. Shortly, the positive operator will be denoted as $A \geq 0$.

The famous Kantorovich inequality is the following [6].

Theorem 1 (Kantorovich inequality) *Let A be a positive operator on a Hilbert space H such that $M \geq A \geq m > 0$. Then the following inequalities hold for every unit vector x in H:*

(i) $\langle Ax, x\rangle \left\langle A^{-1}x, x\right\rangle \leq \frac{(m+M)^2}{4mM}$.

(ii) $\left\langle A^2 x, x\right\rangle \leq \frac{(m+M)^2}{4mM} \left\langle Ax, x\right\rangle^2$.

Notice that the constant $\frac{(m+M)^2}{4mM}$ in Theorem 1 can be expressed as follows: $\frac{(m+M)^2}{4mM} = \left[\frac{\frac{m+M}{2}}{\sqrt{mM}}\right]^2$, that is, inside the bracket $[\ ]$, the numerator is the arithmetic mean and the denominator is the geometric mean of m and M, respectively. This constant $\frac{(m+M)^2}{4mM}$ is called the Kantorovich constant (see Furuta [6]).

In the present article, we will use a weak variant of the Kantorovich inequality by using normalized reproducing kernels of reproducing kernel Hilbert spaces and use the obtained new inequalities in estimating the Berezin number of operators.

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Recall that a reproducing kernel Hilbert space (RKHS) is the Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ of complex-valued functions on some set Ω such that:

(a) the evaluation functional $f \to f(\lambda)$ is continuous for each $\lambda \in \Omega$;

(b) for every $\lambda \in \Omega$ there exists $f_\lambda \in \mathcal{H}$ such that $f_\lambda(\lambda) \neq 0$.

Then by the Riesz theorem for each $\lambda \in \Omega$ there exists a unique function $k_\lambda \in \mathcal{H}$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle$$

for all $f \in \mathcal{H}$. The function k_λ is called the reproducing kernel of the space $\mathcal{H}$. It is well-known that (see [2])

$$k_\lambda(z) = \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z)$$

for any orthonormal basis $\{e_n(z)\}_{n \geq 0}$ of the space $\mathcal{H}(\Omega)$. Let $\widehat{k}_\lambda$ denote the normalized reproducing kernel: $\widehat{k}_\lambda := \frac{k_\lambda}{\|k_\lambda\|_{\mathcal{H}}}$ (note that by (b), we surely have $k_\lambda \neq 0$). For a bounded linear operator A on the RKHS $\mathcal{H}$ (i.e. for $A \in \mathcal{B}(\mathcal{H})$, the Banach algebra of all bounded linear operators on $\mathcal{H}$), its so-called Berezin symbol $\widetilde{A}$ is defined by (see Berezin [3, 4], Nordgren and Rosenthal [15], Engliš [5], and Zhu [17])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_{\mathcal{H}} \ (\lambda \in \Omega).$$

It follows from the Cauchy-Scwartz inequality that $\widetilde{A}$ is a bounded function and

$$\sup_{\lambda \in \Omega} \left| \widetilde{A}(\lambda) \right| \leq \|A\|.$$

The Berezin set of A is the range of the Berezin symbol $\widetilde{A}$:

$$\begin{aligned} Ber(A) = Range\left(\widetilde{A}\right) &= \left\{ \widetilde{A}(\lambda) : \lambda \in \Omega \right\} \\ &= \left\{ \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle : \lambda \in \Omega \right\}. \end{aligned}$$

The Berezin number of operator A is defined by

$$ber(A) := \sup_{\lambda \in \Omega} \left| \widetilde{A}(\lambda) \right|.$$

The numerical range and the numerical radius of operator A on a Hilbert space H are, respectively,

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}$$

and

$$w(A) := \sup\{|\langle Ax, x\rangle| : x \in \mathcal{H} \text{ and } \|x\| = 1\}.$$

It is immediate from the above definitions that $Ber(A) \subseteq W(A)$ and $ber(A) \leq w(A)$. So, it is natural to investigate $Ber(A)$ and $ber(A)$ for operators $A \in \mathcal{B}(\mathcal{H})$. For more detail informations about the Berezin set, Berezin number and their relations with the numerical range and numerical radius, the reader can be found in Karaev [14], Garayev et al. [8], Garayev et al. [9], Garayev et al. [13], Gürdal et al. [11], Gürdal et al. [12], Yamancı et al. [16], Altwaijry et al. [10], and Garayev [7].

2. Kantorovich inequality, Hölder–McCarthy inequality, and Berezin symbols

Here, we give some operator inequalities associated with extensions of Kantorovich inequality in Theorem 1 and Hölder–McCarthy inequality in terms of the Berezin symbols. Namely, we prove the following.

Theorem 2 *Let A be a positive operator on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ satisfying*

$$0 < m \leq \widetilde{A} \leq M,$$

where $\widetilde{A}$ is the Berezin symbol of operator A. Also let $f(t)$ be a real-valued continuous convex function on $[m, M]$. Then

$$\widetilde{f(A)}(\lambda) \leq \frac{(mf(M) - Mf(m))}{(q-1)(M-m)} \left(\frac{(q-1)(f(M) - f(m))}{q(mf(M) - Mf(m))} \right)^q \widetilde{A}(\lambda)^q \tag{1}$$

under any one of the following conditions (2) and (3), respectively:

$$f(M) > f(m), \frac{f(M)}{M} > \frac{f(m)}{m} \text{ and } \frac{f(m)}{m} q \leq \frac{f(M) - f(m)}{M - m} \leq \frac{f(M)}{M} q \tag{2}$$

holds for any real number $q > 1$,

$$f(M) < f(m), \frac{f(M)}{M} < \frac{f(m)}{m} \text{ and } \frac{f(m)}{m} q \leq \frac{f(M) - f(m)}{M - m} \leq \frac{f(M)}{M} q \tag{3}$$

holds for any real number $q < 0$.

The proof of the theorem uses the following lemma (see Furuta [6]).

Lemma 1 *Let $h(t)$ be defined on $[m, M]$ $(M > m > 0)$ by the following formula*

$$h(t) = \frac{1}{t^q} \left(k + \frac{K - k}{M - m}(t - m) \right), \tag{4}$$

where q is any real number such that $q \neq 0, 1$ and K and k are any real numbers.

Then $h(t)$ has the following upper bound on $[m, M]$:

$$\frac{(mK - Mk)}{(q-1)(M-m)} \left(\frac{(q-1)(K-k)}{q(mK - Mk)} \right)^q, \tag{5}$$

where m, M, k, K, and q in (5) satisfy any one of the following conditions (i) and (ii), respectively:

(i) $K > k$, $\frac{K}{M} > \frac{k}{m}$ and $\frac{k}{m} q \leq \frac{K-k}{M-m} \leq \frac{K}{M} q$ holds for any real number $q > 1$;

(ii) $K < k$, $\frac{K}{M} < \frac{k}{m}$ and $\frac{k}{m} q \leq \frac{K-k}{M-m} \leq \frac{K}{M} q$ holds for any real number $q < 0$.

Proof For the sake of completeness, we give a sketch of the proof. Indeed, by an easy differential calculus, we have $h'(t_1) = 0$ when $t_1 = \frac{q}{(q-1)} \frac{(mK - Mk)}{(K-k)}$, and it turns out that t_1 satisfies the required condition $t_1 \in [m, M]$. Also t_1 gives the upper bound (5) of $h(t)$ on the segment $[m, M]$ under any one of the conditions (i) and (ii), respectively. □

Proof [Proof of Theorem 2]Since $f(t)$ is a real-valued continuous convex function on $[m,M]$, we obtain that

$$f(t) \leq f(m) + \frac{f(M)-f(m)}{M-m}(t-m) \text{ for any } [m,M]. \tag{6}$$

By applying the standard operational calculus of positive operator A to (6) since $M \geq \widetilde{A}(\lambda) \geq m$, we obtain that

$$\widetilde{f(A)}(\lambda) \leq f(m) + \frac{f(M)-f(m)}{M-m}\left(\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle - m\right). \tag{7}$$

Multiplying $\widetilde{A}(\lambda)^{-q}$ on both sides of (7), we get

$$\widetilde{A}(\lambda)^{-q}\,\widetilde{f(A)}(\lambda) \leq h(t), \tag{8}$$

where $h(t) = \widetilde{A}(\lambda)^{-q}\left[f(m) + \frac{f(M)-f(m)}{M-m}\left(\widetilde{A}(\lambda)-m\right)\right]$ for all $\lambda \in \Omega$. Then we obtain

$$\widetilde{f(A)}(\lambda) \leq \left[\max_{m \leq t \leq M} h(t)\right]\widetilde{A}(\lambda)^q \tag{9}$$

for all $\lambda \in \Omega$. Putting $K = f(M)$ and $k = f(m)$ in Theorem 2, we see that (i) and (ii) in Theorem 2 just correspond to (i) and (ii) in Lemma 1 so the proof is completed by (9) and Lemma 1. □

Corollary 1 *Let A be an invertible self-adjoint operator on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ satisfying $M \geq \widetilde{A} \geq m > 0$. Then*

$$ber(A^p) \leq \frac{(mM^p - Mm^p)}{(q-1)(M-m)}\left(\frac{(q-1)(M^p-m^p)}{q(mM^p-Mm^p)}\right)^q ber(A)^q \tag{10}$$

under any of the following conditions (i) and (ii), respectively:

(i) $m^{p-1}q \leq \frac{M^p-m^p}{M-m} \leq M^{p-1}q$ holds for real numbers $p > 1$ and $q > 1$;

(ii) $m^{p-1}q \leq \frac{M^p-m^p}{M-m} \leq M^{p-1}q$ holds for real numbers $p < 0$ and $q < 0$.

Proof The proof is similar to the proof in [6]. Put $f(t) = t^p$ for $p \notin [0,1]$ in Theorem 2. As $f(t)$ is a real valued continuous convex function on $[m,M]$, $M^p > m^p$ and $M^{p-1} > m^{p-1}$ hold for any $p > 1$, that is, $f(M) > f(m)$ and $\frac{f(M)}{M} > \frac{f(m)}{m}$ for any $p > 1$. Also, $M^p < m^p$ and $M^{p-1} < m^{p-1}$ hold for any $p < 0$, that is, $f(M) < f(m)$ and $\frac{f(M)}{M} < \frac{f(m)}{m}$ for any $p < 0$, respectively. So by Theorem 2, we have for any $\lambda \in \Omega$ that

$$\left\langle A^p\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \leq \frac{(mM^p - Mm^p)}{(q-1)(M-m)}\left(\frac{(q-1)(M^p-m^p)}{q(mM^p-Mm^p)}\right)^q \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^q,$$

which gives us inequality (10), as desired. □

3. Hölder–McCarthy, Kantorovich, and Berezin number inequalities

Now we give our next result associated with Hölder–McCarthy and Kantorovich inequalities.

Theorem 3 *Let A be a positive operator on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ satisfying $M \geq \widetilde{A} \geq m > 0$. Then for every $p > 1$ we have :*

$$ber(A)^p \leq ber(A^p) \leq K_+(m, M, p)\, ber(A)^p, \tag{11}$$

where $K_+(m, M, p) = \frac{(p-1)^{p-1}}{p^p} \frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}}$.

Proof As $f(t) = t^p$ is a convex function for $p \notin [0,1]$, (i) in Corollary 1 holds in the case where $p \notin [0,1]$ and $q = p$ so that the second inequality in (11) holds by Corollary 1. The first inequality in (11) follows by Hölder–McCarthy inequality. □

Corollary 2 *Let A be a positive operator on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ such that $M \geq A \geq m > 0$. Then*

$$ber(A)^p\, ber\left(A^{-1}\right) \leq \frac{p^p}{(p+1)^{p+1}} \frac{(m+M)^{p+1}}{mM} \tag{12}$$

and

$$ber\left(A^2\right) \leq \frac{p^p}{(p+1)^{p+1}} \frac{(m+M)^{p+1}}{(mM)^p} ber(A)^{p+1} \tag{13}$$

for any p such that $\frac{m}{M} \leq p \leq \frac{M}{m}$.

Proof In (ii) of Corollary 1, we only have to put $p = -1$, and replacing q by $-p$ for $p > 0$, we get (12).

In (i) of Corollary 1, we only have to put $p = 2$ and replace q by $p + 1$ for $p > 0$ to get (13).

When $p = 1$, Corollary 2 becomes the following Kantorovich type inequality :

$$ber(A)\, ber\left(A^{-1}\right) \leq \frac{1}{2} \frac{(m+M)^2}{mM}.$$

It is easy to see that $0 < B \leq A$ ensures $ber(B^p) \leq ber(A^p)$ for any $p \in [0,1]$ (by well known Löwner-Heinz theorem). However, $0 < B \leq A$ does not always ensure $ber(B^p) \leq ber(A^p)$ for any $p > 1$. □

For such consideration, we prove the following theorem.

Theorem 4 *Let A and B be positive operators on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ such that*

$$M_1 \geq A \geq m_1 > 0$$
$$M_2 \geq B \geq m_2 > 0$$
$$A \geq B > 0.$$

Then

$$ber(B^p) \geqslant K_{2,p} ber(A^p) \leq \left(\frac{M_2}{m_2}\right)^{p-1} ber(A^p) \tag{13}$$

and

$$ber(B^p) \geqslant K_{1,p} ber(A^p) \leq \left(\frac{M_1}{m_1}\right)^{p-1} ber(A^p) \tag{14}$$

hold for any $p \geq 1,$ *where* $K_{1,p}$ *and* $K_{2,p}$ *are defined by the following:*

$$K_{1,p} = \frac{(p-1)^{p-1}}{p^p (M_1 - m_1)} \frac{(M_1^p - m_1^p)^p}{(m_1 M_1^p - M_1 m_1^p)^{p-1}}$$

and

$$K_{2,p} = \frac{(p-1)^{p-1}}{p^p (M_2 - m_2)} \frac{(M_2^p - m_2^p)^p}{(m_2 M_2^p - M_2 m_2^p)^{p-1}}.$$

For the proof, we need the following lemma (see Furuta [6, Proposition 2, p. 194]).

Lemma 2 *If* $x \geq 1,$ *then*

$$\frac{(p-1)^{p-1} (x^p - 1)^p}{p^p (x-1) (x^p - x)^{p-1}} \leq x^{p-1} \text{ for } 1 < p < \infty,$$

and the equality holds if and only if $x \downarrow 1.$

Proof [Proof of Theorem 4]For $p = 1,$ the result is trivial. So, we consider only $p > 1.$ First of all, whenever $M \geq m > 0$, we recall the following inequality by putting $x = \frac{M}{m} (\geq 1)$ in Lemma 2 :

$$\frac{(p-1)^{p-1}}{p^p} \frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}} \leq \left(\frac{M}{m}\right)^{p-1} \tag{15}$$

for $p > 1.$ For $p > 1,$ by using Hölder–McCarthy inequality and inequality (15), we have

$$\begin{aligned} \widetilde{B^p}(\lambda) &= \left\langle B^p \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\ &\leq K_{2,p} \left\langle B \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^p \text{ (by (11) of Theorem 3)} \\ &\leq K_{2,p} \left\langle A \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^p \text{ (by } 0 < B \leq A) \\ &\leq K_{2,p} \left\langle A^p \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \leq \left(\frac{M_2}{m_2}\right)^{p-1} \left\langle A^p \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \end{aligned}$$

for all $\lambda \in \Omega.$ Taking supremum from both sides of these inequalities, we have the desired inequalities (13). Now let us prove (14). Indeed, as $0 < A^{-1} \leq B^{-1}$ and $M_1^{-1} \leq A^{-1} \leq m_1^{-1},$ then by applying (13), we have for all $\lambda \in \Omega$ that

$$\begin{aligned}
\widetilde{A^{-p}}(\lambda) &= \left\langle A^{-p}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\
&\leq \frac{(p-1)^{p-1}}{p^p\left(m_1^{-1}-M_1^{-1}\right)} \frac{\left(m_1^p - M_1^{-p}\right)^p}{\left(M_1^{-1}m_1^{-p} - m_1^{-1}M_1^{-p}\right)^{p-1}} \left\langle B^{-p}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\
&\leq \frac{(p-1)^{p-1}}{p^p\left(M_1-m_1\right)} \frac{\left(M_1^p - m_1^{p}\right)^p}{\left(m_1M_1^{p} - M_1m_1^{p}\right)^{p-1}} \left\langle B^{-p}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\
&\leq K_{1,p}\left\langle B^{-p}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \leq \left(\frac{M_1}{m_1}\right)^{p-1}\left\langle B^{-p}\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\
&\leq \left(\frac{M_1}{m_1}\right)^{p-1}\widetilde{B^{-p}}(\lambda),
\end{aligned}$$

which implies (14) by taking inverses in both sides of the inequality above and taking supremum over λ in Ω, so the proof is finished. □

Corollary 3 *If $0 < B \leq A$ and $0 < m \leq B \leq M$, then*

$$ber\left(B^p\right) \leq \left(\frac{M}{m}\right)^p ber\left(A^p\right) \text{ for } p \geq 1. \tag{16}$$

We remark that (13) and (14) of Theorem 4 are more precise estimations than (16) since $K_{j,p} \leq \left(\frac{M_j}{m_j}\right)^{p-1} \leq \left(\frac{M_j}{m_j}\right)^p$ holds for $j = 1, 2$ and $p \geq 1$.

Proposition 1 *Let A be a positive operator such that $M \geq A \geq m > 0$ and B be a positive contraction. Then*

$$ber\left(BAB\right) \leq \frac{\left(M+m\right)^2}{4mM} ber\left(A\right). \tag{17}$$

Proof Since A^{-1} exists, by the Kantorovich inequality, we have

$$\langle ABx, Bx\rangle \left\langle A^{-1}Bx, Bx\right\rangle \leq K\left\|Bx\right\|^4$$

for every $x \in H$, where $K := \frac{(M+m)^2}{4mM}$. In particular,

$$\begin{aligned}
\left\langle AB\widehat{k}_\lambda, B\widehat{k}_\lambda \right\rangle \left\langle A^{-1}B\widehat{k}_\lambda, B\widehat{k}_\lambda \right\rangle &\leq K\left\langle B^2\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^2 \\
&\leq K\left\langle B\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^2 \text{ (by } I \geq B \geq 0) \\
&= K\left\langle A^{-1/2}B\widehat{k}_\lambda, A^{1/2}\widehat{k}_\lambda \right\rangle^2 \\
&\leq K\left\langle A^{-1}B\widehat{k}_\lambda, B\widehat{k}_\lambda \right\rangle\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle
\end{aligned}$$

for all $\lambda \in \Omega$. From this by standard arguments, we get (17). □

4. Selberg inequality, an extension of the Heinz-Kato inequality and Berezin number inequalities

The celebrated Selberg inequality (see Furuta [6]) is very useful in the prime number theory, which is an extension of the classical Bessel's inequality. Namely, if $x_1, x_2, ..., x_n \in H \backslash \{0\}$, then

$$\sum_{i=1}^{n} \frac{|\langle x, x_i \rangle|^2}{\sum_{j=1}^{n} |\langle x_i, x_j \rangle|} \leq \|x\|^2 . \tag{18}$$

The equality in (18) holds if and only if $x = \sum_{i=1}^{n} a_i x_i$ for some complex scalars $a_1, a_2, ..., a_n$ such that for arbitrary $i \neq j$, $\langle x_i, x_j \rangle = 0$ or $|a_i| = |a_j|$ with $\langle a_i x_i, a_j x_j \rangle \geq 0$.

Proposition 2 *Let A be a bounded linear operator on the Hardy–Hilbert space $H^2 = H^2(\mathbb{D})$ of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $\|f\|_{H^2} := \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{1/2} < +\infty$. If $A\widehat{k}_\lambda \neq 0$ for all $\lambda \in \mathbb{D}$, then*

$$ber\left(|A|^2\right) \geq \sup_{\lambda, n} \sum_{i=1}^{n} \frac{\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_{\lambda_i} \right\rangle\right|}{\sum_{j=1}^{n} \frac{\left(1-|\lambda_i|^2\right)^{1/2}\left(1-|\lambda_j|^2\right)}{\left|1-\overline{\lambda_i}\lambda_j\right|}},$$

where $|A| := (A^* A)^{\frac{1}{2}}$ is the module of A.

Proof For the proof, it is enough to put in Selberg inequality (18) $x = A\widehat{k}_\lambda$ and $x_i = \widehat{k}_{\lambda_i}$ $(i = 1, 2, ..., n)$; we omit the details. □

Theorem 5 *Let $T \in \mathcal{B}(\mathcal{H}(\Omega))$ be any operator and A and B be two positive operators such that*

$$\left\|T\widehat{k}_\lambda\right\| \leq \left\|A\widehat{k}_\lambda\right\| \text{ and } \left\|T^*\widehat{k}_\mu\right\| \leq \left\|B\widehat{k}_\mu\right\|$$

for all $\lambda \in \Omega$. Then

$$\sup_{\lambda, \mu \in \Omega} \left|\left\langle T\,|T|^{\alpha+\beta-1}\,\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle\right| \leq ber\left(|A^\alpha|^2\right) ber\left(\left|B^\beta\right|^2\right) \tag{19}$$

for any α and β such that $\alpha, \beta \in [0,1]$ and $\alpha + \beta \geq 1$.

Proof It is a consequence of the Löwner–Heinz inequality [6] that if

$$\widetilde{A} \geq \widetilde{B} \geq 0, \text{ then } \widetilde{A^\alpha} \geq \widetilde{B^\alpha} \tag{20}$$

for each $\alpha \in [0,1]$. On the other hand, the hypothesis $\left\|T\widehat{k}_\lambda\right\| \leq \left\|A\widehat{k}_\lambda\right\|$ for all $\lambda \in \Omega$ is equivalent to

$$\widetilde{|T|^2} \leq \widetilde{|A|^2}, \tag{21}$$

and also the hypothesis $\left\|T^*\widehat{k}_\mu\right\| \leq \left\|B\widehat{k}_\mu\right\|$ for all $\mu \in \Omega$ is equivalent to

$$\widetilde{|T^*|^2} \leq \widetilde{|B|^2}. \tag{22}$$

Hence, applying (20) to (21) and (22), for any $\lambda, \mu \in \Omega$, we obtain :

$$\left\langle |T|^{2\alpha}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \leq \left\langle A^{2\alpha}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \tag{23}$$

for each $\alpha \in [0,1]$;

$$\left\langle |T^*|^{2\beta}\widehat{k}_\mu, \widehat{k}_\mu\right\rangle \leq \left\langle B^{2\beta}\widehat{k}_\mu, \widehat{k}_\mu\right\rangle \tag{24}$$

for each $\beta \in [0,1]$.

Let $T = U|T|$ be the polar decomposition of an operator T, where U is a partial isometric operator and $|T| := (T^*T)^{1/2} = \sqrt{T^*T}$ and $\ker(U) = \ker(|T|)$. In the case where $\alpha, \beta \in [0,1]$ such that $\beta > 0$ and $\alpha + \beta \geq 1$, we recall the following important relation shown in [6] :

$$|T^*|^{2\beta} = U|T|^{2\beta}U^* \text{ holds for any } \beta > 0. \tag{25}$$

Then, we have for all $\lambda, \mu \in \Omega$ that

$$\begin{aligned}
\left|\left\langle T|T|^{\alpha+\beta-1}\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|^2 &= \left|\left\langle U|T|^{\alpha+\beta}\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|^2 \\
&= \left|\left\langle |T|^{\alpha}\widehat{k}_\lambda, |T|^{\beta}U^*\widehat{k}_\mu\right\rangle\right|^2 \\
&\leq \left\||T|^{\alpha}\widehat{k}_\lambda\right\|^2\left\||T|^{\beta}U^*\widehat{k}_\mu\right\|^2 \\
&= \left\langle |T|^{2\alpha}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\left\langle U|T|^{2\beta}U^*\widehat{k}_\mu, \widehat{k}_\mu\right\rangle \\
&= \left\langle |T|^{2\alpha}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\left\langle |T^*|^{2\beta}\widehat{k}_\mu, \widehat{k}_\mu\right\rangle \text{ (by (25))} \\
&\leq \left\langle A^{2\alpha}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\left\langle B^{2\beta}\widehat{k}_\mu, \widehat{k}_\mu\right\rangle \text{ (by (23) and (24))}
\end{aligned}$$

for any α and β such that $\alpha, \beta \in [0,1]$ and $\alpha + \beta \geq 1$; that is, (19) holds because the result is trivial in the case where $\beta = 0$. Whence by taking suprema over λ and n the proof of Theorem 5 is complete. □

Let $\alpha + \beta = 1$ in Theorem 5 in particular. Then, we obtain the following Heinz–Kato-type inequality.

Proposition 3 *Let T be any operator on a RKHS $\mathcal{H} = \mathcal{H}(\Omega)$. If A and B are positive operators such that $\left\|T\widehat{k}_\lambda\right\| \leq \left\|A\widehat{k}_\lambda\right\|$ and $\left\|T^*\widehat{k}_\mu\right\| \leq \left\|B\widehat{k}_\mu\right\|$ for all $\lambda, \mu \in \Omega$, then*

$$\sup_{\lambda,\mu\in\Omega}\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right| \leq ber\left(|A^{\alpha}|^2\right) ber\left(\left|B^{1-\alpha}\right|^2\right) \tag{26}$$

for any $\alpha \in [0,1]$.

Since $\sup_{\lambda\in\Omega}\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| \leq \sup_{\lambda,\mu\in\Omega}\left|\left\langle T\widehat{k}_\lambda, \widehat{k}_\mu\right\rangle\right|$, inequality (26) implies that

$$ber(T) \leq ber\left(|A^{\alpha}|^2\right) ber\left(\left|B^{1-\alpha}\right|^2\right), \ \forall\alpha \in [0,1].$$

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