Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 539 – 560

doi:10.3906/mat-1809-25

On the composition and exterior products of double forms and p-pure manifolds

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Received: 07.09.2018 • **Accepted/Published Online:** 03.01.2019 • **Final Version:** 18.01.2019

Abstract: We translate into double forms formalism the basic Greub and Greub–Vanstone identities that were previously obtained in mixed exterior algebras. In particular, we introduce a second product in the space of double forms, namely the composition product, which provides this space with a second associative algebra structure. The composition product interacts with the exterior product of double forms; we show that the resulting relations provide simple alternative proofs to some classical linear algebra identities as well as to recent results in the exterior algebra of double forms. We define and study a refinement of the notion of pure curvature of Maillot, namely p-pure curvature, and we use one of the basic identities to prove that if a Riemannian n-manifold has k-pure curvature and $n \geq 4k$ then its Pontryagin class of degree $4k$ vanishes.

Key words: Composition product of double forms, exterior product of double forms, interior product of a double form, mixed exterior algebra, Pontrjagin form, Pontrjagin number, p-pure curvature tensor

1. Introduction

Let h be an endomorphism of an Euclidean real vector space (V, g) of dimension $n < \infty$. Recall the classical Girard–Newton identities for $1 \leq r \leq n$

$$rs_r(h) = \sum_{i=1}^{r} (-1)^{i+1} s_{r-i}(h) p_i(h) :$$

where $p_i(h)$ is the trace of the endomorphism $h^{\textcircled{i}} = \underbrace{h \circ ... \circ h}_{i\text{-times}}$ obtained using the composition product. The scalars $s_i(h)$ are the elementary symmetric functions in the eigenvalues of h. It turns out that the invariants $s_i(h)$ are also traces of endomorphisms constructed from h and the metric g using the exterior product; see, for instance, [7].

Another celebrated classical result that also illustrates the interaction between the composition and exterior product is the Cayley–Hamilton theorem:

$$\sum_{r=0}^{n} (-1)^r s_{n-r}(h) h^{\textcircled{r}} = 0.$$

Girard–Newton identities are scalar valued identities while the Cayley–Hamilton theorem is an endomorphism valued identity. Higher double forms valued identities were obtained in [7]. In particular, it is shown that the

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2010 *AMS Mathematics Subject Classification:* Primary 53B20, 53C21; Secondary 15A24, 15A69

infinitesimal version of the general Gauss–Bonnet theorem is a double forms valued identity of Cayley–Hamilton type that again involves the two products. Another illustration of the importance of these interactions is the expression of all Pontryagin numbers of a compact oriented manifold of dimension $n=4k$ as the integral of the following $4k$-form [8]:

$$P_1^{k_1}P_2^{k_2}\cdots P_m^{k_m}=\frac{(4k)!}{[(2k)!]^2(2\pi)^{2k}}\Big(\prod_{i=1}^{m}\frac{[(2i)!]^2}{(i!)^{2k_i}(4i)!}\Big)\mathrm{Alt}\Big[(R\circ R)^{k_1}(R^2\circ R^2)^{k_2}\cdots(R^m\circ R^m)^{k_m}\Big],$$

where R is the Riemann curvature tensor seen as a $(2,2)$ double form; $k_1,k_2,...,k_m$ are nonnegative integers such that $k_1+2k_2+...+mk_m=k$; Alt is the alternating operator; and all the powers over double forms are taken with respect to the exterior product of double forms. The circle $\circ$ denotes the composition product.

In this study, we investigate some other useful relations between these two products. The paper is organized as follows. Sections 2 and 3 provide definitions and basic facts about the exterior and composition products of double forms. In Section 4, we introduce and study the interior products of double forms, which generalize the usual Ricci contractions. Precisely, for a double form ω, the interior product map i_ω, which maps a double form to another double form, is the adjoint of the exterior multiplication map by ω. In particular, if $\omega=g$ we recover the usual Ricci contraction map of double forms.

Section 5 is about some natural extensions of endomorphisms of V onto endomorphisms of the exterior algebra of double forms. We start with an endomorphism $h:V\to V$, and there exists a unique exterior algebra endomorphism $\widehat{h}:\Lambda V\to\Lambda V$ that extends h and such that $\widehat{h}(1)=1$. Next, the space $\Lambda V\otimes\Lambda V$ can be regarded in two ways as ΛV-valued exterior vectors, and therefore the endomorphism $\widehat{h}$ operates on the space $\Lambda V\otimes\Lambda V$ in two natural ways, say $\widehat{h}_R$ and $\widehat{h}_L$. The two obtained endomorphisms are in fact exterior algebra endomorphisms. We prove that the endomorphisms $\widehat{h}_R$ and $\widehat{h}_L$ are nothing but the right and left multiplication maps in the composition algebra; precisely, we prove that

$$\widehat{h}_R(\omega)=e^h\circ\omega,\text{ and }\widehat{h}_L(\omega)=\omega\circ e^{(h^t)},$$

where $e^h:=1+h+\frac{h^2}{2!}+\frac{h^3}{3!}+...$ and the powers are taken with respect to the exterior product of double forms. As a consequence of this discussion we get easy proofs of classical linear algebra, including Laplace expansions of the determinant.

In Section 6, we first state and prove Greub's basic identity relating the exterior and composition products of double forms:

Proposition. *If $h,h_1,...,h_p$ are bilinear forms on V and $h_1...h_p$ is their exterior product, then*

$$\begin{aligned}i_h(h_1...h_p)&=\sum_j\langle h,h_j\rangle h_1...\hat{h}_j...h_p\\&\quad-\sum_{j<k}(h_j\circ h^t\circ h_k+h_k\circ h^t\circ h_j)h_1...\hat{h}_j...\hat{h}_k...h_p.\end{aligned}$$

Consequently, for a bilinear form k on V, the contraction of $\mathrm{c}k^p$ of the exterior power k^p of k is given by

$$\mathrm{c}k^p=p(\mathrm{c}k)k^{p-1}-p(p-1)(k\circ k)k^{p-2}.$$

Using the fact that the diagonal subalgebra (the subspace of all (p,p) double forms, $p \geq 0$) is spanned by exterior products of bilinear forms on V, we obtain the following useful formula as a consequence of the previous identity. This new formula generalizes formula (15) of [5] in Theorem 4.1 to double forms that are not symmetric or do not satisfy the first Bianchi identity:

$$*\Big(\frac{g^{k-p}\omega}{(k-p)!}\Big) = \sum_r (-1)^{r+p} \frac{g^{n-p-k+r}}{(n-p-k+r)!}\frac{c^r}{r!}(\omega^t).$$

where $*$ is the double Hodge star operator on double forms.

Also in Section 6, we state and prove another identity relating the exterior and composition product of double forms, namely the following Greub–Vanstone basic identity:

Theorem. *For $1 \leq p \leq n$, and for bilinear forms $h_1, ..., h_p$ and $k_1, ..., k_p$, we have*

$$(h_1 h_2 ... h_p) \circ (k_1 k_2 ... k_p) = \sum_{\sigma \in S_p} (h_1 \circ k_{\sigma(1)})...(h_p \circ k_{\sigma(p)}) = \sum_{\sigma \in S_p} (h_{\sigma(1)} \circ k_1)...(h_{\sigma(p)} \circ k_p).$$

In particular, when $h = h_1 = ... = h_p$ and $k = k_1 = ... = k_p$, we have the following nice relation:

$$h^p \circ k^p = p!(h \circ k)^p.$$

Section 7 is devoted to the study of p-pure Riemannian manifolds. Letting $1 \leq p \leq n/2$ be a positive integer, a Riemannian n-manifold is said to have a p-pure curvature tensor if at each point of the manifold the curvature operator that is associated to the exterior power R^p of the Riemann curvature tensor R has decomposed eigenvectors. For $p = 1$, we recover the usual pure Riemannian manifolds of Maillot. A pure manifold is always p-pure for $p \geq 1$ and we give examples of p-pure Riemannian manifolds that are p-pure for some $p > 1$ without being pure. The main result of this section is the following:

Theorem. *If a Riemannian n-manifold is k-pure and $n \geq 4k$ then its Pontryagin class of degree $4k$ vanishes.*

The previous theorem refines a result by Maillot in [9], where he proved that all Pontryagin classes of a pure Riemannian manifold vanish.

2. The exterior algebra of double forms

Let (V, g) be an Euclidean real vector space of finite dimension n. In the following we shall identify whenever convenient (via their Euclidean structures) the vector spaces with their duals. Let $\Lambda V^* = \bigoplus_{p \geq 0} \Lambda^p V^*$ (resp. $\Lambda V = \bigoplus_{p \geq 0} \Lambda^p V$) denote the exterior algebra of the dual space V^* (resp. V). Considering tensor products, we define the space of double exterior forms of V (resp. double exterior vectors) as

$$\mathcal{D}(V^*) = \Lambda V^* \otimes \Lambda V^* = \bigoplus_{p,q \geq 0} \mathcal{D}^{p,q}(V^*),$$

$$\text{resp. } \mathcal{D}(V) = \Lambda V \otimes \Lambda V = \bigoplus_{p,q \geq 0} \mathcal{D}^{p,q}(V),$$

where $\mathcal{D}^{p,q}(V^*) = \Lambda^p V^* \otimes \Lambda^q V^*$, resp. $\mathcal{D}^{p,q}(V) = \Lambda^p V \otimes \Lambda^q V$. The space $\mathcal{D}(V^*)$ is naturally a bigraded associative algebra, called the double exterior algebra of V, where for $\omega_1 = \theta_1 \otimes \theta_2 \in \mathcal{D}^{p,q}(V^*)$ and $\omega_2 = \theta_3 \otimes \theta_4 \in \mathcal{D}^{r,s}(V^*)$, the multiplication is given by

$$\omega_1\omega_2 = (\theta_1 \otimes \theta_2)(\theta_3 \otimes \theta_4) = (\theta_1 \wedge \theta_3) \otimes (\theta_2 \wedge \theta_4) \in \mathcal{D}^{p+r,q+s}(V), \tag{1}$$

where $\wedge$ denotes the standard exterior product on the exterior algebra ΛV^*. The product in the exterior algebra of double vectors is defined in the same way.

A double exterior form of degree (p,q) (resp. a double exterior vector of degree (p,q)) is by definition an element of the tensor product $\mathcal{D}^{p,q}(V^*) = \Lambda^p V^* \otimes \Lambda^q V^*$ (resp. $\mathcal{D}^{p,q}(V) = \Lambda^p V \otimes \Lambda^q V$). It can be identified canonically with a bilinear form $\Lambda^p V \times \Lambda^q V \to \mathbf{R}$, which in turn can be seen as a multilinear form that is skew-symmetric in the first p-arguments and also in the last q-arguments.

The above multiplication in $\mathcal{D}(V^*)$ (resp. $\mathcal{D}(V)$) shall be called the exterior product of double forms (resp. exterior product of double vectors).

Recall that the (Ricci) contraction map, denoted by c, maps $\mathcal{D}^{p,q}(V^*)$ into $\mathcal{D}^{p-1,q-1}(V^*)$. For a double form $\omega \in \mathcal{D}^{p,q}(V^*)$ with $p \geq 1$ and $q \geq 1$, we have

$$c\,\omega(x_1 \wedge ... \wedge x_{p-1}, y_1 \wedge ... \wedge y_{q-1}) = \sum_{j=1}^{n} \omega(e_j \wedge x_1 \wedge ...x_{p-1}, e_j \wedge y_1 \wedge ... \wedge y_{q-1}),$$

where $\{e_1, ..., e_n\}$ is an arbitrary orthonormal basis of V and ω is seen as a bilinear form as explained above. If $p = 0$ or $q = 0$, we set $c\,\omega = 0$.

It turns out (see [5]) that the contraction map c on $\mathcal{D}(V^*)$ is the adjoint of the multiplication map by the metric g of V; precisely, we have for $\omega_1, \omega_2 \in \mathcal{D}(V^*)$ the following:

$$< g\,\omega_1, \omega_2 >=< \omega_1, c\,\omega_2 > . \tag{2}$$

Suppose now that we have fixed an orientation on the vector space V. The classical Hodge star operator $* : \Lambda^p V^* \to \Lambda^{n-p} V^*$ can be extended naturally to operate on double forms as follows. For a (p,q)-double form ω (seen as a bilinear form), $*\omega$ is the $(n-p, n-q)$-double form given by

$$*\omega(.,.) = (-1)^{(p+q)(n-p-q)}\omega(*.,*.). \tag{3}$$

Note that $*\omega$ does not depend on the chosen orientation as the usual Hodge star operator is applied twice. The obtained operator is still called the Hodge star operator operating on double forms or the double Hodge star operator. This new operator provides another simple relation between the contraction map c of double forms and the multiplication map by the metric as follows:

$$g\,\omega = *\,c\,*\,\omega. \tag{4}$$

Furthermore, the double Hodge star operator generates the inner product of double forms as follows. For any two double forms $\omega, \theta \in \mathcal{D}^{p,q}$ we have

$$< \omega, \theta >= *\Big(\omega(*\theta)\Big) = (-1)^{(p+q)(n-p-q)} * \Big((*\omega)\theta\Big). \tag{5}$$

The reader is invited to consult the proofs of the above relations in [5].

Definition 2.1. *The subspace*

$$\Delta V^* = \bigoplus_{p\geq 0} \mathcal{D}^{p,p}(V^*), \text{ (resp. } \Delta V = \bigoplus_{p\geq 0} \mathcal{D}^{p,p}(V))$$

of $\mathcal{D}(V^*)$ *(resp.* $\mathcal{D}(V)$*) is a commutative subalgebra and shall be called the diagonal subalgebra.*

3. The composition algebra of double forms

The space $\mathcal{D} = \Lambda V^* \otimes \Lambda V^*$ is canonically isomorphic to the space of linear endomorphisms $L(\Lambda V, \Lambda V)$. Explicitly, we have the following canonical isomorphism:

$$\begin{aligned} \mathcal{T} : \Lambda V^* \otimes \Lambda V^* &\to L(\Lambda V, \Lambda V). \\ \omega_1 \otimes \omega_2 &\to \mathcal{T}(\omega_1 \otimes \omega_2) \end{aligned} \tag{6}$$

is given by

$$\mathcal{T}(\omega_1 \otimes \omega_2)(\theta) = \langle \omega_1^\sharp, \theta\rangle \omega_2^\sharp,$$

where $\omega_i^\sharp$ denotes the exterior vector dual to the exterior form ω_i.

Note that if we look at a double form ω as a bilinear form on ΛV, then $\mathcal{T}(\omega)$ is nothing but the canonical linear operator associated to the bilinear form ω.

It is easy to see that $\mathcal{T}$ maps for each $p \geq 1$ the double form $\frac{g^p}{p!}$ to the identity map in $L(\Lambda^p V, \Lambda^p V)$; in particular, $\mathcal{T}$ maps the double form $1 + g + \frac{g^2}{2!} + ...$ onto the identity map in $L(\Lambda V, \Lambda V)$.

The space $L(\Lambda V, \Lambda V)$ is an algebra under the composition product $\circ$ that is not isomorphic to the algebra of double forms. Pulling back the operation $\circ$ to $\mathcal{D}$ we obtain a second multiplication in $\mathcal{D}$, which we shall call the composition product of double forms or Greub's product of double forms and which will be still denoted by $\circ$.

More explicitly, given two simple double forms $\omega_1 = \theta_1 \otimes \theta_2 \in \mathcal{D}^{p,q}$ and $\omega_2 = \theta_3 \otimes \theta_4 \in \mathcal{D}^{r,s}$, we have

$$\omega_1 \circ \omega_2 = (\theta_1 \otimes \theta_2) \circ (\theta_3 \otimes \theta_4) = \langle \theta_1, \theta_4\rangle \theta_3 \otimes \theta_2 \in \mathcal{D}^{r,q}. \tag{7}$$

It is clear that $\omega_1 \circ \omega_2 = 0$ unless $p = s$.

Alternatively, if we look at ω_1 and ω_2 as bilinear forms, then the composition product reads as follows [7]:

$$\omega_1 \circ \omega_2(u_1, u_2) = \sum_{i_1<i_2<...<i_p} \omega_2(u_1, e_{i_1} \wedge ... \wedge e_{i_p})\omega_1(e_{i_1} \wedge ... \wedge e_{i_p}, u_2), \tag{8}$$

where $\{e_1, ..., e_n\}$ is an arbitrary orthonormal basis of (V, g), $u_1 \in \Lambda^r$ is an r-vector, and $u_2 \in \Lambda^q$ is a q-vector in V.

We list below some properties of this product.

3.1. Transposition of double forms

For a double form $\omega \in \mathcal{D}^{p,q}$, we denote by $\omega^t \in \mathcal{D}^{q,p}$ the transpose of ω, which is defined by

$$\omega^t(u_1, u_2) = \omega(u_2, u_1). \tag{9}$$

Alternatively, if $\omega = \theta_1 \otimes \theta_2$, then

$$\omega^t = (\theta_1 \otimes \theta_2)^t = \theta_2 \otimes \theta_1. \tag{10}$$

A double form ω is said to be a symmetric double form if $\omega^t = \omega$.

Proposition 3.1. *Letting ω_1, ω_2 be two arbitrary elements of $\mathcal{D}$, then:*

(1) $(\omega_1 \circ \omega_2)^t = \omega_2^t \circ \omega_1^t$ and $(\omega_1\omega_2)^t = \omega_1^t\omega_2^t$.

(2) $\mathcal{T}(\omega_1^t) = (\mathcal{T}(\omega_1))^t$.

(3) If ω_3 is a third double form then $\langle \omega_1 \circ \omega_2, \omega_3\rangle = \langle \omega_2, \omega_1^t \circ \omega_3\rangle = \langle \omega_1, \omega_3 \circ \omega_2^t\rangle$.

Proof Without loss of generality, we may assume that $\omega_1 = \theta_1 \otimes \theta_2$ and $\omega_2 = \theta_3 \otimes \theta_4$, and then

$$\begin{aligned}(\omega_1 \circ \omega_2)^t = \Big((\theta_1 \otimes \theta_2) \circ (\theta_3 \otimes \theta_4)\Big)^t =< \theta_1, \theta_4 > (\theta_3 \otimes \theta_2)^t =< \theta_1, \theta_4 > (\theta_2 \otimes \theta_3)\\ = (\theta_4 \otimes \theta_3) \circ (\theta_2 \otimes \theta_1) = (\theta_3 \otimes \theta_4)^t \circ (\theta_1 \otimes \theta_2)^t = \omega_2^t \circ \omega_1^t.\end{aligned}$$

Similarly.

$$(\omega_1\omega_2)^t = \theta_2 \wedge \theta_4 \otimes \theta_1 \wedge \theta_3 = \omega_1^t\omega_2^t.$$

This proves (1). Next, we prove prove relation (2) as follows:

$$\begin{aligned}< \mathcal{T}(\omega_1^t)(u_1), u_2 > &=< \mathcal{T}((\theta_1 \otimes \theta_2)^t)(u_1), u_2 >=< \mathcal{T}(\theta_2 \otimes \theta_1)(u_1), u_2 >\\ &=< \theta_2^\sharp, u_1 >< \theta_1^\sharp, u_2 >=< u_1, < \theta_1^\sharp, u_2 > \theta_2^\sharp >=< u_1, \mathcal{T}(\theta_1 \otimes \theta_2)u_2 >\\ &= \langle\Big(\mathcal{T}(\theta_1 \otimes \theta_2)\Big)^t(u_1), u_2\rangle = \langle\Big(\mathcal{T}(\omega_1)\Big)^t(u_1), u_2\rangle.\end{aligned}$$

Finally we prove (3). Without loss of generality assume as above that the three double forms are simple. Let $\omega_3 = \theta_5 \otimes \theta_6$ and then a simple computation shows that:

$$\langle \omega_1 \circ \omega_2, \omega_3\rangle = \langle \theta_1, \theta_4\rangle\langle \theta_3 \otimes \theta_2, \theta_5 \otimes \theta_6\rangle = \langle \theta_1, \theta_4\rangle\langle \theta_3, \theta_5\rangle\langle \theta_2, \theta_6\rangle.$$

$$\langle \omega_2, \omega_1^t \circ \omega_3\rangle = \langle \theta_3 \otimes \theta_4, \langle \theta_2, \theta_6\rangle\theta_5 \otimes \theta_1 = \langle \theta_1, \theta_4\rangle\langle \theta_3, \theta_5\rangle\langle \theta_2, \theta_6\rangle.$$

$$\langle \omega_1, \omega_3 \circ \omega_2^t\rangle = \langle \theta_1 \otimes \theta_2, \langle \theta_5, \theta_3\rangle\theta_4 \otimes \theta_6 = \langle \theta_1, \theta_4\rangle\langle \theta_3, \theta_5\rangle\langle \theta_2, \theta_6\rangle.$$

This completes the proof of the proposition.

The composition product provides another useful formula for the inner product of the double forms as follows:

Proposition 3.2 ([7])**.** *The inner product of two double forms $\omega_1, \omega_2 \in \mathcal{D}^{p,q}$ is the full contraction of the composition product $\omega_1^t \circ \omega_2$ or $\omega_2^t \circ \omega_1$. Precisely, we have:*

$$\langle \omega_1, \omega_2\rangle = \frac{1}{p!}\mathrm{c}^p(\omega_2^t \circ \omega_1) = \frac{1}{p!}\mathrm{c}^p(\omega_1^t \circ \omega_2.) \tag{11}$$

Proof We use the fact that the contraction map c is the adjoint of the exterior multiplication map by g and the above proposition as follows:

$$\frac{1}{p!}\mathrm{c}^p(\omega_2^t\circ\omega_1)=\langle\omega_2^t\circ\omega_1,\frac{g^p}{p!}\rangle=\langle\omega_1,\big(\omega_2^t\big)^t\circ\frac{g^p}{p!}=\langle\omega_1,\omega_2\rangle,$$

where we use the fact that $\frac{g^p}{p!}$ is a unit element in the composition algebra. The proof of the second relation is similar.

Remark 3.1. *The inner product used by Greub and Vanstone in [2,3,12] is the pairing product, which can be defined by*

$$\langle\langle\omega_1,\omega_2\rangle\rangle=\frac{1}{p!}\mathrm{c}^p(\omega_2\circ\omega_1)=\frac{1}{p!}\mathrm{c}^p(\omega_1\circ\omega_2.)$$

This is clearly different from the inner product that we are using in this paper. The two products coincide if ω_1 or ω_2 is a symmetric double form.

4. Interior product for double forms

Recall that for a vector $v\in V$, the interior product map $i_v:\Lambda^pV^*\to\Lambda^{p-1}V^*$, for $p\geq 1$, is defined by declaring

$$i_v\alpha(x_2,...,x_p)=\alpha(v,x_2,...,x_p).$$

There are two natural ways to extend this operation to double forms seen as bilinear maps as above. Precisely we define the inner product map $i_v:\mathcal{D}^{p,q}\to\mathcal{D}^{p-1,q}$ for $p\geq 1$ and the adjoint inner product map $\tilde{i}_v:\mathcal{D}^{p,q}\to\mathcal{D}^{p,q-1}$ for $q\geq 1$ by declaring

$$i_v\omega(x_2,...,x_p;y_1,...,y_q))=\omega(v,x_2,...,x_p;y_1,...,y_q)$$

and

$$\tilde{i}_v(\omega)(x_1,...,x_p;y_2,...,y_q))=\omega(x_1,...,x_p;v,y_2,...,y_q).$$

Note that the first map is nothing but the usual interior product of vector valued p-forms. The second map can be obtained from the first one via transposition as follows:

$$\tilde{i}_v(\omega)=\Big(i_v(\omega^t)\Big)^t.$$

In particular, the maps i_v and $\tilde{i}_v$ satisfy the same algebraic properties as the usual interior product of usual forms.

Next, we define a new natural (diagonal) interior product on double forms as follows. Letting $v\otimes w\in V\otimes V$ be a decomposable $(1,1)$ double vector, we define $i_{v\otimes w}:\mathcal{D}^{p,q}\to\mathcal{D}^{p-1,q-1}$ for $p,q\geq 1$ by

$$i_{v\otimes w}=i_v\circ\tilde{i}_w.$$

Equivalently,

$$i_{v\otimes w}\omega(x_2,...,x_p;y_2,...,y_q))=\omega(v,x_2,...,x_p;w,y_2,...,y_q).$$

The previous map is obviously bilinear with respect to v and w and therefore can be extended and defined for any $(1,1)$ double vector in $V\otimes V$.

Let h be a $(1,1)$ double form, i.e. a bilinear form on V. Then in a basis of V we have $h = \sum_i h(e_i,e_j)e_i^* \otimes e_j^*$. The dual $(1,1)$ double vector associated to h via the metric g denoted by $h^\sharp$ is by definition

$$h^\sharp = \sum_i h(e_i,e_j)e_i \otimes e_j.$$

We then define the interior product i_h to be the interior product $i_{h^\sharp}$.

Proposition 4.1. *Letting h be an arbitrary $(1,1)$ double form, then:*

1. *for any $(1,1)$ double form k we have*
$$i_h k = i_k h = \langle h,k\rangle.$$
2. *For any $(2,2)$ double form R we have*
$$i_h R = \overset{\circ}{R}h,$$
where for a $(1,1)$ double form h, $\overset{\circ}{R}h$ denotes the operator defined, for instance, in [1], by
$$\overset{\circ}{R}h(a,b) = \sum_{i,j} h(e_i,e_j)R(e_i,a;e_j,b).$$
3. *The exterior multiplication map by h in $\mathcal{D}(V^*)$ is the adjoint of the interior product map i_h, that is*
$$\langle i_h\omega_1,\omega_2\rangle = \langle \omega_1, h\omega_2\rangle.$$
4. *For $h=g$, we have that $i_g = c$ is the contraction map in $\mathcal{D}(V^*)$ as defined in the introduction.*

Proof To prove the first assertion, assume that $h = \sum_{i,j} h(e_i,e_j)e_i^* \otimes e_j^*$ and $k = \sum_{r,s} k(e_r,e_s)e_r^* \otimes e_s^*$, where (e_i^*) is an orthonormal basis of V^*. Then

$$\begin{aligned} i_h k &= \sum_{i,j,r,s} h(e_i,e_j)k(e_r,e_s)i_{e_i\otimes e_j}(e_r^*\otimes e_s^*) = \sum_{i,j,r,s} h(e_i,e_j)k(e_r,e_s)\langle e_i,e_r\rangle\langle e_j,e_s\rangle \\ &= \sum_{i,j} h(e_i,e_j)k(e_i,e_j) = \langle h,k\rangle. \end{aligned}$$

Next, we have

$$i_h R(a,b) = \sum_{i,j} h(e_i,e_j)i_{e_i\otimes e_j}R(a,b) = \sum_{i,j} h(e_i,e_j)R(e_i,a;e_j,b) = \overset{\circ}{R}h(a,b).$$

This proves statement 2. To prove the third one, assume without loss of generality that $h = v^*\otimes w^*$ is decomposed, and then

$$\begin{aligned} \langle i_h(\omega_1),\omega_2\rangle &= \langle i_v \circ \tilde{i}_w(\omega_1),\omega_2\rangle = \langle \tilde{i}_w(\omega_1),(v^*\otimes 1)\omega_2\rangle \\ &= \langle \omega_1,(1\otimes w^*)(v^*\otimes 1)\omega_2\rangle = \langle \omega_1,(v^*\otimes w^*)\omega_2\rangle \\ &= \langle \omega_1, h\omega_2\rangle. \end{aligned}$$

To prove relation 4, let (e_i^*) be an orthonormal basis of V^*, and then $g=\sum_{i=1}^n e_i^*\otimes e_i^*$ and

$$\begin{aligned} i_g\omega(x_1,...,x_{p-1};y_1,...,y_{q-1}) &= \sum_{i=1}^n i_{e_i}\circ \tilde{i}_{e_i}\omega(x_1,...,x_{p-1};y_1,...,y_{q-1}) \\ &= \sum_{i=1}^n \omega(e_i,x_1,...,x_{p-1};e_i,y_1,...,y_{q-1}) = \mathrm{c}\omega(x_1,...,x_{p-1};y_1,...,y_{q-1}). \end{aligned}$$

This completes the proof of the proposition.

More generally, for a fixed double form $\psi\in\mathcal{D}(V^*)$, following Greub we denote by $\mu_\psi:\mathcal{D}(V^*)\to\mathcal{D}(V^*)$ the left exterior multiplication map by ψ; precisely,

$$\mu_\psi(\omega)=\psi\omega.$$

We then define the map $i_\psi:\mathcal{D}\to\mathcal{D}$ as the adjoint map of μ:

$$\langle i_\psi(\omega_1),\omega_2\rangle=\langle\omega_1,\mu_\psi(\omega_2)\rangle.$$

Note that part (3) of Proposition 4.1 shows that this general interior product i_ψ coincides with the above one in the case where ψ is a $(1,1)$ double form.

Remark 4.1. *Let us remark at this stage that the interior product of double forms defined here differs by a transposition from the inner product of Greub. This is due to the fact that he is using the pairing product as explained in Remark 3.1. Precisely, an interior product $i_\psi\omega$ in the sense of Greub will be equal to the interior product $i_{\psi^t}\omega$ as defined here in this paper.*

It results directly from the definition that for any two double forms ψ,φ we have

$$\mu_\psi\circ\mu_\varphi=\mu(\psi\varphi).$$

Consequently, one immediately gets

$$i_\psi\circ i_\varphi=i_{\varphi\psi}. \tag{12}$$

Note that for $\omega\in\mathcal{D}^{p,q}$ and $\psi\in\mathcal{D}^{r,s}$ we have $i_\psi(\omega)\in\mathcal{D}^{p-r,q-s}$ if $p\geq r$ and $q\geq s$. Otherwise $i_\psi(\omega)=0$. Furthermore, it results immediately from formula (12) and statement (4) of Proposition (4.1) that

$$i_{g^k}(\omega)=\mathrm{c}^k(\omega) \tag{13}$$

for any $\omega\in\mathcal{D}$, where c is the contraction map, $c^k=\underbrace{c\circ...\circ c}_{k\text{-times}}$, and g^k is the exterior power of the metric g.

In particular, for $\omega=g^p$, we get $i_{g^k}g^p=c^k(g^p)$. Then by direct computation or by using the general formula in Lemma 2.1 in [5], one gets the following simple but useful identity:

Proposition 4.2. *For $1\leq k\leq p\leq n=\dim(\mathrm{V})$ we have*

$$i_{g^k}(\frac{g^p}{p!})=\frac{(n+k-p)!}{(p-k)!}\frac{g^{p-k}}{(n-p)!}. \tag{14}$$

We now state and prove some other useful facts about the interior product of double forms.

Proposition 4.3. *Let $\omega \in \mathcal{D}^{p,q}(V^*)$, and the double Hodge star operator $*$ is related to the interior product via the following relation:*

$$*\omega = i_\omega \frac{g^n}{n!}. \tag{15}$$

More generally, for any integer k such that $1 \leq k \leq n$ we have

$$*\frac{g^{n-k}}{(n-k)!}\omega = i_\omega \frac{g^k}{k!}. \tag{16}$$

Proof Let $\omega \in \mathcal{D}^{p,q}(V^*)$ and $\theta \in \mathcal{D}^{n-p,n-q}(V^*)$ be arbitrary double forms. To prove the previous proposition, it is sufficient to prove that

$$\langle *\omega, \theta\rangle = \langle i_\omega(\frac{g^n}{n!}), \theta\rangle.$$

Using equation (5), we have $\langle *\omega, \theta\rangle = (-1)^{(2n-p-q)(p+q-n)} * (*^2\omega\theta) = *(\omega\theta)$.

Since $\omega\theta \in \mathcal{D}^{n,n}$ and $\dim(\mathcal{D}^{n,n}) = 1$, then

$$\omega\theta = \langle \omega\theta, \frac{g^n}{n!}\rangle \frac{g^n}{n!} = \langle i_\omega(\frac{g^n}{n!}), \theta\rangle.$$

This proves the first part of the proposition. The second part results from the first one and equation (14) as follows:

$$*\frac{g^{n-k}}{(n-k)!}\omega = i_{\frac{g^{n-k}}{(n-k)!}\omega}(\frac{g^n}{n!}) = i_\omega \circ i_{\frac{g^{n-k}}{(n-k)!}}\left(\frac{g^n}{n!}\right) = i_\omega\left(\frac{g^k}{k!}\right).$$

Proposition 4.4.

1. *For any two double forms $\omega_1, \omega_2 \in \mathcal{D}(V^*)$, we have*

   $$*(\omega_1 \circ \omega_2) = *\omega_1 \circ *\omega_2.$$

   In other words, $*$ is a composition algebra endomorphism.

2. *On the diagonal subalgebra $\Delta(V^*)$, we have (formulas (11a) and (11b) in [3])*

   $$* \circ \mu_\omega = i_\omega \circ * \text{ and } \mu_\omega \circ * = * \circ i_\omega.$$

   In particular, we get the relations

   $$*\mu_\omega * = i_\omega \text{ and } * i_\omega * = \mu_\omega, \tag{17}$$

   where μ_ω is the left exterior multiplication map by ω in $\Delta(V^*)$.

Proof To prove statement 1, we assume that $\omega_1, \omega_2 \in \mathcal{D}(V)$
$*(\omega_1 \circ \omega_2) = *[(\theta_1 \otimes \theta_2) \circ (\theta_3 \otimes \theta_4)] =< \theta_1, \theta_4 > *\theta_3 \otimes *\theta_2$.

As $*$ is an isometry, we have:

$$< \theta_1, \theta_4 > *\theta_3 \otimes *\theta_2 =< *\theta_1, *\theta_4 > *\theta_3 \otimes *\theta_2 = *\omega_1 \circ *\omega_2.$$

To prove statement 2, let $\omega \in \mathcal{D}^{p,q}$ and $\varphi \in \mathcal{D}^{r,s}$, and then

$$\begin{aligned} * \circ \mu_\omega(\varphi) &= *(\omega\varphi) = i_{\omega\varphi}(\frac{g^n}{n!}) = (-1)^{pr+qs} i_{\varphi\omega}(\frac{g^n}{n!}) \\ &= (-1)^{pr+qs} i_\omega \circ i_\varphi(\frac{g^n}{n!}) = (-1)^{pr+qs} i_\omega \circ *\varphi. \end{aligned}$$

If $\omega, \varphi \in \Delta(V^*)$ then $p = q$ and $r = s$ and the result follows. To prove the second statement in (2), just apply to the previous equation the double Hodge star operator twice, once from the left and once from the right, and then use the fact that on the diagonal subalgebra we have that $*^2$ is the identity map.

As a direct consequence of the previous formula (17), applied to $\omega = g$, we recover the following result (Theorem 3.4 of [5]):

$$*\mathrm{c}* = \mu_g \text{ and } *\mu_g* = \mathrm{c}.$$

5. Exterior extensions of the endomorphisms on V

Let $h \in \mathcal{D}^{1,1}(V^*)$ be a $(1,1)$ double form on V, and let $\bar{h} = \mathcal{T}(h)$ be its associated endomorphism on V via the metric g.

There exists a unique exterior algebra endomorphism $\widehat{h}$ of ΛV that extends $\bar{h}$ and such that $\widehat{h}(1) = 1$. Explicitly, for any set of vectors $v_1, ..., v_p$ in V, the endomorphism is defined by declaring

$$\widehat{h}(v_1 \wedge ... \wedge v_p) = \overline{h}(v_1) \wedge ... \wedge \overline{h}(v_p).$$

Then one can obviously extend the previous definition by linearity.

Proposition 5.1. *The double form that is associated to the endomorphism* $\widehat{h}$ *is* $e^h := 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \dots$. *In other words, we have*

$$\mathcal{T}(e^h) = \mathcal{T}\left(\sum_{i=0}^{\infty} \frac{h^p}{p!}\right) = \widehat{h},$$

where $h^0 = 1$ *and* $h^p = 0$ *for* $p > n$. *In particular, we have* $T_V\left(\frac{g^p}{p!}\right) = \mathrm{Id}_{\Lambda^p V}$.

Proof Letting v_i and w_i be arbitrary vectors in V and $1 \le p \le n$, then

$$\begin{aligned} \langle \widehat{h}(v_1 \wedge ... \wedge v_p), w_1 \wedge ... \wedge w_p\rangle &= \langle \overline{h}(v_1) \wedge ... \wedge \overline{h}(v_p), w_1 \wedge ... \wedge w_p\rangle \\ &= \frac{1}{p!} \sum_{\sigma \in S_p} \epsilon(\sigma) \langle \overline{h}(v_{\sigma(1)}) \wedge ... \wedge \overline{h}(v_{\sigma(p)}), w_1 \wedge ... \wedge w_p\rangle \\ &= \frac{1}{p!} \sum_{\sigma,\rho \in S_p} \epsilon(\sigma)\epsilon(\rho) \langle \overline{h}(v_{\sigma(1)}), w_{\rho(1)}\rangle ... \langle \overline{h}(v_{\sigma(p)}), w_{\rho(p)}\rangle \\ &= \frac{1}{p!} \sum_{\sigma,\rho \in S_p} \epsilon(\sigma)\epsilon(\rho) h(v_{\sigma(1)}, w_{\rho(1)}) ... h(v_{\sigma(p)}, w_{\rho(p)}) \\ &= \frac{h^p}{p!}\big(v_1, ..., v_p; w_1, ..., w_p\big). \end{aligned}$$

This completes the proof of the proposition.

We can now extend the exterior algebra endomorphism $\widehat{h}$ on ΛV to an exterior algebra endomorphism on the space $\mathcal{D}(V)$ of double vectors. In the same way as we did for the interior product in the previous paragraph, we can perform this extension in two natural ways as follows:

We define the right endomorphism

$$\widehat{h}_R : \mathcal{D}(V) \to \mathcal{D}(V),$$

for a simple double vector $\omega = \theta_1 \otimes \theta_2$ by

$$\widehat{h}_R(\omega) = h_R(\theta_1 \otimes \theta_2) = \theta_1 \otimes \widehat{h}(\theta_2).$$

Then one extends the definition using linearity. Similarly, we define the left extension endomorphism

$$\widehat{h}_L : \mathcal{D}(V) \to \mathcal{D}(V)$$

by:

$$\widehat{h}_L(\omega) = h_L(\theta_1 \otimes \theta_2) = \widehat{h}(\theta_1) \otimes \theta_2.$$

Proposition 5.2. *The endomorphisms $\widehat{h}_L$ and $\widehat{h}_R$ are double exterior algebra endomorphisms.*

Proof

1. Without loss of generality, let $\omega = \theta_1 \otimes \theta_2$ and $\theta = \theta_3 \otimes \theta_4$ be simple double forms, and then

$$\begin{aligned}\widehat{h}_R(\omega\theta) &= \widehat{h}_R\big((\theta_1 \otimes \theta_2)(\theta_3 \otimes \theta_4)\big) = \widehat{h}_R\big(\theta_1 \wedge \theta_3 \otimes \theta_2 \wedge \theta_4\big)\\ &= (\theta_1 \wedge \theta_3) \otimes \widehat{h}(\theta_2 \wedge \theta_4) = (\theta_1 \wedge \theta_3) \otimes (\widehat{h}(\theta_2) \wedge \widehat{h}(\theta_4))\\ &= (\theta_1 \otimes \widehat{h}(\theta_2))(\theta_3 \otimes \widehat{h}(\theta_4)) = \widehat{h}_R(\theta_1 \otimes \theta_2)\widehat{h}_R(\theta_3 \otimes \theta_4)\\ &= \widehat{h}_R(\omega)\widehat{h}_R(\theta).\end{aligned}$$

Proposition 5.3. *Letting $\widehat{h}_R, \widehat{h}_L$ be as above and $1 \leq p \leq n$, then*

$$\widehat{h}_R(\frac{g^p}{p!}) = \widehat{h}_L(\frac{g^p}{p!}) = \frac{h^p}{p!}, \tag{18}$$

where the metric g is seen here as a $(1,1)$ double exterior vector.

Proof Letting (e_i) be an orthonormal basis for (V, g), then the double vector g splits to $g = \sum_{i=1}^n e_i \otimes e_i$ and therefore

$$\widehat{h}_R(g) = \widehat{h}_R(\sum_{i=1}^n e_i \otimes e_i) = \sum_{i=1}^n e_i \otimes \widehat{h}(e_i) = \sum_{i,j=1}^n h(e_i, e_j) e_i \otimes e_j = h.$$

Next, Proposition (5.2) shows that

$$\widehat{h}_R(g^p) = \big(\widehat{h}_R(g)\big)^p = h^p.$$

The proof for $\widehat{h}_L$ is similar.

A special case of the previous proposition deserves more attention; namely, when $p = n$, we have

$$\widehat{h}_R(\frac{g^n}{n!}) = \widehat{h}_L(\frac{g^n}{n!}) = \frac{h^n}{n!} = \det h.\frac{g^n}{n!}. \tag{19}$$

The next proposition shows that the endomorphisms $\widehat{h}_R$ and $\widehat{h}_L$ are nothing but the the right and left multiplication maps in the composition algebra.

Proposition 5.4. *With the above notations we have*

$$\widehat{h}_R(\omega) = e^h \circ \omega, \text{ and } \widehat{h}_L(\omega) = \omega \circ e^{(h^t)}. \tag{20}$$

Proof As $\widehat{h}_R$ is linear in ω, we may assume, without loss of any generality, that the double (p,q) vector ω is simple; that is, $\omega = e_{i_1} \wedge ... \wedge e_{i_p} \otimes e_{j_1} \wedge ... \wedge e_{j_q}$. Let us use multiindex notation and write $\omega = e_I \otimes e_J$. On one hand, we have

$$\widehat{h}_R(\omega) = e_I \otimes \widehat{h}(e_J) = \sum_K \langle \widehat{h}(e_J), e_K \rangle e_I \otimes e_K =$$

$$= \sum_K \frac{h^q}{q!}(e_J, e_K) e_I \otimes e_K = \sum_{K,L} \frac{h^q}{q!}(e_L, e_K)\langle e_L, e_J \rangle e_I \otimes e_K$$

$$= \sum_{K,L} \frac{h^q}{q!}(e_L, e_K)(e_L \otimes e_K) \circ (e_I \otimes e_J) = \frac{h^q}{q!} \circ \omega.$$

To prove the second assertion we proceed as follows:

$$\widehat{h}_L(\omega) = \left(\widehat{h}_R(\omega^t)\right)^t = (e^h \circ \omega^t)^t = \omega \circ (e^h)^t = \omega \circ e^{(h^t)}.$$

The fact that $(e^h)^t = e^{(h^t)}$ results from Proposition 3.1.

Corollary 5.5. *The adjoint endomorphism of $\widehat{h}_R$ (resp. $\widehat{h}_L$) is $\widehat{(h^t)}_R$ (resp. $\widehat{(h^t)}_L$).*

Proof Proposition 3.1 shows that $(e^h)^t = e^{(h^t)}$ and

$$\langle \widehat{h}_R(\omega_1), \omega_2 \rangle = \langle e^h \circ \omega_1, \omega_2 \rangle = \langle \omega_1, e^{(h^t)} \circ \omega_2 \rangle = \langle \omega_1, \widehat{h^t}_R(\omega_2) \rangle.$$

The proof for $\widehat{h}_L$ is similar.

Using the facts that both $\widehat{h}_L$ and $\widehat{h}_R$ are exterior algebra homomorphisms and the previous corollary, we can easily prove the following technical but useful identities.

Corollary 5.6. *Let $\omega \in \mathcal{D}(V)$ and h be an endomorphism of V. Then we have*

$$i_\omega \circ \widehat{h}_R = \widehat{h}_R \circ i_{\widehat{(h^t)}_R(\omega)}, \text{ and } i_\omega \circ \widehat{h}_L = \widehat{h}_L \circ i_{\widehat{(h^t)}_L(\omega)}. \tag{21}$$

Proof Since $\widehat{h}_R$ is an exterior algebra endomorphism, then for any double vectors ω and θ we have

$$\widehat{h}_R(\omega\theta) = \widehat{h}_R(\omega)\widehat{h}_R(\theta).$$

That is,

$$\widehat{h}_R \circ \mu_\omega = \mu_{\widehat{h}_R(\omega)} \circ \widehat{h}_R.$$

Next, take the adjoint of both sides of the previous equation to get

$$i_\omega \circ \widehat{(h^t)}_R = \widehat{(h^t)}_R \circ i_{\widehat{h}_R(\omega)}.$$

The proof of the second identity is similar.

Now we have enough tools to easily prove delicate results of linear algebra, including the general Laplace expansions of the determinant, as follows.

Proposition 5.7 (Laplace Expansion of the determinant, Proposition 7.2.1 in [2]). *For* $1 \leq p \leq n$*, we have*

$$\frac{(h^t)^{n-p}}{(n-p)!} \circ (*\frac{h^p}{p!}) = \det h \frac{g^{n-p}}{(n-p)!} \text{ and } \big(*\frac{(h^t)^p}{(p)!}\big) \circ \frac{h^{n-p}}{(n-p)!} = \det h \frac{g^{n-p}}{(n-p)!}. \tag{22}$$

Proof Using identities (21) we have

$$\begin{aligned}\frac{(h^t)^{n-p}}{(n-p)!} \circ (*h^p) =& \widehat{(h^t)}_R \circ i_{h^p} \frac{g^n}{n!} = \widehat{(h^t)}_R \circ i_{\widehat{h}_R(g^p)} \frac{g^n}{n!} = i_{g^p} \circ \widehat{(h^t)}_R(\frac{g^n}{n!}) \\ &= i_{g^p} \circ \frac{(h^t)^n}{n!} = \det(h^t) i_{g^p} \circ \frac{g^n}{n!} = (\det h) \frac{p!}{(n-p)!} g^{n-p}.\end{aligned}$$

The second identity can be proved in the same way as the first one by using $\widehat{h}_L$ instead of $\widehat{h}_R$, or simply just by taking the transpose of the first identity.

To see why the previous identity coincides with the classical Laplace expansion of the determinant we refer the reader to, e.g., [7].

Proposition 5.8 (Proposition 7.2.2, [2]). *Letting* h *be a bilinear form on the vector space* V*, then*

$$\begin{aligned} i_{*h^p} h^q =& \binom{2n-p-q}{n-p} p!q!(\det h) \frac{h^{p+q-n}}{(p+q-n)!}, \\ (*h^p)(*h^q) =& \binom{2n-p-q}{n-p} p!q!(\det h)(*\frac{h^{p+q-n}}{(p+q-n)!}). \end{aligned}$$

Proof We use in succession identities 14, 18, and 20–22 to get

$$\begin{aligned} i_{*h^p} h^q &= i_{*h^p} \widehat{h}_R(g^q) = \widehat{h}_R \circ i_{\widehat{(h^t)}_R(*h^p)}(g^q) \\ &= \widehat{h}_R \circ i_{\frac{(h^t)^{n-p}}{(n-p)!} \circ (*h^p)}(g^q) = \frac{p! \det h}{(n-p)!} \widehat{h}_R \circ i_{g^{n-p}}(g^q) \\ &= \frac{p!q!(2n-p-q)!}{(n-q)!(n-p)!} \widehat{h}_R(\frac{g^{p+q-n}}{(p+q-n)!}) = \frac{p!q!(2n-p-q)!}{(n-q)!(n-p)!} \frac{h^{p+q-n}}{(p+q-n)!}. \end{aligned}$$

This proves the first identity. The second one results from the first one by using identity 17 as follows:

$$(*h^p)(*h^q) = *i_{*h^p} * (*h^q) = *\big(i_{*h^p}(h^q)\big).$$

The interior product provides a simple formulation of the Newton (or cofactor) transformations $t_p(h)$ of a bilinear form h and also for its characteristic coefficients $s_k(h)$ [7] as follows:

Proposition 5.9. *Letting h be a bilinear form on the vector space V, then:*

1. *For $0 \leq p \leq n$, the pth invariant of h is given by*

$$s_p(h) := *\frac{g^{n-p}h^p}{(n-p)!p!} = i_{\frac{h^p}{p!}}\frac{g^p}{p!}.$$

2. *For $0 \leq p \leq n-1$, the pth Newton transformation of h is given by*

$$t_p(h) := *\frac{g^{n-p-1}h^p}{(n-p-1)!p!} = i_{\frac{h^p}{p!}}\frac{g^{p+1}}{(p+1)!}.$$

3. *More generally, for $0 \leq p \leq n-r$, the (r,p) cofactor transformation [7] of h is given by*

$$s_{(r,p)}(h) := *\frac{g^{n-p-r}h^p}{(n-p-r)!p!} = i_{\frac{h^p}{p!}}\frac{g^{p+r}}{(p+r)!}.$$

Proof First we use formula (14) to prove (1) as follows. For $0 \leq p \leq n-1$ we have

$$p!t_p(h) = *\frac{g^{n-p-1}h^p}{(n-p-1)!} = i_{\frac{g^{n-p-1}h^p}{(n-p-1)!}}(\frac{g^n}{n!}) = i_{h^p} \circ i_{\frac{g^{n-p-1}}{(n-p-1)!}}(\frac{g^n}{n!}) = i_{h^p}(\frac{g^{p+1}}{p+1}).$$

In the same way, we prove together relation (2) and its generalization, relation (3), as follows:

$$p!s_{(r,p)}(h) = *\frac{g^{n-p-r}h^p}{(n-p-r)!} = i_{\frac{g^{n-p-r}h^p}{(n-p-r)!}}(\frac{g^n}{n!}) = i_{h^p} \circ i_{\frac{g^{n-p-r}}{(n-p-r)!}}(\frac{g^n}{n!}) = i_{h^p}(\frac{g^{p+r}}{p+r}).$$

Remark 5.1. *According to [7], for $0 \leq r \leq n-pq$, the (r,pq) cofactor transformation of a (p,p) double form ω is defined by*

$$h_{(r,pq)}(\omega) := *\frac{g^{n-pq-r}\omega^q}{(n-pq-r)!}.$$

Using the same arguments as above, it is easy to see that

$$h_{(r,pq)}(\omega) = i_{\omega^q}\frac{g^{pq-r}}{(pq-r)!}.$$

6. Greub and Greub–Vanstone basic identities

6.1. Greub's basic identities

We now state and prove Greub's basic identities relating the exterior and composition products of double forms.

Proposition 6.1 (Proposition 6.5.1 in [2])**.** *If $h, h_1, ..., h_p$ are $(1,1)$-forms, then*

$$\begin{aligned} i_h(h_1...h_p) = &\sum_j \langle h, h_j\rangle h_1...\hat{h}_j...h_p \\ &- \sum_{j<k}(h_j \circ h^t \circ h_k + h_k \circ h^t \circ h_j)h_1...\hat{h}_j...\hat{h}_k...h_p. \end{aligned} \tag{23}$$

In particular, if $k = h_1 = ... = h_p$, *we have*

$$i_h k^p = p\langle h, k\rangle k^{p-1} - p(p-1)(k \circ h^t \circ k)k^{p-2}.$$

Proof Assume without loss of generality that $h = \theta \otimes \vartheta$ and $h_i = \theta_i \otimes \vartheta_i$, where $\theta, \vartheta, \theta_i$, and ϑ_i are in V^*. Then

$$\begin{aligned} i_h(h_1...h_p) &= i_{(\theta\otimes\vartheta)}(\theta_1 \wedge ... \wedge \theta_p \otimes \vartheta_1 \wedge ... \wedge \vartheta_p) \\ &= i_\theta \circ \tilde{i}_\vartheta(\theta_1 \wedge ... \wedge \theta_p \otimes \vartheta_1 \wedge ... \wedge \vartheta_p) \\ &= i_\theta\Big(\theta_1 \wedge ... \wedge \theta_p\Big) \otimes i_\vartheta\Big(\vartheta_1 \wedge ... \wedge \vartheta_p\Big) \\ &= \sum_{j,k}(-1)^{j+k}\langle \vartheta, \vartheta_j\rangle\langle\theta,\theta_k\rangle(\theta_1 \wedge ... \wedge \hat{\theta}_k \wedge ... \wedge \theta_p \otimes \vartheta_1 \wedge ... \wedge \hat{\vartheta}_j \wedge ... \wedge \vartheta_p), \end{aligned}$$

where we have used the fact that the ordinary interior product in the exterior algebra $\Lambda(V^*)$ is an antiderivation of degree –1. Next, write the previous sum in three parts for $j = k$, $j < k$, and $j > k$ as follows:

$$\begin{aligned} i_h(h_1...h_p) &= \sum_j \langle \vartheta, \vartheta_j\rangle\langle\theta,\theta_j\rangle(\theta_1 \wedge ... \wedge \hat{\theta}_j \wedge ... \wedge \theta_p \otimes \vartheta_1 \wedge ... \wedge \hat{\vartheta}_j \wedge ... \wedge \vartheta_p) \\ &- \sum_{j<k}\langle \vartheta, \vartheta_j\rangle\langle\theta,\theta_k\rangle(\theta_j \otimes \vartheta_k)[\theta_1 \wedge ... \wedge \hat{\theta}_j \wedge ... \wedge \hat{\theta}_k \wedge ... \wedge \theta_p] \otimes [\vartheta_1 \wedge ... \wedge \hat{\vartheta}_j \wedge ... \wedge \hat{\vartheta}_k \wedge ... \wedge \vartheta_p] \\ &- \sum_{k<j}\langle \vartheta, \vartheta_j\rangle\langle\theta,\theta_k\rangle(\theta_j \otimes \vartheta_k)[\theta_1 \wedge ... \wedge \hat{\theta}_j \wedge ... \wedge \hat{\theta}_k \wedge ... \wedge \theta_p] \otimes [\vartheta_1 \wedge ... \wedge \hat{\vartheta}_j \wedge ... \wedge \hat{\vartheta}_k \wedge ... \wedge \vartheta_p]. \end{aligned}$$

Using the definition of the composition product, one can easily check that

$$\langle \vartheta, \vartheta_j\rangle\langle\theta,\theta_k\rangle(\theta_j \otimes \vartheta_k) = h_k \circ h^t \circ h_j.$$

Consequently, we can write

$$\begin{aligned} i_h(h_1...h_p) &= \sum_j \langle h, h_j\rangle h_1...\hat{h}_j...h_p \\ &- \sum_{j<k}(h_k \circ h^t \circ h_j)h_1...\hat{h}_i...\hat{h}_j...h_p \\ &- \sum_{k<j}(h_k \circ h^t \circ h_j)h_1...\hat{h}_i...\hat{h}_j...h_p. \end{aligned}$$

This completes the proof of the proposition.

Corollary 6.2. *If* $h_1...h_p$ *are* $(1,1)$ *double forms, then*

$$c(h_1...h_p) = \sum_i (ch_i)h_1...\hat{h}_i....h_p - \sum_{i<j}(h_j \circ h_i + h_i \circ h_j)h_1...\hat{h}_i...\hat{h}_j...h_p. \tag{24}$$

In particular, for a $(1,1)$ *double form* k, *the contraction of* k^p *is given by*

$$ck^p = p(ck)k^{p-1} - p(p-1)(k \circ k)k^{p-2}.$$

Proof Recall that $h=g$ is a unit element for the composition product and that the contraction map c is the adjoint of the exterior multiplication map by g. The corollary follows immediately from the previous.

As a corollary to Greub's basic identity (23), Vanstone proved the following formula, which is in fact the main result of his paper [12] (formula (27)):

$$i_{\omega^t}\frac{g^{q+2p}}{(q+2p)!}=(-1)^p\sum_r(-1)^r\mu_{\frac{g^{r+q}}{(r+q)!}}\circ i_{\frac{g^r}{r!}}(\omega),$$

where ω is any (p,p) double form, and p,q are arbitrary integers.

In view of formula (16) of this paper, the previous identity can be reformulated as follows:

$$*\frac{g^{n-q-2p}\omega^t}{(n-q-2p)!}=\sum_r(-1)^{r+p}\frac{g^{r+q}}{(r+q)!}\frac{c^r}{r!}(\omega).$$

Letting $k=n-q-p$, the previous formula then reads

$$*\Big(\frac{g^{k-p}\omega}{(k-p)!}\Big)=\sum_r(-1)^{r+p}\frac{g^{n-p-k+r}}{(n-p-k+r)!}\frac{c^r}{r!}(\omega^t). \tag{25}$$

We then recover formula (15) of [5] in Theorem 4.1. Note that Vanstone's proof of this identity does not require the (p,p) double form ω to satisfy the first Bianchi identity or to be a symmetric double form.

6.2. Greub–Vanstone basic identities

Greub–Vanstone basic identities are stated in the following theorem.

Theorem 6.3 ([3]). *For $1\le p\le n$, and for bilinear forms $h_1,...,h_p$ and $k_1,...,k_p$, we have*

$$(h_1h_2...h_p)\circ(k_1k_2...k_p)=\sum_{\sigma\in S_p}(h_1\circ k_{\sigma(1)})...(h_p\circ k_{\sigma(p)})=\sum_{\sigma\in S_p}(h_{\sigma(1)}\circ k_1)...(h_{\sigma(p)}\circ k_p).$$

In particular, when $h=h_1=...=h_p$ and $k=k_1=...=k_p$, we have the following nice relation:

$$h^p\circ k^p=p!(h\circ k)^p. \tag{26}$$

Proof We assume that $h_i=\theta_i\otimes\vartheta_i$ and $k_i=\theta_i'\otimes\vartheta_i'$, where $\theta_i,\vartheta_i,\theta_i'$, and ϑ_i' are in V^*, and then by definition of the exterior product of double forms, we have

$$h_1...h_p=\theta_1\wedge...\wedge\theta_p\otimes\vartheta_1\wedge...\wedge\vartheta_p$$

and

$$k_1...k_p=\theta_1'\wedge...\wedge\theta_p'\otimes\vartheta_1'\wedge...\wedge\vartheta_p'.$$

It follows from the definition of the composition product that

$$(h_1h_2...h_p)\circ(k_1k_2...k_p)=\det(\langle\theta_i,\vartheta_j'\rangle)[\theta_1'\wedge...\wedge\theta_p'\otimes\vartheta_1\wedge...\wedge\vartheta_p].$$

Now the determinant here can be expanded in two different ways:

$$\det(\langle\theta_i,\vartheta_j'\rangle)=\sum_{\sigma\in S_p}\varepsilon_\sigma\langle\theta_1,\vartheta_{\sigma(1)}'\rangle...\langle\theta_p,\vartheta_{\sigma(p)}'\rangle=\sum_{\sigma\in S_p}\varepsilon_\sigma\langle\theta_{\sigma(1)},\vartheta_1'\rangle...\langle\theta_{\sigma(p)},\vartheta_p'\rangle.$$

Therefore, using the definition of the composition product, we get:

$$(h_1h_2...h_p)\circ(k_1k_2...k_p) = \sum_{\sigma\in S_p} \varepsilon_\sigma \langle\theta_1, \vartheta'_{\sigma(1)}\rangle...\langle\theta_p, \vartheta'_{\sigma(p)}\rangle[\theta'_1 \wedge ... \wedge \theta'_p \otimes \vartheta_1 \wedge ... \wedge \vartheta_p]$$

$$= \sum_{\sigma\in S_p} \langle\theta_1, \vartheta'_{\sigma(1)}\rangle...\langle\theta_p, \vartheta'_{\sigma(p)}\rangle[\theta'_{\sigma(1)} \wedge ... \wedge \theta'_{\sigma(p)} \otimes \vartheta_1 \wedge ... \wedge \vartheta_p]$$

$$= \sum_{\sigma\in S_p} \Big(\prod_{i=1}^{p}(\theta_i \otimes \vartheta'_i) \circ (\theta'_{\sigma(i)} \otimes \vartheta'_{\sigma(i)})\Big) = \sum_{\sigma\in S_p} \Big(\prod_{i=1}^{p} h_1 \circ k_{\sigma(i)}\Big).$$

If we use the second expansion of the determinant we get the second formula using the same arguments.

7. Pontryagin classes and p-pure curvature tensors

7.1. Alternating operator, Bianchi map

For each $p \geq 1$, we define the alternating operator as follows:

$$\begin{aligned}\mathrm{Alt} :&\mathcal{D}^{p,p}(V^*) \longrightarrow \Lambda^{2p}(V^*)\\ &\omega \mapsto \mathrm{Alt}(\omega)(v_1, ..., v_p, v_{p+1}, ...v_{2p})\\ &= \frac{1}{(2p)!}\sum_{\sigma\in S_{2p}} \varepsilon(\sigma)\omega(v_{\sigma(1)} \wedge ... \wedge v_{\sigma(p)}, v_{\sigma(p+1)} \wedge ... \wedge v_{\sigma(2p)}).\end{aligned}$$

Another basic map in $\mathcal{D}(V^*)$ is the first Bianchi map, denoted by $\mathfrak{S}$. It maps $\mathcal{D}^{p,q}(V^*)$ into $D^{p+1,q-1}(V^*)$ and is defined as follows. Let $\omega \in \mathcal{D}^{p,q}(V^*)$ and set $\mathfrak{S}\omega = 0$ if $q = 0$. Otherwise, set

$$\mathfrak{S}\omega(e_1 \wedge ... \wedge e_{p+1}, e_{p+2} \wedge ... \wedge e_{p+q}) = \frac{1}{p!}\sum_{\sigma\in S_{p+1}} \varepsilon(\sigma)\omega(e_{\sigma(1)} \wedge ... \wedge e_{\sigma(p)}, e_{\sigma(p+1)} \wedge e_{p+2} \wedge ... \wedge e_{p+q}). \tag{27}$$

In other terms, $\mathfrak{S}$ is a partial alternating operator with respect to the first $(p+1)$ arguments. If we assume that $p = q$, then the composition

$$\mathfrak{S}^p := \mathfrak{S} \circ ... \circ \mathfrak{S} \tag{28}$$

is up to a constant factor, the alternating operator Alt. In particular, we have the following relation first observed by Thorpe [11] and Stehney [10].

Lemma 7.1. *If* $\omega \in \ker \mathfrak{S}$, *then* $\mathrm{Alt}(\omega) = 0$.

Lemma 7.2. *The linear application* Alt *is surjective.*

Proof If ω is a $(2p)$-form in $\Lambda^{2p}(V^*)$, then ω is also a (p,p) double form whose image under the alternating operator is the $(2p)$-form ω itself.

Lemma 7.3. *We have the following isomorphism:*

$$\mathcal{D}^{p,p}(V)/\ker \mathrm{Alt} \cong \Lambda^{2p}(V),$$

In particular, we have the following orthogonal decomposition:

$$\mathcal{D}^{p,p}(V) = \ker \mathrm{Alt} \oplus \Lambda^{2p}(V).$$

7.2. p-Pure Riemannian manifolds

According to Maillot [9], a Riemannian n-manifold has a pure curvature tensor if at each point of the manifold there exists an orthonormal basis (e_i) of the tangent space at this point such that the Riemann curvature tensor R belongs to $\mathrm{Span}\{e_i^*\wedge e_j^*\otimes e_i^*\wedge e_j^* : 1\leq i<j\leq n\}$. This class contains all conformally flat manifolds, hypersurfaces of space forms, and all three-dimensional Riemannian manifolds. Maillot proved in [9] that all Pontryagin forms of a pure Riemannian manifold vanish. In this section we are going to refine this result.

Definition 7.1. *Let $1\leq p\leq n/2$ be a positive integer. A Riemannian n-manifold is said to have a p-pure curvature tensor if at each point of the manifold there exists an orthonormal basis (e_i) of the tangent space at this point such that the exterior power R^p of R belongs to*

$$\mathrm{Span}\{e_{i_1}^*\wedge ...\wedge e_{i_{2p}}^*\otimes e_{i_1}^*\wedge ...\wedge e_{i_{2p}}^* : 1\leq i_1<...<i_{2p}\leq n\}.$$

The previous definition can be reformulated using the exterior product of double forms as follows.

Proposition 7.4. *Let $1\leq p\leq n/2$ be a positive integer. A Riemann n-manifold with Riemann curvature tensor R is p-pure if and only if at each point of the manifold, there exists a family $\{h_i : i\in I\}$ of simultaneously diagonalizable symmetric bilinear forms on the tangent space such that the exterior power R^p of R at that point belongs to*

$$\mathrm{Span}\{h_{i_1}...h_{i_{2p}} : i_1,..i_{2p}\in I\}.$$

We notice that the condition that the family $\{h_i : i\in I\}$ consists of simultaneously diagonalizable symmetric bilinear forms is equivalent to the fact that $h_i^t=h_i$ and $h_i\circ h_j=h_j\circ h_i$ for all $i,j\in I$.

Proof Assuming that R is p-pure, then by definition we have

$$\begin{aligned}R^p &= \sum_{1\leq i_1<...<i_{2p}\leq n}\lambda_{i_1...i_{2p}}e_{i_1}^*\wedge ...\wedge e_{i_{2p}}^*\otimes e_{i_1}^*\wedge ...\wedge e_{i_{2p}}^*\\ &= \sum_{1\leq i_1<...<i_{2p}\leq n}\lambda_{i_1...i_{2p}}\big(e_{i_1}^*\otimes e_{i_1}^*\big)\big(e_{i_2}^*\otimes e_{i_2}^*\big)...\big(e_{i_{2p}}^*\otimes e_{i_{2p}}^*\big)\\ &= \sum_{1\leq i_1<...<i_{2p}\leq n}\lambda_{i_1...i_{2p}}h_{i_1}...h_{i_{2p}},\end{aligned}$$

where $h_i=e_i^*\otimes e_i^*$. It is clear that $h_i^t=h_i$ and $h_i\circ h_j=\delta_{ij}e_j^*\otimes e_i^*=h_j\circ h_i$.

Conversely, assume that there exists a family $\{h_i : i\in I\}$ of simultaneously diagonalizable symmetric bilinear forms such that

$$R^p=\sum_{i_1,...,i_{2p}\in I}\lambda_{i_1...i_{2p}}h_{i_1}...h_{i_{2p}}.$$

Let (e_i) be an orthonormal basis of the tangent space at the point under consideration that diagonalizes simultaneously all the bilinear forms in the family $\{h_i : i\in I\}$. Then if $h_{i_k}=\sum_{j_k=1}^n\rho_{i_kj_k}e_{j_k}^*\otimes e_{j_k}^*$ for each

$k = 1, ..., 2p$ we have

$$\begin{aligned} R^p &= \sum_{i_1,...,i_{2p}\in I} \lambda_{i_1...i_{2p}} h_{i_1}...h_{i_{2p}} \\ &= \sum_{i_1,...,i_{2p}\in I} \sum_{j_1,...,j_{2p}=1}^{n} \lambda_{i_1...i_{2p}} \rho_{i_1 j_1}...\rho_{i_{2p} j_{2p}} \left(e^*_{j_1} \otimes e^*_{j_1}\right)...\left(e^*_{j_{2p}} \otimes e^*_{j_{2p}}\right) \\ &= \sum_{i_1,...,i_{2p}\in I} \sum_{j_1,...,j_{2p}=1}^{n} \lambda_{i_1...i_{2p}} \rho_{i_1 j_1}...\rho_{i_{2p} j_{2p}} e^*_{j_1} \wedge ... \wedge e^*_{j_{2p}} \otimes e^*_{j_1} \wedge ... \wedge e^*_{j_{2p}}. \end{aligned}$$

This completes the proof.

In the next proposition we provide several classes of examples and properties about p-pure manifolds

Proposition 7.5. *1. Every pure Riemannian manifold is p-pure for any $p \geq 1$. More generally, if a Riemannian manifold is p-pure for some p then it is pq-pure for any $q \geq 1$.*

However the converse it is not always true. A Riemannian manifold can be p-pure for some $p > 1$ without being pure.

2. *A Riemannian manifold of dimension $n = 2p$ is always p-pure.*

3. *A Riemannian manifold of dimension $n = 2p + 1$ is always p-pure.*

4. *A Riemannian manifold with constant p-sectional curvature, in the sense of Thorpe [11], is always p-pure.*

Proof The first and second statements in (1) are straightforward to prove. The next three properties provide examples of p-pure manifolds ($p > 1$) without being necessarily pure. Property (2) follows from the fact that in this case the Riemann tensor R is such that R^p is proportional to g^n. To prove (3), we use Proposition 2.1 in [6], which shows that in this case we have

$$R^p = \omega_1 g^{2p-1} + \omega_0 g^{2p},$$

where ω_1 is a symmetric bilinear form and ω_0 is a scalar. Finally, (4) follows from the fact that constant p-sectional curvature is equivalent to the fact that its Riemann tensor R satisfying R^p is proportional to g^{2p}.

We are now ready to state and prove the following theorem.

Theorem 7.6. *If a Riemannian n-manifold is k-pure and $n \geq 4k$, then its Pontryagin class of degree $4k$ vanishes.*

Proof Denote by R the Riemann curvature tensor of the given Riemannian manifold. Then the following differential form is a representative of the Pontryagin class of degree $4k$ of the manifold [10]:

$$P_k(R) = \frac{1}{(k!)^2(2\pi)^{2k}} \mathrm{Alt}\big(R^k \circ R^k\big). \tag{29}$$

We are going to show that $P_k(R)$ vanishes.

According to Proposition 7.4 there exists a family $\{h_i : i \in I\}$ of simultaneously diagonalizable symmetric bilinear forms such that

$$R^k = \sum_{i_1,...,i_{2k}\in I} \lambda_{i_1...i_{2k}} h_{i_1}...h_{i_{2k}}.$$

Therefore, we have

$$R^k \circ R^k = \sum_{\substack{i_1,\ldots,i_{2k}\in I \\ j_1,\ldots,j_{2k}\in I}} \lambda_{i_1\ldots i_{2k}}\lambda_{j_1\ldots j_{2k}} h_{i_1}\ldots h_{i_{2k}} \circ h_{j_1}\ldots h_{j_{2k}}.$$

Next, Proposition 6.3, shows that each term of the previous sum is an exterior product of double forms of the form $hi \circ h_j$, each of which is a symmetric bilinear form and therefore belongs to the kernel of the first Bianchi sum $\mathfrak{S}$. On the other hand, the kernel of $\mathfrak{S}$ is closed under exterior products [4], and consequently $R^k \circ R^k$ belongs to the kernel of $\mathfrak{S}$ and therefore $\mathrm{Alt}\big(R^k \circ R^k\big) = 0$ by Lemma 7.1.

Remark 7.1. *We remark that the previous theorem can alternatively be proved directly without using identity 7.4 as follows.*

Let us use a multiindex and write $R^k = \sum_I \lambda_I e_I \otimes e_I$ *as in the definition. Then*

$$\mathrm{Alt}\big(R^k \circ R^k\big) = \mathrm{Alt}\Big(\sum_I \lambda_I^2 e_I \otimes e_I\Big) = 0.$$

As a direct consequence of the previous theorem, we obtain the following, equivalent to a result of Stehney (Theorem 3.3, [10]).

Corollary 7.7. *Let M be a Riemannian manifold and p an integer such that $4p \le n = \dim M$. If at any point $m \in M$ the Riemann curvature tensor R satisfies*

$$R^p = c_p A^{2p},$$

where $A : T_mM \longrightarrow T_mM$ is symmetric bilinear form and c_p is a constant, then the differential form $\mathrm{Alt}(R^p \circ R^p)$ *is* 0.

Proof Since A is symmetric then $R^p = c_p A\ldots A$ is p-pure; the result follows directly from the previous theorem.

References

[1] Besse AL. Einstein Manifolds. Classics in Mathematics. Berlin, Germany: Springer-Verlag, 2002.

[2] Greub WH. Multilinear Algebra, Second Edition. New York, NY, USA: Springer-Verlag, 1978.

[3] Greub WH, Vanstone JR. A basic identity in mixed exterior algebra. Linear & Multilinear Algebra 1987; 21(1): 41-61. doi: 10.1080/03081088708817778.

[4] Kulkarni RS. On the Bianchi identities. Mathematische Annalen 1972; 199(4): 175-204. doi: 10.1007/BF01429873.

[5] Labbi ML. Double forms, curvature structures and the (p,q)-curvatures. Transactions of the American Mathematical Society 2005; 357(10): 3971-3992.

[6] Labbi ML. On generalized Einstein metrics. Balkan Journal of Geometry and Its Applications 2010; 15(2): 61-69.

[7] Labbi ML. On some algebraic identities and the exterior product of double forms. Archivum Mathematicum (Brno) 2013; 49(4): 241-271.

[8] Labbi ML. Remarks on Bianchi sums and Pontrjagin classes. Journal of the Australian Mathematical Society 2014; 97(3): 365-382. doi: 10.1017/S1446788714000366.

[9] Maillot H. Sur les variétés riemanniennes a opérateur de courbure pur. Comptes Rendus de l'Académie des Sciences Paris A 1974; 278: 1127-1130 (in French).

[10] Stehney A. Courbure d'ordre p et les classes de Pontrjagin. Journal of Differential Geometry 1973; 8: 125-134 (in French). doi: 10.4310/jdg/1214431487.

[11] Thorpe JA. Sectional curvatures and characteristic classes. Annals of Mathematics 1964; 80: 429-443.

[12] Vanstone JR. The Poincaré map in mixed exterior algebra. Canadian Mathematical Bulletin 1983; 26(2): 129-136. doi: 10.4153/CMB-1983-021-2.