Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 44 – 62
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doi:10.3906/mat-1805-103

Extensions and topological conditions of NJ rings

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Received: 21.05.2018 • **Accepted/Published Online:** 12.10.2018 • **Final Version:** 18.01.2019

Abstract: A ring R is said to be NJ if $J(R) = N(R)$. This paper mainly studies the relationship between NJ rings and related rings, and investigates the Dorroh extension, the Nagata extension, the Jordan extension, and some other extensions of NJ rings. At the same time, we also prove that if R is a weakly 2-primal α-compatible ring with an isomorphism α of R, then $R[x;\alpha]$ is NJ; if R is a weakly 2-primal δ-compatible ring with a derivation δ of R, then $R[x;\delta]$ is NJ. Moreover, we consider some topological conditions for NJ rings and show for a NJ ring R that R is *J-pm* if and only if $J\text{-}Spec(R)$ is a normal space if and only if $Max(R)$ is a retract of $J\text{-}Spec(R)$.

Key words: Jacobson radical, nilpotent element, weakly 2-primal ring, polynomial extension, topological space

1. Introduction

Throughout this paper, all rings are associative with identity unless otherwise stated. Given a ring R, we use the symbol $N(R)$ to denote the set of all nilpotent elements of R, $U(R)$ its unit group. The prime radical, the Levitzki radical, the upper nil-radical, the Jacobson radical, and the Brown–McCoy radical of a ring R are denoted by $P(R)$, $L(R)$, $N^*(R)$, $J(R)$, and $BM(R)$, respectively. The symbol $M_n(R)$ denotes the ring of $n \times n$ matrices over a ring R, $T_n(R)$ denotes the ring of $n \times n$ upper triangular matrices over R, $D_n(R)$ denotes the subring $\{A \in T_n(R)|$ the diagonal entries of A are all equal$\}$ of $T_n(R)$, and $V_n(R)$ denotes the ring of all matrices (a_{ij}) in $D_n(R)$ such that $a_{ij} = a_{(i+1)(j+1)}$ for $i = 1, 2, \cdots, n-2$ and $j = 1, 2, \cdots, n-1$.

In recent years, a growing number of articles have studied the class of rings that is associated with the set $N(R)$ of all nilpotent elements of a ring R. In 1973, Shin [21] proved that the prime radical $P(R)$ coincides with the set of all nilpotent elements of R if and only if every minimal prime ideal is completely prime. In 1993, the term 2-primal, which satisfies $P(R) = N(R)$, was created by Birkenmeier et al. [5]. At the same time, they also studied some fundamental properties of 2-primal rings and proved that the subring, direct sum of 2-primal rings and the polynomial ring $R[x]$, and the ring of all $n \times n$ upper triangular matrices $T_n(R)$ over a 2-primal ring R are also 2-primal. Since then, many papers have further researched 2-primal rings. For example, Marks [16] investigated conditions on ideals of a 2-primal ring R that will ensure that the skew polynomial ring $R[x;\alpha]$ and the differential polynomial $R[x;\delta]$ be 2-primal.

In 2001, Marks [17] proposed NI rings. A ring R is called NI if the upper nilradical $N^*(R)$ coincides with

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2010 *AMS Mathematics Subject Classification:* 16N20; 16W80

the set of all nilpotent elements of R. Note that a ring R is NI if and only if $N(R)$ forms an ideal. Obviously, every 2-primal ring is NI, but the converse is negative by Example 1.2 of [12]. Hwang et al. [12] studied the basic structure of NI rings and showed that R is NI if and only if every subring (possibly without identity) of R is NI if and only if every minimal strongly prime ideal is completely prime if and only if $R/N^*(R)$ is reduced. They also proved that the direct sum and the direct limit of a direct system of NI rings are also NI. In addition, they investigated topological conditions for NI rings relating to the space $SSpec(R)$ of strongly prime ideals of R and proved for an NI ring R that R is *weakly pm* if and only if $Max(R)$ is a retract of $SSpec(R)$ if and only if $SSpec(R)$ is normal, where $Max(R)$ is the space of maximal ideals of R.

More generally, in 2011, Chen et al. [7] called a ring R weakly 2-primal if the set of nilpotent elements in R coincided with its Levitzki radical $L(R)$. By the definition, we have the following implication: 2-primal $\Rightarrow$ weakly 2-primal $\Rightarrow$ NI. The converse is negative by [7]. Meanwhile, they proved that R is weakly 2-primal if and only if $T_n(R)$ is weakly 2-primal and if R is a weakly 2-primal α-compatible ring, then $R[x;\alpha]$ is weakly 2-primal. Moreover, Wang et al. [24] further showed that if R is $(\alpha,\ \delta)$-compatible, then R is weakly 2-primal if and only if the Ore extension $R[x;\alpha,\delta]$ is weakly 2-primal.

Naturally, in this paper, we will study the relationship between the Jacobson radical $J(R)$ and the set of nilpotent elements of a ring R and put forward the concept of NJ rings. A ring R is called NJ if the Jacobson radical coincides with the set of all nilpotent elements, that is, $J(R) = N(R)$. First, we research the relationship between 2-primal rings, weakly 2-primal rings, NI rings, and NJ rings and we also give some examples of NJ rings. Second, we study some fundamental properties of NJ rings and prove that, for some conditions, the trivial extension, the direct product, the Dorroh extension, the Nagata extension, the Jordan extension, and the direct limit are also NJ. Third, we investigate the polynomial extension of NJ rings and show that if R is weakly 2-primal α-compatible with an automorphism α of R, then $R[x;\alpha]$ is NJ; if R is weakly 2-primal δ-compatible or δ-Armendariz NJ with a derivation δ of R, then $R[x;\delta]$ is NJ. Lastly, we apply the topological methods to study some topological conditions for NJ rings and prove for an NJ ring R that R is *J-pm* if and only if $J\text{-}Spec(R)$ is a normal space if and only if $Max(R)$ is a retract of $J\text{-}Spec(R)$.

2. Some properties of NJ-rings

Definition 2.1 *A ring R is called an NJ ring if the set of nilpotent elements in R coincides with its Jacobson radical, that is, $N(R) = J(R)$.*

Clearly, every NJ ring is NI, but the converse is negative by the following example.

Example 2.2 *Let $R = \mathbb{Z}[[x]]$. Then R is a domain (hence NI) and $N(R) = 0$, but $J(\mathbb{Z}[[x]]) = x\mathbb{Z}[[x]] \neq 0$ and so R is not an NJ ring.*

Proposition 2.3 *(1) A ring R is NJ if and only if R is NI and $J(R)$ is nil.*

(2) A ring R is an NJ ring if and only if R is an NI ring and $R/N^*(R)$ is J-semisimple.

Proof (1) Suppose that R is NJ. Then $J(R) = N(R)$ is nil and R is NI. Conversely, suppose that R is NI and $J(R)$ is nil. Then we have $N^*(R) = N(R) \subseteq J(R) \subseteq N(R)$ and so R is NJ.

(2) Every NJ ring is NI. Since $N^*(R) \subseteq J(R)$, we get $J(R/N^*(R)) = J(R)/N^*(R)$. On the other hand, $J(R) = N(R) = N^*(R)$ implies $J(R)/N^*(R) = 0$ and so $R/N^*(R)$ is J-semisimple.

Conversely, since $N(R) = N^*(R) \subseteq J(R)$, we have $J(R)/N^*(R) = J(R/N^*(R)) = \bar{0}$ and so $J(R) = N^*(R) = N(R)$. It is proved that R is an NJ ring. □

Example 2.2 also proves that a 2-primal ring (hence weakly 2-primal) is not NJ. Meanwhile, we can also find an NJ ring that it is not 2-primal by Example 1.7 of [14].

Proposition 2.4 *If R is an NJ ring and satisfies a polynomial identity, then R is weakly 2-primal.*

Proof Since R is an NJ ring, we have $L(R) \subseteq J(R) = N(R)$. As it is well known that if a ring R satisfies a polynomial identity, then every nil ideal of R is locally nilpotent. Hence, $N(R) \subseteq L(R)$, completed. □

In the following, we supply several examples of NJ rings. We can see that NJ rings are abundant.

Example 2.5 *(1) Recall that a ring R (without 1) is called nil if every element in R is nilpotent, i.e. $R = N(R)$. Every nil ring is NJ.*

(2) Every division ring is an NJ ring. In fact, $N(R) = J(R) = 0$.

(3) Every Boolean ring is an NJ ring. Note that the Jacobson radical of a ring R contains no idempotent elements except for 0. Then the Jacobson radical of the Boolean ring R is 0. On the other hand, for every $a \in N(R)$, there exists a positive integer n such that $a^n = 0$. Since R is a Boolean ring, we have $a^2 = a$ and so $N(R) = 0$.

(4) Recall that a ring R is a Jacobson ring if every prime ideal is an intersection of primitive ideals. Every commutative Jacobson ring is NJ. Let $R \subseteq A$ be a commutative ring such that A is finitely generated as an R-algebra and R is a Jacobson ring. Then, by Corollary 5.4 of [15], A is also a Jacobson ring. In particular, A is an NJ ring.

(5) Let R be a commutative affine algebra over a field K. By ([15], p60), the Jacobson radical of R is exactly the set of nilpotent elements in R. Thus, R is an NJ ring.

(6) Every semi-Abelian π-regular ring is NJ by Corollary 3.13 of [6].

(7) Every locally finite Abelian ring is NJ by Proposition 2.5 of [11].

Considering Example 2.5 (1, 2, 3), one may naturally ask whether the converse of Example 2.5 (1, 2, 3) also holds. The answer is negative, as can be seen by $R = \mathbb{Z}$. In fact, $J(R) = N(R) = 0$ and so R is an NJ ring, but R is not a division ring. At the same time, R is also not a nil-ring and a Boolean ring.

As is well known, every division ring is local, and every division ring is NJ by Example 2.5 (2), so it is natural to ask whether local rings are related to NJ rings. However, there is no implication between the classes of local rings and NJ rings by the following.

Example 2.6 *(1) Let $R = \mathbb{Z}$. Obviously, R is an NJ ring, but R is not local.*

(2) If R is a commutative local domain, then $N(R) = 0$, but $J(R)$ is the unique maximal ideal of R, which is nonzero if R is not a field in ([15, p 60]).

Proposition 2.7 *Supposing that $R \neq 0$, and every $a \notin U(R)$ is nilpotent, then R is a NJ ring, where $U(R)$ is the unit group of R.*

Proof By hypothesis, we have $R \backslash U(R) = N(R)$. The following is similar to the proof of ([15], Proposition 19.3). If $a \notin N(R) = R \backslash U(R)$, then $a \in U(R)$ and there exists $r \in R$ such that $ar = ra = 1$. Thus,

$1-ra=0 \notin U(R)$ and so $a \notin J(R)$. It is implied that $J(R) \subseteq R \backslash U(R) = N(R)$. For the converse inclusion, if $a \in N(R) = R\backslash U(R)$, then $a \notin U(R)$. Let k be the smallest positive integer such that $a^k = 0$. We claim that $Ra \subseteq R\backslash U(R) = N(R)$. Assume that there exists some $ra \in Ra$ such that $ra \in U(R)$. Then $(ra)a^{k-1} = 0$ implies $a^{k-1} = 0$. This is a contradiction. Thus, Ra is a nil left ideal, and so $a \in Ra \subseteq J(R)$. This implies $N(R) \subseteq J(R)$, and R is an NJ ring. □

Recall that a ring R is called reduced if there are no nonzero nilpotent elements in R. Note that a division ring is a domain and a domain is reduced. However, there is no relationship between the classes of reduced rings (domains) and NJ rings by the following examples.

Example 2.8 *Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then $J(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} = N(R)$ and so R is an NJ ring. However, R is not reduced (hence, R is not a domain). In fact, take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; then $A^2 = 0$, but $A \neq 0$.*

Proposition 2.9 *If R is a reduced ring and $J(R)$ is nil, then R is an NJ ring.*

Proof Since R is reduced, we have $N(R) = 0$. On the other hand, $J(R)$ is nil, that is, $J(R) \subseteq N(R) = 0$. Hence, $J(R) = N(R) = 0$ and so R is an NJ ring. □

Recall that a ring R is feckly reduced if $R/J(R)$ is reduced. A ring R is feckly Armendariz if $R/J(R)$ is Armendariz. Obviously, every feckly reduced ring is feckly Armendariz. A ring R is directly finite if $ab = 1$ implies $ba = 1$ for $a, b \in R$. By Proposition 2.6 (2) of [14], every feckly Armendariz ring is directly finite.

Proposition 2.10 *Every NJ ring is feckly reduced. Hence, every NJ ring is directly finite. Conversely, if R is a feckly reduced and $J(R)$ is nil, then R is NJ.*

Proof Let R be an NJ ring. If $\bar{r}^2 = 0$ for every $\bar{r} \in R/J(R)$, then $r^2 \in J(R)$. We have $r \in N(R) = J(R)$ since R is NJ. Hence, $\bar{r} = 0$ and so R is a feckly reduced ring. Conversely, it is easy to see that $N(R) \subseteq J(R)$ since R is feckly reduced. Therefore, R is NJ by $J(R) \subseteq N(R)$. □

However, the converse of Proposition 2.10 is not correct by Example 2.2 (1). In fact, $R/J(R) \cong \mathbb{Z}$ is a reduced ring. Recall that a ring R is regular if for every $a \in R$ there exists $x \in R$ such that $a = axa$. Note that $J(R) = 0$ if R is a regular ring. Moreover, we can see that reduced, regular, and NJ are mutually independent by the following examples and Example 2.2. Let $R = \mathbb{Z}$. Then R is reduced and R is also NJ, but R is not regular. If $R = M_2(F)$ and F is a field, we have that R is a regular ring since F is regular. However, R is not reduced. At the same time, R is not NJ since $J(R) = 0 \neq N(R)$. This means that there exists a regular ring such that it is not NJ.

Proposition 2.11 *(1) Every reduced regular ring is NJ.*
(2) If R is a regular ring, then we have the following equivalent:

(i) R is reduced;

(ii) R is NJ;

(iii) R is NI.

Proposition 2.12 *(1) Let R be a ring. Then we have the following equivalent:*

(i) A ring R is NJ;

(ii) Every ideal I of R is an NJ ring;

(iii) R is an NI ring and every proper ideal of R is NJ.

(2) Let $I \subseteq J(R)$ be an ideal of R. If R is NJ, then R/I is NJ.

(3) Let I be a nil ideal of R. If R/I is an NJ ring, then R is NJ.

(4) Let I be an ideal of R and I NJ as a ring. If $a \in N(R)$, then ab, $ba \in N(R)$ for any $b \in I$.

(5) Let I be an ideal of R. Then the following are equivalent:

(i) R/I is an NJ ring;

(ii) $x^n \in I$ if and only if $x \in p(I)$, that is, $p(I) = \{x \in R | x^n \in I \text{ for some } n \in \mathbb{N}\}$, where $p(I) = \cap\{J | J \text{ is a maximal right ideal of } R \text{ containing } I\}$.

Proof (1) (i) $\Leftrightarrow$ (ii) and (i) $\Rightarrow$ (iii) are clear by $J(I) = J(R) \cap I = N(R) \cap I = N(R)$. For (iii) $\Rightarrow$ (i), we only need to show $J(R) \subseteq N(R)$ since R is NI. If $a \in J(R)$, then we have $J(RaR) = N(RaR)$ by the hypothesis. On the other hand, since $a \in RaR$ and $a \in J(R)$, we can obtain $a \in J(R) \cap RaR = J(RaR)$ and so $a \in N(RaR) = N(R) \cap RaR$. This implies $a \in N(R)$ and R is NJ.

(2) Since $I \subseteq J(R)$, we have $J(R/I) = J(R)/I$. If $\bar{r} \in N(R/I)$, then there exists a positive integer n such that $\bar{r}^n = \bar{0}$ and so $r^n \in I \subseteq J(R) = N(R)$. Hence, $r \in N(R) = J(R)$ and $\bar{r} \in J(R)/I = J(R/I)$. For the converse inclusion, if $\bar{a} \in J(R/I) = J(R)/I$, then $a \in J(R) = N(R)$ and so $\bar{a} \in N(R/I)$. This implies that R/I is an NJ ring.

(3) If $r \in N(R)$, then there exists a positive integer n such that $r^n = 0$. Thus, we have $\bar{r}^n = \bar{0}$ in R/I and so $\bar{r} \in N(R/I) = J(R/I)$. By hypothesis, we have $I \subseteq J(R)$ and $\bar{r} \in J(R/I) = J(R)/I$. Hence, $r \in J(R)$ and $N(R) \subseteq J(R)$. On the other hand, if $a \in J(R)$, then $\bar{r} \in J(R)/I = J(R/I) = N(R/I)$. Thus, there exists a positive integer m such that $\bar{r}^m = \bar{0}$ and so $r^m \in I \subseteq N(R)$. Hence, $r \in N(R)$ and $J(R) \subseteq N(R)$, as desired.

(4) Suppose $a^n = 0$. We will show that $a^{n-k}b \in N(R)$ for every $0 \leqslant k \leqslant n-1$, by induction on k. Then the $k = n-1$ case will complete the proof. When $k = 0$, $a^n b = 0 \in N(R)$. Assume that $a^{n-k}b \in N(R)$ with $0 \leqslant k < n-1$. Then there exists a positive integer m such that $(a^{n-k}b)^m = 0$ and so $(a^{n-k-1}ba)^{m+1} = 0$. Thus, we have $a^{n-k-1}ba \in N(R) \cap I = N(I) = J(I)$ since I is NJ, and so $a^{n-k-1}ba^{n-k-1}b \in J(I) = N(I)$. Therefore, $a^{n-k-1}b$ is nilpotent and the induction goes through.

(5) (i)$\Rightarrow$ (ii) For any $y \in \{x \in R | x^n \in I \ for\ some\ n \in \mathbb{N}\}$, then there exists a positive integer n such that $y^n \in I$. In $\bar{R} = R/I$, we have $\bar{y}^n = \bar{0}$ and so $\bar{y} \in N(R/I) = J(R/I)$. Note that J/I is a maximal right ideal of R/I if and only if J is a maximal right ideal of R containing I. This implies $\bar{y} \in J/I$ and $y \in J$. Thus, $x \in p(I)$. Let $x \in p(I)$. Then, for every maximal right ideal J of R containing I, we have $x \in J$ and so $\bar{x} \in J/I$. That is, $\bar{x} \in J(R/I) = N(R/I)$. There exists $n \in \mathbb{N}$ such that $x^n \in I$.

(ii) $\Rightarrow$ (i) For any $\bar{r} \in N(R/I)$, there exists $n \in \mathbb{N}$ such that $r^n \in I$ and so $r \in p(I)$. By the above proof, we can obtain $\bar{r} \in J(R/I)$. Conversely, if $\bar{a} \in J(R/I)$, then $a \in p(I)$. Thus, we have $a^n \in I$ for some n. Therefore, $\bar{a} \in N(R/I)$. □

Considering Proposition 2.12 (2)(3), if R/I is an NJ ring and I is also NJ as a ring without 1, then R is also NJ. However, the following example gives a negative answer.

Example 2.13 *Let F be a field and $R = M_2(F)$. Then $J(R) = M_2(J(F)) = 0$, but $N(R) \neq 0$. Thus, R is not NJ. Take $I = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$. Obviously, we have $J(I) = N(I) = 0$ and so I is NJ as a ring. On the other hand, $R/I \cong F$ and we can obtain $J(R/I) = N(R/I) = 0$. Hence, R/I is NJ.*

Proposition 2.14 *Let e be an idempotent element of R. If R is an NJ ring, then eRe is also NJ.*

Proof Notice that $J(eRe) = J(R) \cap eRe = N(R) \cap eRe = N(eRe)$. □

Corollary 2.15 *If there exists a ring R such that $M_n(R)$ is an NJ ring for all $n \geq 2$, then R is NJ.*

Proof Since $E_{11}M_n(R)E_{11} = RE_{11} \cong R$, we have that R is NJ by Proposition 2.14. □

The index of nilpotency of a nilpotent element a in a ring R is the least positive integer n such that $a^n = 0$. The index of nilpotency of a subset X of R is the supremum of the indices of nilpotency of all nilpotent elements in X. If such a supremum is finite, then X is said to be of bounded index of nilpotency.

Proposition 2.16 *(1) Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a direct sum of rings R_γ and Γ an indexed set. Then R is an NJ ring if and only if R_γ is an NJ ring for every $\gamma \in \Gamma$.*

(2) Let $R = \prod_{\gamma\in\Gamma} R_\gamma$ be a direct product of rings R_γ and Γ an indexed set. If R is of bounded index of nilpotency, then R is an NJ ring if and only if R_γ is a NJ ring for every $\gamma \in \Gamma$.

Proof (1) Assume that R_γ is an NJ ring for each $\gamma \in \Gamma$. It comes from $J(\bigoplus_{\gamma\in\Gamma} R_\gamma) = \bigoplus_{\gamma\in\Gamma} J(R_\gamma) = \bigoplus_{\gamma\in\Gamma} N(R_\gamma) = N(\bigoplus_{\gamma\in\Gamma} R_\gamma)$. Conversely, suppose that R is an NJ ring. If $a_\gamma \in J(R_\gamma)$ for every $\gamma \in \Gamma$, then $(0, \ldots, a_\gamma, 0, \ldots) \in \bigoplus_{\gamma\in\Gamma} J(R_\gamma) = J(\bigoplus_{\gamma\in\Gamma} R_\gamma) = N(\bigoplus_{\gamma\in\Gamma} R_\gamma)$ and so $a_\gamma \in N(R_\gamma)$. On the other hand, if $a_\gamma \in N(R_\gamma)$, then $(0, \ldots, a_\gamma, 0, \ldots) \in N(\bigoplus_{\gamma\in\Gamma} R_\gamma) = J(\bigoplus_{\gamma\in\Gamma} R_\gamma) = \bigoplus_{\gamma\in\Gamma} J(R_\gamma)$. Hence, $a_\gamma \in J(R_\gamma)$. This completes the proof.

(2) Since R is of bounded index of nilpotency, we have $N(\prod_{\gamma\in\Gamma} R_\gamma) = \prod_{\gamma\in\Gamma} N(R_\gamma)$. This implies that $J(\prod_{\gamma\in\Gamma} R_\gamma) = \prod_{\gamma\in\Gamma} J(R_\gamma) = \prod_{\gamma\in\Gamma} N(R_\gamma) = N(\prod_{\gamma\in\Gamma} R_\gamma)$. Hence, R is an NJ ring. For the converse, the proof is similar to the proof of (1). □

Corollary 2.17 *For a central idempotent element $e \in R$, eR and $(1-e)R$ are NJ if and only if R is NJ.*

Proof If R is an NJ ring, then eR and $(1-e)R$ are NJ since e is central. Conversely, since $R = eR \bigoplus (1-e)R$, it comes from Proposition 2.16 (1). □

Proposition 2.18 *Let R be a ring. Then the following are equivalent:*

(1) R is an NJ ring;

(2) $T_n(R)$ is an NJ ring for all $n \geq 2$;

(3) $D_n(R)$ *is an NJ ring for all* $n \geq 2$*;*

(4) $V_n(R)$ *is an NJ ring for all* $n \geq 2$*.*

Proof We apply the method in the proof of Proposition 2.5 in [14] to prove this proposition.

(1) $\Leftrightarrow$ (2) Let $I = \{A \in T_n(R) \mid \text{each diagonal entry of } A \text{ is zero}\}$. Then I is a nil ideal of $T_n(R)$ and $I \subseteq J(T_n(R))$. On the other hand, we can obtain $T_n(R)/I \cong \bigoplus_{i=1}^{n} R_i$ where $R_i = R$. The proof is completed by Proposition 2.12 (3) and Proposition 2.16 (1).

(1) $\Leftrightarrow$ (3) It is similar to the above proof. Let $I = \{B \in D_n(R) \mid \text{each diagonal entry of } B \text{ is zero}\}$. Then I is a nil ideal of $D_n(R)$ and $D_n(R)/I \cong R$.

(1) $\Leftrightarrow$ (4) Let $I = \{C \in V_n(R) \mid \text{each diagonal entry of } C \text{ is zero}\}$. Then I is a nil ideal of $V_n(R)$ and $V_n(R)/I \cong R$. □

Considering Corollary 2.15 and Proposition 2.18, it is natural to ask whether $M_n(R)$ is also NJ when R is an NJ ring for all $n \geq 2$. However, the answer is negative by the following.

Example 2.19 *Let* $R = \mathbb{Z}_2$ *and* $S = M_2(\mathbb{Z}_2)$*. Clearly,* R *is an NJ ring from* $J(R) = N(R) = 0$*. On the other hand,* $J(S) = M_2(J(\mathbb{Z}_2)) = 0$*, but* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(S) \neq 0 = J(S)$*. Therefore,* S *is not an NJ ring.*

Recall that for a ring R and an (R,R)-bimodule M, the trivial extension of R by M is the ring $T(R,M) = R \bigoplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$ where $r \in R$, $m \in M$ and the usual matrix operations are used.

Corollary 2.20 *(1) Let* $R = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$*, where* S *and* T *are rings and* M *is an* (S,T)*-bimodule. Then* S *and* T *are both NJ if and only if* R *is NJ.*

(2) Let R *be a ring and* M *an* (R,R)*-bimodule. Then the trivial extension* $T(R,M)$ *is an NJ ring if and only if* R *is an NJ ring.*

3. Extensions of NJ rings

In this section, we focus on some extensions of NJ rings, such as the Dorroh extension, the ideal-extension, the Nagata extension, the Jordan extension, and so on.

Let R_0 be an algebra with identity over a commutative ring S. Due to [8], the Dorroh extension of R_0 by S is the Abelian group $S \bigoplus R_0$ with multiplication given by $(s_1, r_1)(s_2, r_2) = (s_1s_2, s_1r_2 + s_2r_1 + r_1r_2)$ for $s_i \in S$, $r_i \in R_0$.

Proposition 3.1 *Let* R_0 *be an algebra with identity over a field* F*. Then* R_0 *is an NJ ring if and only if the Dorroh extension* R *of* R_0 *by* F *is an NJ ring.*

Proof By the proof of Proposition 2.7 in [14], we obtain that $J(R) = 0 \bigoplus J(R_0)$. On the other hand, since F is a field, we have $N(R) = 0 \bigoplus N(R_0)$. The proof is completed by the condition. □

Let R be a ring and V an (R,R)-bimodule that is a general ring (possibly with no unity) in which $(vw)r = v(wr)$, $(vr)w = v(rw)$ and $(rv)w = r(vw)$ hold for all $v, w \in V$ and $r \in R$. Then ideal-extension $I(R;V)$ of R by V is defined to be the additive Abelian group $I(R;V) = R \bigoplus V$ with multiplication $(r,v)(s,w) = (rs, rw + vs + vw)$.

Proposition 3.2 *Suppose that for any $v \in V$, there exists $w \in V$ such that $v + w + vw = 0$. Then we have the following:*

(1) If an ideal-extension $S = I(R;V)$ is an NJ ring, then R and V are both NJ;

(2) If R is a reduced NJ ring and V is NJ, then the ideal-extension $S = I(R;V)$ is NJ.

Proof (1) By hypothesis, we can obtain $V = J(V)$ and $(0,V) \subseteq J(S)$. If $v \in J(V) = V$, then $(0,v) \in J(S) = N(S)$ and so there exists a positive integer n such that $(0,v)^n = (0,0)$. Thus, we have $v^n = 0$ and $v \in N(V)$. Therefore, V is an NJ ring. If $a \in J(R)$, then we have $(a,0) \in S$. Next we claim that $(a,0) \in J(S)$. For any $(r,v) \in S$, we have $(1,0) - (a,0)(r,v) = (1-ar, -av) = (1-ar, 0)(1, (1-ar)^{-1}(-av))$. Since $(1, (1-ar)^{-1}(-av)) = (1,0) + (0, (1-ar)^{-1}(-av)) \in U(S)$ by $(0,v) \subseteq J(V)$ and $a \in J(R)$, we can obtain $(1,0) - (a,0)(r,v) \in U(S)$ and so $(a,0) \in J(S) = N(S)$. Then there exists a positive integer n such that $(a,0)^n = (0,0)$. Thus, we have $a^n = 0$ and $a \in N(R)$. If $a \in N(R)$, then $(a,0) \in N(S) = J(S)$. For any $r \in R$, we have $(1,0) - (a,0)(r,0) = (1-ar, 0) \in U(S)$ and so $1 - ar \in U(R)$. Thus, $a \in J(R)$ and so R is an NJ ring.

(2) If $(a,v) \in J(S)$, then we have $(a,0) \in J(S)$ since $(0,v) \in J(S)$ and $(a,v) = (a,0) + (0,v)$. For any $r \in R$, $(1,0) - (a,0)(r,0) = (1-ar, 0) \in U(S)$ and so $1 - ar \in U(R)$. Thus, $a \in J(R) = N(R) = 0$. This implies that $(a,v) = (0,v)$. Therefore, we have $v \in V = J(V) = N(V)$ and so $(a,v) \in N(S)$. For the converse, if $(a,v) \in N(S)$, then there exists a positive integer n such that $(a,v)^n = (0,0)$. Thus, we have $a^n = 0$ and so $s = 0$ by the multiplication and reduced property. This implies that $(a,v) = (0,v) \in J(S)$. Therefore, the ideal-extension $S = I(R;V)$ is NJ. □

Due to Nagata [19], let R be a commutative ring, M be a left R-module, and σ be an endomorphism of R. Given $R \bigoplus M$ a (possibly noncommutative) ring structure with multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, \sigma(r_1) m_2 + r_2 m_1)$, where $r_i \in R$ and $m_i \in M$, we call this extension the Nagata extension of R by M and σ.

Proposition 3.3 *Let R be a commutative ring and σ an endomorphism of R. Then R is an NJ ring if and only if the Nagata extension S of R by R and σ is also NJ.*

Proof In order to prove this proposition, we only need to show that $N(S) = N(R) \bigoplus R$ and $J(S) = J(R) \bigoplus R$. If $(a,b) \in N(S)$, then there exists a positive integer n such that $(a,b)^n = (0,0)$ and so $a^n = 0$. This implies that $(a,b) \in N(R) \bigoplus R$. If $(a,b) \in N(R) \bigoplus R$, then there is a positive integer m such that $a^m = 0$. Moreover, we can imply that $[(a,b)^m]^2 = (a^m, *)^2 = (0, *)^2 = (0,0)$ and so $(a,b) \in N(S)$. Therefore, it is implied that $N(S) = N(R) \bigoplus R$. If $(a,b) \in J(S)$, then we have $(1,0) - (a,b)(s,r) = (1 - as, -\sigma(a)r - sb) \in U(S)$ for any $(s,r) \in S$. By the multiplication, this implies that $1 - as$ is invertible in R. That is, we have $a \in J(R)$ and so $(a,b) \in J(R) \bigoplus R$. If $(a,b) \in J(R) \bigoplus R$, then we have $1 - ax \in U(R)$ for any $x \in R$. For any $(s,r) \in S$, we consider $(1,0) - (a,b)(s,r) = (1 - as, -\sigma(a)r - sb)$. Since R is commutative, we can find $y = \sigma[(1-as)^{-1}] \cdot (1-as)^{-1} \cdot [\sigma(a)r + sb]$ such that $((1-as)^{-1}, y)(1-as, -\sigma(a)r - sb) = (1-as, -\sigma(a)r - sb)((1-as)^{-1}, y) = (1,0)$. Thus, $(a,b) \in J(S)$, completed. □

Let α be an injective homomorphism of a ring R and A an extension ring of R. If α can be extended to an isomorphism of A and $A = \bigcup_{n=0}^{\infty} \alpha^{-n}(R)$, then we call this extension ring A the Jordan extension of R.

Proposition 3.4 *Let α be an isomorphism of R. If R is an NJ ring, then the Jordan extension A of R is also NJ.*

Proof If $a \in N(A)$, then there exists a positive integer n such that $a^n = 0$ and there is an integer $m \geq 0$ such that $\alpha^m(a) \in R$. Then we have $[\alpha^m(a)]^n = \alpha^m(a^n) = 0$ and so $\alpha^m(a) \in N(R) = J(R)$ since R is NJ. For any $r \in A$, there exists an integer $k \geq 0$ such that $\alpha^k(r) \in R$. Taking $l = m + k$, it is well known that $\alpha(J(R)) \subseteq J(R)$ since α is an isomorphism of R. This implies $\alpha^l(a) \in J(R)$ and $\alpha^l(r) \in R$. Then we can obtain that there exists $b \in R$ such that $\alpha^l(a)\alpha^l(r) + b + \alpha^l(a)\alpha^l(r)b = 0$ and so $\alpha^l(ar + \alpha^{-l}(b) + ar\alpha^{-l}(b)) = 0$. Since α is an isomorphism, we have $ar + \alpha^{-l}(b) + ar\alpha^{-l}(b) = 0$. Therefore, it is implied that $a \in J(A)$ and $N(A) \subseteq J(A)$. If $a \in J(A)$, then there exists an integer $n \geq 0$ such that $\alpha^n(a) \in R$. For any $r \in R$, $a\alpha^{-n}(r) \in J(A)$ and there is $b \in A$ such that $a\alpha^{-n}(r) + b + a\alpha^{-n}(r)b = 0$. We can find an integer $m \geq 0$ such that $\alpha^m(b) \in R$ for $b \in A$. Thus, we have $\alpha^n(a\alpha^{-n}(r) + b + a\alpha^{-n}(r)b) = 0$ and so $\alpha^n(a)r + \alpha^n(b) + \alpha^n(a)r\alpha^n(b) = 0$. This implies $\alpha^n(a) \in J(R) = N(R)$ since $\alpha^n(b) = \alpha^{n-m}(\alpha^m(b)) \in R$. Hence, there exists a positive integer k such that $\alpha^n(a^k) = [\alpha^n(a)]^k = 0$ and so $a^k = 0$. This implies $a \in N(A)$ and $J(A) \subseteq N(A)$. Therefore, we show that A is NJ. □

Let D be a ring and C a subring of D with $1_D \in C$ and $R[D,C]$ denotes the set $\{(a_1, \ldots, a_n, b, b, \ldots) | a_i \in D, b \in C, n \geq 1, 1 \leq i \leq n\}$. Then $R[D,C]$ is a ring under the componentwise addition and multiplication. It is well known that $J(R[D,C]) = R[J(D), J(D) \cap J(C)]$.

Proposition 3.5 *Let D be a ring and C a subring of D with $1_D \in C$. Then $R[D,C]$ is an NJ ring if and only if D and C are both NJ.*

Proof Assume that D and C are both NJ. It is sufficient that $N(R[D,C]) = R[N(D), N(C)]$. In fact, if $(d_1, d_2, \ldots, d_n, c, c, \ldots) \in N(R[D,C])$, then there exists a positive integer k such that $(d_1, d_2, \ldots, d_n, c, c, \ldots)^k = 0$. Thus, for every $1 \leq i \leq n$, we have $d_i^k = 0$ and $c^k = 0$. That is, $d_i \in N(D)$ and $c \in N(C)$. Conversely, if $(d_1, d_2, \ldots, d_n, c, c, \ldots) \in R[N(D), N(C)]$, then we always find a positive integer m such that $d_i^m = c^m = 0$ for $1 \leq i \leq n$. Thus, we have $(d_1, d_2, \ldots, d_n, c, c, \ldots) \in N(R[D,C])$. By the hypothesis, we can obtain $J(R[D,C]) = R[J(D), J(D) \cap J(C)] = R[N(D), N(C)] = N(R[D,C])$, completely.

Suppose that $R[D,C]$ is an NJ ring. Then we have $R[J(D), J(D) \cap J(C)] = J(R[D,C]) = N(R[D,C]) = R[N(D), N(C)]$. This implies that $J(D) = N(D)$ and $N(C) = J(D) \cap J(C)$. Thus, we can obtain that D is an NJ ring and $N(C) \subseteq J(C)$. If $c \in J(C)$, then $1 - cr \in U(C)$ for every $r \in C$. For every $(d_1, d_2, \ldots, d_n, r, r, \ldots) \in R[D,C]$, we have $(1, \ldots, 1, 1, \ldots) - (0, \ldots, 0, c, c, \ldots)(d_1, d_2, \ldots, d_n, r, r, \ldots) = (1, \ldots, 1, 1-cr, 1-cr, \ldots) \in U(R[D,C])$. It is implied that $(0, \ldots, 0, c, c, \ldots) \in J(R[D,C]) = R[J(D), J(D) \cap J(C)]$ and so $c \in J(D) \cap J(C) = N(C)$. Moreover, we can get $N(C) = J(C)$ and so c is also an NJ ring. □

Proposition 3.6 *Let $(I, \leqslant)$ be a strictly ordered set and $\{A_\alpha | \alpha \in I\}$ a family of NJ rings. Suppose that*

$(A_\alpha, (\varphi_{\alpha\beta})_{\alpha\leqslant\beta})$ is a direct limit system over I and $(A, (\eta_\alpha)_{\alpha\in I})$ is a direct limit of the direct system. If $\varphi_{\alpha\beta}:\ A_\alpha \to A_\beta$ is an isomorphism for all $\alpha \leqslant \beta$ and $\eta_\alpha:\ A_\alpha \to A$ is a monomorphism for all $\alpha \in I$, then the direct limit $A = \lim\limits_{\rightarrow} A_\alpha$ is also NJ.

Proof Note that since A is a direct limit of $\{A_\alpha | \alpha \in I\}$, we have $A = \bigcup\limits_{\alpha\in I} Im\eta_\alpha$. Let $a \in N(A) \subseteq A = \bigcup\limits_{\alpha\in I} Im\eta_\alpha$. Then there exists a positive integer n such that $a^n = 0$ and there is $a_\alpha \in A_\alpha$ such that $\eta_\alpha(a_\alpha) = a$. Hence, we can obtain $0 = a^n = (\eta_\alpha(a_\alpha))^n = \eta_\alpha(a_\alpha^n)$. Since η_α is a monomorphism, we have $a_\alpha^n = 0$ and so $a_\alpha \in N(A_\alpha) = J(A_\alpha)$. For any $r \in A$, there exists $r_\beta \in A_\beta$ such that $\eta_\beta(r_\beta) = r$. Next we claim that ar is a quasiregular element in A. For α and β, we can find $k \in I$ such that $\alpha \leqslant k$ and $\beta \leqslant k$ because I is a direct set. Then a and r can be expressed by $a = \eta_\alpha(a_\alpha) = \eta_k\varphi_{\alpha k}(a_\alpha)$ and $r = \eta_\beta(r_\beta) = \eta_k\varphi_{\beta k}(r_\beta)$. Since $a_\alpha \in N(A_\alpha)$ and η_k is monomorphism, we can imply $\varphi_{\alpha k}(a_\alpha) \in N(A_k) = J(A_k)$ and $\varphi_{\beta k}(r_\beta) \in A_k$. Thus, $\varphi_{\alpha k}(a_\alpha)\varphi_{\beta k}(r_\beta)$ is a quasiregular element in A_k. That is, there exists $b_k \in A_k$ such that $\varphi_{\alpha k}(a_\alpha)\varphi_{\beta k}(r_\beta) + b_k + \varphi_{\alpha k}(a_\alpha)\varphi_{\beta k}(r_\beta)b_k = 0$. Moreover, $\eta_k\varphi_{\alpha k}(a_\alpha)\eta_k\varphi_{\beta k}(r_\beta) + \eta_k(b_k) + \eta_k\varphi_{\alpha k}(a_\alpha)\eta_k\varphi_{\beta k}(r_\beta)\eta_k(b_k) = 0$ and so $ar + \eta_k(b_k) + ar\eta_k(b_k) = 0$. This implies that ar is quasiregular in A and so $a \in J(A)$. It is proved that $N(A) \subseteq J(A)$.

For the converse inclusion, suppose $a \in J(A)$. Then there exists $a_\alpha \in A_\alpha$ such that $\eta_\alpha(a_\alpha) = a$. Next we prove $a_\alpha \in J(A_\alpha)$. For any $r_\alpha \in A_\alpha$, we can get that $\eta_\alpha(a_\alpha r_\alpha) = a\eta_\alpha(r_\alpha)$ is quasiregular in A and so $a\eta_\alpha(r_\alpha) + b + a\eta_\alpha(r_\alpha)b = 0$ for some $b \in A$. For the element $b \in A$, we can find $b_\beta \in A_\beta$ such that $\eta_\beta(b_\beta) = b$. Then, by the condition, there is $k \in I$ such that $\alpha \leqslant k$ and $\beta \leqslant k$ and we can obtain $\eta_\alpha(a_\alpha r_\alpha) = \eta_k\varphi_{\alpha k}(a_\alpha r_\alpha)$ and $\eta_\beta(b_\beta) = \eta_k\varphi_{\beta k}(b_\beta)$. Since $\eta_\alpha(a_\alpha r_\alpha) + b + \eta_\alpha(a_\alpha r_\alpha)b = 0$, we have the equation $\eta_k\varphi_{\alpha k}(a_\alpha r_\alpha) + \eta_k\varphi_{\alpha k}(b_\beta) + \eta_k\varphi_{\alpha k}(a_\alpha r_\alpha)\eta_k\varphi_{\alpha k}(b_\beta) = 0$ and $\varphi_{\alpha k}(a_\alpha r_\alpha) + \varphi_{\alpha k}(b_\beta) + \varphi_{\alpha k}(a_\alpha r_\alpha)\varphi_{\alpha k}(b_\beta) = 0$. Because I is a strictly ordered set, we have the following two cases:

If $\beta \leqslant \alpha \leqslant k$, then $\varphi_{\beta k} = \varphi_{\alpha k}\varphi_{\beta\alpha}$. By the fact that $\varphi_{\alpha k}$ is isomorphism, we have $a_\alpha r_\alpha + \varphi_{\beta\alpha}(b_\beta) + a_\alpha r_\alpha\varphi_{\beta\alpha}(b_\beta) = 0$. Hence, $a_\alpha \in J(A_\alpha)$ because of $\varphi_{\beta\alpha}(b_\beta) \in A_\alpha$.

If $\alpha \leqslant \beta \leqslant k$, then $\varphi_{\alpha k} = \varphi_{\beta k}\varphi_{\alpha\beta}$. Similarly, we have $\varphi_{\alpha\beta}(a_\alpha r_\alpha) + b_\beta + \varphi_{\alpha\beta}(a_\alpha r_\alpha)b_\beta = 0$. Since $\varphi_{\alpha\beta}$ is isomorphism, it implies that $a_\alpha r_\alpha + \varphi_{\alpha\beta}^{-1}(b_\beta) + a_\alpha r_\alpha\varphi_{\alpha\beta}^{-1}(b_\beta) = 0$, so $a_\alpha \in J(A_\alpha)$ by $\varphi_{\alpha\beta}^{-1}(b_\beta) \in A_\alpha$.

According to the above cases, we always have $a_\alpha \in J(A_\alpha) = N(A_\alpha)$. Then there exists a positive integer n such that $a_\alpha^n = 0$. It is further implied that $a^n = (\eta_\alpha(a_\alpha))^n = \eta_\alpha(a_\alpha^n) = 0$ and so $a \in N(A)$. This proves that $J(A) \subseteq N(A)$. Therefore, $A = \lim\limits_{\rightarrow} A_\alpha$ is NJ. □

In the following, we study the polynomial extension of NJ rings including the polynomial ring $R[x]$, the skew polynomial ring $R[x;\alpha]$, and the differential polynomial ring $R[x;\delta]$ of a ring R.

Recall that a ring R is Armendariz if $f(x)g(x) = 0$ for any $f(x) = \sum\limits_{i=0}^{n} a_i x^i, g(x) = \sum\limits_{j=0}^{m} b_j x^j \in R[x]$, then $a_i b_j = 0$ for any i, j. Since there is an Armendariz ring such that it is not feckly Armendariz by Example 1.2 of [14], it is also not NJ. Conversely, we can also find an NJ ring such that it is not Armendariz by Example 1.7 of [14]. Therefore, there is no relationship between Armendariz rings and NJ rings.

Lemma 3.7 ([1], Theorem 1) *$J(R[x]) = N[x]$ for a ring R, where $N = J(R[x]) \cap R$ is a nil ideal containing the locally nilpotent radical $L(R)$ of R.*

Proposition 3.8 *(1) If R is a weakly 2-primal ring, then $R[x]$ is NJ.*

(2) Let R be an Armendariz ring. If R is an NJ ring, then $R[x]$ is also NJ.

(3) Let $J(R)$ be a nil ideal. If $R[x]$ is an NJ ring, then R is also NJ.

Proof (1) If R is a weakly 2-primal ring, then $R[x]$ is also weakly 2-primal and $N(R[x]) = N(R)[x]$ by Theorem 3.1 and Corollary 2.3 of [24]. Hence, we have $N(R[x]) = L(R[x]) \subseteq J(R[x]) = (J(R[x]) \cap R)[x] \subseteq N^*(R)[x] = N(R)[x] = N(R[x])$ by Lemma 3.7. Therefore, $R[x]$ is NJ.

(2) Note that if R is an Armendariz ring, then we have $J(R[x]) = N^*(R[x]) = N^*(R)[x]$ by Theorem 1.3 of [14] and $N(R[x]) = N(R)[x]$ by Lemma 2.6 and Lemma 5.1 of [3]. Moreover, we can imply $J(R[x]) = N^*(R[x]) = N^*(R)[x] = N(R)[x] = N(R[x])$ since R is NJ.

(3) Since $R[x]$ is an NJ ring, $R[x]$ is an NI ring. Thus, it implies that R is NJ by Proposition 2.4 (2) of [12]. Therefore, R is NJ by Proposition 2.3. □

Let α be an endomorphism of R. We denote by $R[x;\alpha]$ the skew polynomial ring whose elements are the polynomials over R, the addition is defined as usual, and the multiplication is subject to the reaction $xr = \alpha(r)x$ for any $r \in R$. According to Annin [2], a ring R is said to be α-compatible if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. The work in [18] called a ring R nil-semicommutative if for any $a, b \in nil(R)$, $ab = 0$ implies $aRb = 0$. In [4], an automorphism α of R is said to be of locally finite order if for every $r \in R$ there exists integer $n(r) \geqslant 1$ such that $\alpha^{n(r)}(r) = r$.

Lemma 3.9 *(1)([4], Theorem 3.1) Let R be a ring and α an automorphism of R. Then $J(R[x;\alpha]) = I \cap J(R) + I_0[x;\alpha]$, where $I = \{r \in R | rx \in J(R[x;\alpha])\}$ and $I_0[x;\alpha] = \{\sum_{i\geqslant 1}^{n} r_i x^i | r_i \in I,\ n \in \mathbb{N}\}$.*

(2)([4], Corollary 3.3) If α is an automorphism of R of locally finite order and $J(R)$ is locally nilpotent, then $J(R[x;\alpha]) = J(R)[x;\alpha]$.

Theorem 3.10 *(1) Let R be a ring and α an automorphism of R. If R is a weakly 2-primal α-compatible ring, then $R[x;\alpha]$ is NJ.*

(2) Let α be an automorphism of R of locally finite order and $J(R)$ locally nilpotent. Then we have the following:
(i) If R is a nil-semicommutative α-compatible NJ ring, then $R[x;\alpha]$ is NJ;
(ii) If $R[x;\alpha]$ is NJ, then R is also NJ.

Proof (1) Suppose that R is a weakly 2-primal α-compatible ring. By Theorem 3.1 and Corollary 2.1 of [24], $R[x;\alpha]$ is a weakly 2-primal ring and $N(R[x;\alpha]) = N(R)[x;\alpha]$. Then we have $N(R[x;\alpha]) = L(R[x;\alpha]) \subseteq J(R[x;\alpha])$. According to Lemma 3.9 (1), $J(R[x;\alpha]) = I \cap J(R) + I_0[x;\alpha]$, where $I = \{r \in R | rx \in J(R[x;\alpha])\}$ and $I_0[x;\alpha] = \{\sum_{i\geqslant 1}^{n} r_i x^i | r_i \in I,\ n \in \mathbb{N}\}$, and in the following, we prove that $J(R[x;\alpha]) \subseteq N(R)[x;\alpha] = N(R[x;\alpha])$. For any $f(x) = \sum_{i=0}^{n} a_i x^i \in J(R[x;\alpha])$, then $a_0 \in I \cap J(R)$ and $a_1 x + a_2 x^2 + \cdots + a_n x^n \in I_0[x;\alpha]$. Further, we can imply $a_i \in I$ for all $i \geqslant 0$. For every $i \geqslant 0$, we have $a_i x \in J(R[x;\alpha])$. There exists $g_i(x) = \sum_{j=0}^{m} b_{ij} x^j \in R[x;\alpha]$ such that $a_i x + g_i(x) + a_i x g_i(x) = 0$. We can obtain the following equations:

$b_{i0} = 0$, $a_i + b_{i1} + a_i\alpha(b_{i0}) = 0$, $b_{i2} + a_i\alpha(b_{i1}) = 0$, $b_{i3} + a_i\alpha(b_{i2}) = 0$, $\cdots$, $b_{im} + a_i\alpha(bi(m-1)) = 0$, and $a_i\alpha(b_{im}) = 0$. Because $b_{i0} = 0$, the equation $a_i + b_{i1} + a_i\alpha(b_{i0}) = 0$ can become $b_{i1} = -a_i$. We plug this into the equation $b_{i2} + a_i\alpha(b_{i1}) = 0$, and then it becomes $b_{i2} = a_i\alpha(a_i)$. We also substitute it into the equation $b_{i3} + a_i\alpha(b_{i2}) = 0$, and similarly, it implies $b_{i3} = -a_i\alpha(a_i)\alpha^2(a_i)$. Continuing this progress, for the equation $b_{im} + a_i\alpha(bi(m-1)) = 0$, we have $b_{im} = (-1)^m a_i\alpha(a_i)\alpha^2(a_i)\cdots\alpha^{m-1}(a_i)$. Then the equation $a_i\alpha(b_{im}) = 0$ can become $a_i\alpha(a_i)\alpha^2(a_i)\cdots\alpha^m(a_i) = 0$. Thus, $a_i^{m+1} = 0$ since R is α-compatible. Therefore, $a_i \in N(R)$ and so $f(x) \in N(R)[x;\alpha] = N(R[x;\alpha])$. We prove that $J(R[x;\alpha]) \subseteq N(R[x;\alpha])$ and $R[x;\alpha]$ is NJ.

(2)(i) By Lemma 3.9 (2) and Theorem 2.5 of [23], we have $J(R[x;\alpha]) = J(R)[x;\alpha]$ $= N(R)[x;\alpha] = N(R[x;\alpha])$. Therefore, $R[x;\alpha]$ is NJ.

(ii) Clearly, $J(R) \subseteq N(R)$. For any $a \in N(R)$, then $a \in N(R[x;\alpha])$. Since $N(R[x;\alpha]) = J(R[x;\alpha])$, we have $a \in J(R[x;\alpha]) = J(R)[x;\alpha]$. Hence, $N(R) \subseteq J(R)$. Therefore, R is NJ. $\square$

Let δ be a derivation of R; that is, δ is an additive map such that $\delta(ab) = \delta(a)b + a\delta(b)$, for $a, b \in R$. We denote by $R[x;\delta]$ the differential polynomial ring whose elements are the polynomials over R, the addition is defined as usual, and the multiplication is subject to the reaction $xr = rx + \delta(r)$ for any $r \in R$. According to Annin [2], R is called δ-compatible if for each $a, b \in R$, $ab = 0 \Rightarrow a\delta(b) = 0$. A ring is called *locally finite* if every finite subset in it generates a finite semigroup multiplicatively. In [20], a ring R is δ-Armendariz if for each $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\delta]$, $f(x)g(x) = 0$ implies $a_i b_j = 0$ for each $0 \leqslant i \leqslant n$, $0 \leqslant j \leqslant m$.

Lemma 3.11 ([9], Theorem 3.2) *Let R be a ring and δ a derivation of R. Then $J(R[x;\delta]) = (J(R[x;\delta]) \cap R)[x;\delta]$.*

Theorem 3.12 *Let R be a ring and δ a derivation of R. Then we have the following:*

(1) If R is a weakly 2-primal δ-compatible ring, then $R[x;\delta]$ is NJ;

(2) If R is a locally finite δ-Armendariz ring, then $R[x;\delta]$ is NJ;

(3) If R is a δ-Armendariz NJ ring, then $R[x;\delta]$ is NJ.

Proof (1) Since R is a weakly 2-primal δ-compatible ring, by Theorem 3.1 and Corollary 2.1 of [24], $R[x;\delta]$ is a weakly 2-primal ring and $N(R[x;\delta]) = N(R)[x;\delta]$. This implies that $N(R[x;\delta]) = L(R[x;\delta]) \subseteq J(R[x;\delta]) = (J(R[x;\delta]) \cap R)[x;\delta]$ by Lemma 3.11. Next we claim that if R is δ-compatible, then $J(R[x;\delta]) \cap R$ is a nil ideal. For any $a \in J(R[x;\delta]) \cap R$, then $1 - ax \in U(R[x;\delta])$. There exists $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x;\delta]$ such that $(1-ax)f(x) = 1$. It implies the following equations: $a_0 - a\delta(a_0) = 1$, $a_1 - aa_0 - a\delta(a_1) = 0$, $a_2 - aa_1 - a\delta(a_2) = 0$, $\cdots$, $a_{n-1} - aa_{n-2} - a\delta(a_{n-1}) = 0$, $a_n - aa_{n-1} - a\delta(a_n) = 0$, and $aa_n = 0$. Since $aa_n = 0$ and R is δ-compatible, we get $a\delta(a_n) = 0$ and so $a_n = aa_{n-1}$. Hence, $a^2 a_{n-1} = aa_n = 0$ and $a^2\delta(a_{n-1}) = 0$. Thus, $aa_{n-1} = a^2 a_{n-2}$ and so $a^3 a_{n-2} = a^2 a_{n-1} = 0$. Continuing this progress, we can obtain $a^n a_1 = 0$. Moreover, since $a_1 - aa_0 - a\delta(a_1) = 0$, we have $a^{n-1}a_1 = a^n a_0$ and $a^{n+1}a_0 = a^n a_1 = 0$, so $a^{n+1}\delta(a_0) = 0$. Hence, $a^n a_0 = a^n$ and $a^{n+1} = a^{n+1}a_0 = 0$. Therefore, $a \in N(R)$ and $J(R[x;\delta]) \cap R$ is nil. Moreover, we have $N(R[x;\delta]) = L(R[x;\delta]) \subseteq J(R[x;\delta]) = (J(R[x;\delta]) \cap R)[x;\delta] \subseteq N^*(R)[x;\delta] \subseteq N(R)[x;\delta] = N(R[x;\delta])$. Hence, $R[x;\delta]$ is NJ.

(2) By Corollary 3.15 of [20], we have $J(R[x;\delta]) = N(R[x;\delta])$. Hence, $R[x;\delta]$ is NJ.

(3) Since R is a δ-Armendariz ring, by Corollary 3.4 of [20], we have $J(R[x;\delta]) = N^*(R[x;\delta]) \subseteq N(R[x;\delta])$. On the other hand, according to Proposition 2.9 of [20], $N(R[x;\delta]) \subseteq N(R)[x;\delta] = N^*(R)[x;\delta] = N^*(R[x;\delta]) = J(R[x;\delta])$. Hence, $R[x;\delta]$ is NJ. □

4. Topological conditions for NJ rings

In [12], a ring R is called *weakly pm* if every strongly prime ideal is contained in a unique maximal ideal in it. They also showed that if R is an NI ring, then R is *weakly pm* if and only if $SSpec(R)$ is a normal space if and only if $Max(R)$ is a retract of $SSpec(R)$, where $SSpec(R)$ is the space of all strongly prime ideals of R and $Max(R)$ is the subspace of all maximal ideals of R. In this section, we will apply the topological methods of Sun [22] and Hwang et al. [12] to analyze these conditions for NJ rings relating to the space of J-prime ideals in place of $SSpec(R)$.

In [10], an ideal I of R is called J-prime if I is a prime ideal and I is an intersection of primitive ideals (equivalently, $J(R/I) = 0$). Clearly, every maximal ideal is primitive and every primitive ideal is J-prime. We write $J\text{-}Spec(R)$ for the space of all J-prime ideals of R and denote the lattice of all ideals of R by $Idl(R)$. Let A be a subset of a ring R. Define $D(A) = \{P \in J\text{-}Spec(R) | A \nsubseteq P\}$ and $S(A) = \cap\{P \in J\text{-}Spec(R) | A \subseteq P\}$. It is well known that the following hold.

Lemma 4.1 *Let R be a ring and A be a subset of R.*

(1) $D(A) = \bigcup_{a\in A} D(a) = D(S(A))$*;*

(2) $\bigcup_{i\in\Gamma} D(A_i) = D(\sum_{i\in\Gamma} A_i)$*, where A_i is a subset of R containing 0 for all $i \in \Gamma$;*

(3) $D(I) \cap D(J) = D(I \cap J) = D(IJ)$ *for ideals I, J of R;*

(4) $(J\text{-}Spec(R), \{D(I) | I \lhd R\})$ *is a topological space with a base $\{D(a) | a \in R\}$;*

(5) $S(I)S(J) \subseteq S(IJ) = S(I) \cap S(J)$ *for ideals I, J of R.*

Lemma 4.2 *(1) If R is an NJ ring, then $D(I) \cap D(J) = \emptyset$ if and only if $IJ \subseteq J(R)$ for ideals I, J in R;*

(2) If F is a closed set and $D(K)$ is a open set in $J\text{-}Spec(R)$ satisfying $(F \cap Max(R)) \subseteq D(K)$, then $F \subseteq D(K)$.

Proof (1) Let $D(I) \cap D(J) = \emptyset$. By Lemma 4.1 (3), we have $D(IJ) = \emptyset$. That is, for any $P \in J\text{-}Spec(R)$, $IJ \subseteq P$. Thus, $IJ \subseteq \cap\{P | P \in J\text{-}Spec(R)\}$. Since every primitive ideal is J-prime, we can get $IJ \subseteq \cap\{P | P \in J\text{-}Spec(R)\} \subseteq \cap\{I | I \textit{ is a primitive ideal of } R\} = J(R)$. Let $IJ \subseteq J(R)$. Since every J-prime ideal is strongly prime and R is NJ, we have $N^*(R) = \cap\{P | P \in SSpec(R)\} \subseteq \cap\{P | P \in J\text{-}Spec(R)\} \subseteq J(R) = N(R) = N^*(R)$ and so $J(R) = \cap\{P | P \in J\text{-}Spec(R)\}$. For any $P \in J\text{-}Spec(R)$, we have $IJ \subseteq P$. Thus, $D(I) \cap D(J) = \emptyset$.

(2) Suppose that $F \nsubseteq D(K)$. Then there exists a J-prime ideal $P \in J\text{-}Spec(R)$ such that $P \in F$, but $P \notin D(K)$ and so $K \subseteq P$. Since F is a closed set in $J\text{-}Spec(R)$, we can find an ideal L of R such that $F = J\text{-}Spec(R) \backslash D(L)$. Therefore, $P \notin D(L)$ and so $L \subseteq P$. Moreover, $K + L \subseteq P$. For any maximal ideal M containing P, we have $K + L \subseteq M$. Hence, $K \subseteq M$ and $L \subseteq M$. Since every maximal ideal is primitive, we can get $M \in J\text{-}Spec(R)$ and $M \notin D(L)$. Thus, $M \in F \cap Max(R) \subseteq D(K)$ and so $K \nsubseteq M$, a contradiction. □

Lemma 4.3 *(1) J-Spec(R) is a compact space;*

(2) If J-Spec(R) is a normal space, then $Max(R)$ is a Hausdorff space;

(3) Let R be an NJ ring and $J(R) = BM(R)$. If $Max(R)$ is a Hausdorff space, then J-Spec(R) is a normal space, where $BM(R)$ is the Brown–McCoy radical of R.

Proof (1) Let $\{U_i | i \in \Omega\}$ be a open cover of the topological space J-$Spec(R)$. Then we have J-$Spec(R) = \bigcup_{i\in\Omega} U_i$. Hence, there exists a subset A of R such that $\bigcup_{i\in\Omega} U_i = \bigcup_{a\in A} D(a)$. Take $I = (A) = RAR$. Next we claim that $\bigcup_{a\in A} D(a) = \bigcup_{b\in I} D(b)$. For any $P \in \bigcup_{a\in A} D(a)$, then there exists $a \in A$ such that $P \in D(a)$ and so $a \notin P$. Thus, $I \nsubseteq P$ by $a \in I$. Therefore, $P \in D(I) = \bigcup_{b\in I} D(b)$ and so $\bigcup_{a\in A} D(a) \subseteq \bigcup_{b\in I} D(b)$. For any $P \in \bigcup_{b\in I} D(b)$, we have $a' \notin P$ for some $a' \in I$. Assume that $P \notin \bigcup_{a\in A} D(a)$. Then, for each $a \in A$, we have $a \in P$, a contradiction. Therefore, $\bigcup_{a\in A} D(a) = \bigcup_{b\in I} D(b) = D(I)$. This implies that J-$Spec(R) = \bigcup_{i\in\Omega} U_i = \bigcup_{a\in A} D(a) = \bigcup_{b\in I} D(b) = D(I)$. Hence, $I = (A) = R$. There exist $n \in \mathbb{N}$, r_i, $s_i \in R$, and $a_i \in A$ such that $\sum_{i=1}^{n} r_i a_i s_i = 1$. It implies that $(a_1, a_2, \cdots, a_n) = R$ and so $\bigcup_{i=1}^{n} D(a_i) = D((a_1, a_2, \cdots, a_n)) = D(R) = J$-$Spec(R)$. For every $1 \leqslant i \leqslant n$, we take $U_i \in \{U_i | i \in \Omega\}$ such that U_i contains $D(a_i)$. Then $\bigcup_{i=1}^{n} U_i = J$-$Spec(R)$. Therefore, J-$Spec(R)$ is a compact space.

(2) Let M_1, $M_2 \in Max(R)$ with $M_1 \neq M_2$. Then M_1, $M_2 \in J$-$Spec(R)$ and $M_1 \notin D(M_1)$, $M_2 \notin D(M_2)$. Hence, $\{M_1\} \subseteq J$-$Spec(R)\backslash D(M_1)$ and $\{M_2\} \subseteq J$-$Spec(R)\backslash D(M_2)$. For any $P \in J$-$Spec(R)\backslash D(M_1)$, we have $M_1 \subseteq P$. Since M_1 is a maximal ideal, we can obtain $M_1 = P$ and so $P \in \{M_1\}$. Thus, $\{M_1\} \subseteq J$-$Spec(R)\backslash D(M_1)$. Similarly, we also have $\{M_2\} \subseteq J$-$Spec(R)\backslash D(M_2)$. That is, $\{M_1\}$ and $\{M_2\}$ are closed sets in J-$Spec(R)$. Because J-$Spec(R)$ is a normal space, there exist open neighborhoods $D(I)$ and $D(J)$ such that $\{M_1\} \subseteq D(I)$, $\{M_2\} \subseteq D(J)$, and $D(I) \cap D(J) = \emptyset$. Hence, $M_1 \in D(I) \cap Max(R)$, $M_2 \in D(J) \cap Max(R)$, and $(D(I) \cap Max(R)) \cap (M_2 \in D(J) \cap Max(R)) = \emptyset$. Therefore, $Max(R)$ is Hausdorff.

(3) Let F_1 and F_2 be closed sets in J-$Spec(R)$ and $F_1 \cap F_2 = \emptyset$. Then $F_1 \cap Max(R)$ and $F_2 \cap Max(R)$ are closed sets in $Max(R)$ and $(F_1 \cap Max(R)) \cap (F_2 \cap Max(R)) = \emptyset$. Since $Max(R)$ is a compact Hausdorff space, $Max(R)$ is a normal space. There exist two open neighborhoods A, B in $Max(R)$ such that $F_1 \cap Max(R) \subseteq A$, $F_2 \cap Max(R) \subseteq B$, and $A \cap B = \emptyset$. For A and B, we can find ideals I and J such that $A = D(I) \cap Max(R)$ and $B = D(J) \cap Max(R)$. At the same time, we also have $\emptyset = A \cap B = (D(I) \cap Max(R)) \cap (D(J) \cap Max(R)) = D(I) \cap D(J) \cap Max(R) = D(IJ) \cap Max(R)$. For any $M \in Max(R)$, it can imply $IJ \subseteq M$. Thus, $IJ \subseteq BM(R) = J(R)$. By Lemma 4.2 (1), we can obtain $D(I) \cap D(J) = \emptyset$. Since $F_1 \cap Max(R) \subseteq A = D(I) \cap Max(R)$ and $F_2 \cap Max(R) \subseteq B = D(J) \cap Max(R)$, we have $F_1 \cap Max(R) \subseteq D(I)$ and $F_2 \cap Max(R) \subseteq D(J)$. Moreover, $F_1 \subseteq D(I)$ and $F_2 \subseteq D(J)$ by Lemma 4.2 (2). Therefore, J-$Spec(R)$ is normal. □

According to Koh [13], a ring R is strongly harmonic provided that for each pair of distinct maximal ideals M_1, M_2 there are ideals I_1, I_2 such that $I_1 \nsubseteq M_1$, $I_2 \nsubseteq M_2$, and $I_1 I_2 = 0$. Obviously, if R is a strongly harmonic ring, then $Max(R)$ is a Hausdorff space. Associated to Lemma 4.3 (2), we will prove the following proposition.

Proposition 4.4 *If R is a strongly harmonic ring, then J-$Spec(R)$ is a normal space.*

Proof Let F_1 and F_2 be closed sets in J-$Spec(R)$ with $F_1 \cap F_2 = \emptyset$. Then $F_1 \cap Max(R)$ and $F_2 \cap Max(R)$ are closed sets in $Max(R)$ and $(F_1 \cap Max(R)) \cap (F_2 \cap Max(R)) = \emptyset$. Let $M \in F_1 \cap Max(R)$. Then $M \notin F_2 \cap Max(R)$. It is well known that every closed subset in a compact space is a compact subset. That is, $F_1 \cap Max(R)$ and $F_2 \cap Max(R)$ are both compact subsets in $Max(R)$. According to Theorem 3.2 of [13], we can find ideals I_M and J_M such that $M \in D(I_M)$, $F_2 \cap Max(R) \subseteq D(J_M)$, and $I_M J_M = 0$. Repeating the above procedure and by the compactness of $F_1 \cap Max(R)$, we can find a finite number of ideals $I_1, I_2, \cdots, I_n, J_1, J_2, \cdots, J_n$ such that $F_1 \cap Max(R) \subseteq \bigcup\limits_{i=1}^{n} D(I_i) = D(\sum\limits_{i=1}^{n} I_i)$, $F_2 \cap Max(R) \subseteq \bigcap\limits_{i=1}^{n} D(J_i) = D(J_1 J_2 \cdots J_n)$, and $I_i J_i = 0$. Taking $I = \sum\limits_{i=1}^{n} I_i$ and $J = J_1 J_2 \cdots J_n$, then $IJ = 0$. By Lemma 4.2 (2), we have $F_1 \subseteq D(I)$, $F_2 \subseteq D(J)$ and $D(I) \cap D(J) = D(IJ) = \emptyset$. Therefore, J-$Spec(R)$ is normal. □

In Sun's work in [22], $Idl(R)$ is *normal* if for each pair I_1, $I_2 \in Idl(R)$ with $I_1 + I_2 = R$, there are ideals J_1, J_2 such that $I_1 + J_1 = R = I_2 + J_2$ and $J_1 J_2 = 0$.

Proposition 4.5 *Let R be an NJ ring. Then we have the following:*

(1) J-$Spec(R)$ is a normal space if and only if for each pair I_1, $I_2 \in Idl(R)$ with $I_1 + I_2 = R$, there are ideals J_1, J_2 such that $I_1 + J_1 = R = I_2 + J_2$ and $S(J_1)S(J_2) \subseteq J(R)$;

(2) $Idl(R)$ is normal, and then J-$Spec(R)$ is a normal space.

Proof (1) For the sufficiency, let F_1 and F_2 be closed sets in J-$Spec(R)$ with $F_1 \cap F_2 = \emptyset$. Then there exist I_1, $I_2 \in Idl(R)$ such that $F_1 = J\text{-}Spec(R) \backslash D(I_1)$ and $F_2 = J\text{-}Spec(R) \backslash D(I_2)$. Since $F_1 \cap F_2 = \emptyset$, we have $D(I_1) \cup D(I_2) = J\text{-}Spec(R)$ and so $I_1 + I_2 = R$. By the hypothesis, there are ideals J_1, J_2 such that $I_1 + J_1 = R = I_2 + J_2$ and $J_1 J_2 = 0$. Hence, $J\text{-}Spec(R) = D(R) = D(I_1 + J_1) = D(I_1) \cup D(J_1)$ and $J\text{-}Spec(R) = D(R) = D(I_2 + J_2) = D(I_2) \cup D(J_2)$. Moreover, we can get $F_1 \subseteq D(J_1)$, $F_2 \subseteq D(J_2)$ and $D(S(J_1)) = D(J_1)$, $D(S(J_2)) = D(J_2)$ by Lemma 4.1 (1). This implies that $D(J_1) \cap D(J_2) = D(S(J_1)) \cap D(S(J_2)) = D(S(J_1)S(J_2))$. Since $S(J_1)S(J_2) \subseteq J(R)$, we have $D(J_1) \cap D(J_2) = D(S(J_1)S(J_2)) = \emptyset$. Therefore, J-$Spec(R)$ is normal.

For the necessity, assume that I_1, $I_2 \in Idl(R)$ with $I_1 + I_2 = R$. Let $F_1 = J\text{-}Spec(R) \backslash D(I_1)$ and $F_2 = J\text{-}Spec(R) \backslash D(I_2)$. Then F_1 and F_2 are closed sets in J-$Spec(R)$ and $D(I_1) \cup D(I_2) = D(I_1 + I_2) = D(R) = J\text{-}Spec(R)$. Hence, $F_1 \cap F_2 = J\text{-}Spec(R) \backslash (D(I_1) \cup D(I_2)) = \emptyset$. Since J-$Spec(R)$ is normal, there exist J_1, $J_2 \in Idl(R)$ such that $F_1 \subseteq D(J_1)$, $F_2 \subseteq D(J_2)$, and $D(J_1) \cap D(J_2) = \emptyset$. Because $J\text{-}Spec(R) = F_1 \cup D(I_1) \subseteq D(J_1) \cup D(I_1) = D(J_1 + I_1) \subseteq J\text{-}Spec(R)$, we have $D(J_1 + I_1) = J\text{-}Spec(R)$ and so $J_1 + I_1 = R$. Similarly, we also conclude $J_2 + I_2 = R$. R is NJ and $D(J_1) \cap D(J_2) = \emptyset$, so, by Lemma 4.2 (1), we can obtain $J_1 J_2 \subseteq J(R)$. On the other hand, if R is NJ, we have $J(R) = \{P | P \in J\text{-}Spec(R)\}$. Hence, $S(J_1)S(J_2) \subseteq S(J_1 J_2) \subseteq J(R)$ by Lemma 4.1 (5).

(2) Let $Idl(R)$ be normal. Then for each pair I_1, $I_2 \in Idl(R)$ with $I_1 + I_2 = R$, there are ideals J_1, J_2 such that $I_1 + J_1 = R = I_2 + J_2$ and $J_1 J_2 = 0$. Hence, $S(J_1)S(J_2) \subseteq S(J_1 J_2) = S(0) = \{P | P \in J\text{-}Spec(R)\} = J(R)$ by Lemma 4.1 (5). Therefore, according to (1), we can prove that J-$Spec(R)$ is normal. □

In [22], a ring R is called *pm* if every prime ideal is contained in a unique maximal ideal in it. Hwang et al. in [12] extended to *weakly pm*. In this section, we call a ring R *J-pm* if every J-prime ideal is contained in a unique maximal ideal in it. Clearly, every *pm* ring is *weakly pm* and every *weakly pm* is *J-pm*.

Proposition 4.6 *If $Max(R)$ is a retract of $J\text{-}Spec(R)$, then R is J-pm.*

Proof For any $P \in J\text{-}Spec(R)$, let $\mu : J\text{-}Spec(R) \to Max(R)$ be a continuous retraction with $\mu(P) = M$. From the proof of Lemma 4.3 (2), we can obtain that $\{M\}$ is a closed set in $Max(R)$, so $\mu^{-1}(\{M\})$ is also a closed set in $J\text{-}Spec(R)$. By Observation 1.4 of [22], the closed set $\mu^{-1}(\{M\})$ contains the closure of $\{P\}$, and so any maximal ideal M' containing P. Moreover, we have $M' = \mu(M') = M$. Therefore, R is *J-pm*. □

Lemma 4.7 *Let R be an NJ ring and X be a multiplicative monoid in $R\backslash 0$. Suppose that $P \subseteq J(R)$ is an ideal of R maximal with respect to the property $P \cap X = \emptyset$. Then P is J-prime.*

Proof Let A, B be ideals of R with $AB \subseteq P$. Since $P \cap X = \emptyset$, we have $AB \cap X = \emptyset$. Assume that there exist $x \in A \cap X$ and $y \in B \cap X$. Then we can get $xy \in AB \cap X$, a contradiction. Therefore, P is a prime ideal. Since R is NJ and $P \subseteq J(R)$, we have $J(R/P) = N(R/P) = N^*(R/P)$ by Proposition 2.12 (2). According to Lemma 2.2 of [12], we can obtain $J(R/P) = 0$. Thus, P is J-prime. □

Proposition 4.8 *If R is a NJ J-pm ring, then $Max(R)$ is a Hausdorff space.*

Proof We apply the proof of ([12], Lemma 3.4) and ([22], Lemma 2.1). Let M_1, $M_2 \in Max(R)$ with $M_1 \neq M_2$. Consider the multiplicative subset $S = \{a_1b_1a_2b_2\cdots a_nb_n | a_i \in M_1,\ b_i \in M_2,\ i = 1, 2, \cdots, n,\ n \in \mathbb{N}\}$. If $0 \notin S$, then there would be a J-prime ideal $P \subseteq J(R)$ such that $P \cap S = \emptyset$ by Lemma 4.7. It implies $P \subseteq M_1$ and $P \subseteq M_2$, which is a contradiction because R is *J-pm*. Therefore, $0 \in S$ and there are $a_i \in M_1$, $b_i \in M_2$, and $n \in \mathbb{N}$ such that $a_1b_1a_2b_2\cdots a_nb_n = 0$. Then there are c_i, $d_i \in R$ such that $x_1 = a_1c_1\cdots a_{n-1}c_{n-1}a_n \notin M_1$ and $x_2 = b_1d_1\cdots b_{n-1}d_{n-1}b_n \notin M_2$. Otherwise, we have $a_1b_1\cdots a_{n-1}b_{n-1}a_n \in M_1$ and so $(a_1b_1\cdots a_{n-1}b_{n-1}a_n)b_n \notin S$, a contradiction. Since R is NJ, $R/J(R)$ is reduced. In $R/J(R)$, we have $\overline{a_1b_1a_2b_2\cdots a_nb_n} = \bar{0}$. For any $r \in R$, we can imply $\overline{a_1b_1a_2b_2\cdots a_nb_nc_1c_2\cdots c_{n-1}d_1d_2\cdots d_{n-1}} = \bar{0}$ and so $\overline{x_1rx_2} = \bar{0}$. Thus, $x_1Rx_2 \subseteq J(R)$. In an NJ ring, $J(R) = \cap\{P | P \in J\text{-}Spec(R)\}$. That is, for any $P \in J\text{-}Spec(R)$, we have $x_1Rx_2 \subseteq P$. Assume that $D(x_1) \cap D(x_2) \neq \emptyset$. Then we can find $P \in D(x_1) \cap D(x_2)$ and so $x_1 \notin P$, $x_2 \notin P$. It implies $x_1Rx_2 \nsubseteq P$ because P is a prime ideal, a contraction. This proves that $D(x_1) \cap D(x_2) = \emptyset$. On the other hand, since $x_1 \in M_1$ and $x_2 \in M_2$, we have $M_1 \in D(x_1)$ and $M_2 \in D(x_2)$. Therefore, $Max(R)$ is Hausdorff. □

Theorem 4.9 *Let R be an NJ ring. Then the following are equivalent:*

(1) $J\text{-}Spec(R)$ is a normal space;

(2) $Max(R)$ is a retract of $J\text{-}Specc(R)$ and $Max(R)$ is a Hausdorff space.

Proof (1) ⇒ (2) If $J\text{-}Spec(R)$ is a normal space, then $Max(R)$ is a Hausdorff space by Lemma 4.3 (2). Next we only prove that $Max(R)$ is a retract of $J\text{-}Spec(R)$. For any $P \in J\text{-}Spec(R)$, we let $F_P = \{I \triangleleft R | I + P = R\}$. In the proof of Theorem 1.6 of [22], we know that F_P has the following properties:

(i) if $I_1 + I_2 \in F_p$, then either $I_1 \in F_p$ or $I_2 \in F_P$;

(ii) if $I \in F_P$, and $I \subseteq J$, then $J \in F_P$. Meanwhile, for each $I \triangleleft R$, we define $M_P = \sum\{I \triangleleft | I \notin F_P\}$. If $1 \in M_P$, then we have $\sum\{I \triangleleft | I \notin F_P\} = R$ and so $P + \sum\{I \triangleleft | I \notin F_P\} = R$. This implies $\sum\{I \triangleleft | I \notin F_P\} \in F_P$ and so $J \in F_P$ for each $J \in \{I \triangleleft R | I \notin F_P\}$, a contradiction. Thus, $1 \notin M_P$. On the other hand, if $P \in F_P$, then we have $P + P = R$ and so $P = R$, a contradiction. Hence, $P \notin F_P$ and $P \subseteq M_P$. Assume that M_P is not a maximal ideal. There exists $M \in Max(R)$ with $M \neq R$ such that $M_P \subsetneqq M$ and so $M \nsubseteq M_P$. Then $M \in F_P$ and so $M + P = R$, which implies $M = M + M_P \supseteq M + P = R$, a contradiction. Therefore, M_P is a maximal ideal. Now we define a map $\mu : J\text{-}Spec(R) \to Max(R)$ with $\mu(P) = M_P$ for each $P \in J\text{-}Spec(R)$. Obviously, when $P \in Max(R)$, $P \subseteq M_P$; on the other hand, for any $I \triangleleft R$ with $I \notin F_P$, then $I + P \neq R$. Since $P \subseteq I + P$ and so $I \subseteq P$, therefore, $M_P \subseteq P$. That is, if $P \in Max(R)$, then $P = M_P$. Moreover, μ is extended from $1_{Max(R)} : Max(R) \to Max(R)$. Next we prove that μ is a continuous map. For any $I \triangleleft R$, $D(I) \cap Max(R)$ is a open set in $Max(R)$. We will claim that $\mu^{-1}(D(I) \cap Max(R)) = \{P \in J\text{-}Spec(R) | \mu(P) \in D(I) \cap Max(R)\}$ is an open set in $J\text{-}Spec(R)$. For each $P \in \mu^{-1}(D(I) \cap Max(R))$, we have $\mu(P) \in D(I) \cap Max(R)$. That is, $\mu(P) \in D(I)$ and $\mu(P) \in Max(R)$. This implies $I \nsubseteq \mu(P) = M_P$ and $I \in F_P$, so we have $I + P = R$. By Lemma 4.5 (1), there exist ideals J_1, J_2 such that $I + J_1 = R = P + J_2$ and $S(J_1)S(J_2) \subseteq J(R)$. Since $R = P + J_2$, we can obtain $J_2 \nsubseteq P$ and so $P \in D(J_2)$. It can imply $\mu^{-1}(D(I) \cap Max(R)) \subseteq D(J_2)$. For the converse inclusion, if $P' \in D(J_2)$, then $J_2 \nsubseteq P'$ and so $S(J_2) \nsubseteq P'$ by $J_2 \subseteq S(J_2)$. Since $S(J_1)S(J_2) \subseteq J(R) \subseteq P'$ and P' is a prime ideal, we have $S(J_1) \subseteq P'$ and so $J_1 \subseteq P'$. Hence, we get $I + P' = R$ by $I + J_1 = R$. Thus, $I \in F_{P'}$ and $I \nsubseteq \mu(P') = M_{P'}$. Therefore, we can obtain $M_{P'} = \mu(P') \in D(I) \cap Max(R)$ and so $P' \in \mu^{-1}(D(I) \cap Max(R))$. That is, $\mu^{-1}(D(I) \cap Max(R)) = D(J_2)$ and so $\mu^{-1}(D(I) \cap Max(R))$ is a open set.

(2) $\Rightarrow$ (1) Let $\mu : J\text{-}Spec(R) \to Max(R)$ be a continuous retraction by the hypothesis. Suppose that F_1 and F_2 are closed sets in $J\text{-}Spec(R)$ with $F_1 \cap F_2 = \emptyset$. Then $F_1 \cap Max(R)$ and $F_2 \cap Max(R)$ are closed sets in $Max(R)$ and $(F_1 \cap Max(R)) \cap (F_2 \cap Max(R)) = \emptyset$. Since every compact Hausdorff space is normal, there are I, $J \triangleleft R$ such that $F_1 \cap Max(R) \subseteq D(I) \cap Max(R)$, $F_2 \cap Max(R) \subseteq D(J) \cap Max(R)$ and $(D(I) \cap Max(R)) \cap (D(J) \cap Max(R)) = \emptyset$. Because μ is continuous, $\mu^{-1}(D(I) \cap Max(R))$ and $\mu^{-1}(D(J) \cap Max(R))$ are open sets in $J\text{-}Spec(R)$. Assume that $\mu^{-1}(D(I) \cap Max(R)) \cap \mu^{-1}(D(J) \cap Max(R)) \neq \emptyset$. Then there exists $P \in \mu^{-1}(D(I) \cap Max(R)) \cap \mu^{-1}(D(J) \cap Max(R))$ and so $\mu(P) \in D(I) \cap D(J) \cap Max(R)$, a contradiction. Therefore, $\mu^{-1}(D(I) \cap Max(R)) \cap \mu^{-1}(D(J) \cap Max(R)) = \emptyset$. By Observation 1.4 of [22] and $F_1 \cap Max(R) \subseteq D(I) \cap Max(R)$, we have $F_1 = \mu^{-1}(F_1 \cap Max(R)) \subseteq \mu^{-1}(D(I) \cap Max(R))$. Similarly, we also have $F_2 \subseteq \mu^{-1}(D(J) \cap Max(R))$. Therefore, $J\text{-}Spec(R)$ is normal. □

Theorem 4.10 *Let R be an NJ ring. Then the following are equivalent:*

(1) R is J-pm;

(2) $J\text{-}Spec(R)$ is a normal space;

(3) $Max(R)$ is a retract of $J\text{-}Spec(R)$.

Proof (3) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) are proved by Proposition 4.6 and Theorem 4.9, respectively.

(1) $\Rightarrow$ (2) If R is an NJ *J-pm* ring, then $Max(R)$ is a Hausdorff space by Proposition 4.8. According to Theorem 4.9, we only prove that $Max(R)$ is a retract of $J\text{-}Spec(R)$. Since R is *J-pm*, we can obtain a retraction $\mu : J\text{-}Spec(R) \to Max(R)$ by sending each J-prime ideal to the unique maximal ideal containing it. We only

prove that μ is continuous by showing that $\mu^{-1}(\mathfrak{F})$ is closed in J-$Spec(R)$ for a closed set $\mathfrak{F}$ of $Max(R)$. In the following, we apply the proof of ([22], Theorem 2.2). Let $B = \bigcup\{M|M \in \mathfrak{F}\}$, $F = \bigcap\{M|M \in \mathfrak{F}\}$, and $I = \bigcap\{P \in J\text{-}Spec(R)|\mu(P) \in \mathfrak{F}\}$. We will claim that $\mu^{-1}(\mathfrak{F}) = J\text{-}Spec(R)\backslash D(I)$. For each $P \in \mu^{-1}(\mathfrak{F})$, then $\mu(P) \in \mathfrak{F}$ and so $I \subseteq P$. Hence, we have $P \in J\text{-}Spec(R)\backslash D(I)$. Therefore, $\mu^{-1}(\mathfrak{F}) \subseteq J\text{-}Spec(R)\backslash D(I)$.

If $Q \in J\text{-}Spec(R)$ with $Q \subseteq B$, then clearly $Q + F \subseteq B$ and so there exists a maximal ideal M with $Q + F \subseteq M$. Since $F \subseteq M$ and $\mathfrak{F}$ is a closed set, we have $M \in \mathfrak{F}$. Because R is *J-pm*, M is the unique maximal ideal containing Q.

For each $P \in J\text{-}Spec(R)\backslash D(I)$, then we have $I \subseteq P$. Considering the multiplicative monoid $X = \{s_1t_1s_2t_2\cdots s_nt_n|s_i \notin B,\ t_i \notin P,\ i-1,2,\cdots,n,\ n \in \mathbb{N}\}$ in Theorem 3.7 of [12]. Similarly, we also can prove that $0 \notin X$. Then there exists a J-prime ideal $Q \subseteq J(R)$ such that $Q \cap X = \emptyset$ by Lemma 4.7. Hence, we can obtain $Q \subseteq P \cap B$ and so $\mu(P) \in \mathfrak{F}$. □

Acknowledgments

This work was supported by the National Natural Science Foundation of China (11071097) and the Natural Science Foundation of the Jiangsu Higher Education Institutions (BK20181406).

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