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On a pair of Ramanujan's modular equations and P-Q theta functions of level 35

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Abstract: S. Ramanujan recorded several modular equations and P-Q theta function identities in his notebooks and lost notebook without recording the proofs. In this paper, we provide an elementary proof of two modular equations and two P-Q theta function identities of level 35, which have been proved by B.C. Berndt using the theory of modular forms.

Key words: Theta function, modular equation

1. Introduction

The Gauss series or the ordinary hypergeometric series ${}_2F_1\left[a, b; c; z\right]$ [16–18, 21, 26, 27] is defined by

$$ {}_2F_1\left[a, b; c; z\right] = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots . $$

It is well known that the Gauss hypergeometric function ${}_2F_1\left[a, b; c; z\right]$ has many important applications in mathematics, physics, and engineering and many special functions are the particular cases or limiting values of the Gauss hypergeometric function. For example, the perimeter of an ellipse with semi-axes a and b and eccentricity $e = \frac{\sqrt{a^2-b^2}}{a}$ can be expressed by $4a{}_2F_1\left[-\frac{1}{2}, \frac{1}{2}; 1; e^2\right]$; the conformal modulus $\mu(r)$ [10, 22, 23] of the Grötzsch ring in the plane can be given by

$$ \mu(r) = \frac{\pi}{2}\frac{{}_2F_1\left[\frac{1}{2}, \frac{1}{2}; 1; 1-r^2\right]}{{}_2F_1\left[\frac{1}{2}, \frac{1}{2}; 1; r^2\right]}, $$

which plays a very important role in the conformal and quasi-conformal mapping theory. The classical complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ [5–9, 19, 20, 24, 25] of the first and second kinds are given by

$$ \mathcal{K}(r) = \frac{\pi}{2}\,{}_2F_1\left[\frac{1}{2}, \frac{1}{2}; 1; r^2\right] $$

and

$$ \mathcal{E}(r) = \frac{\pi}{2}\,{}_2F_1\left[-\frac{1}{2}, \frac{1}{2}; 1; r^2\right]. $$

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Let

$$K(\alpha) = {}_2F_1\left[\frac{1}{2},\frac{1}{2};1;\alpha\right]$$

and

$$K'(\alpha) = {}_2F_1\left[\frac{1}{2},\frac{1}{2};1;1-\alpha\right] \quad .$$

Suppose that

$$k\frac{K'(\alpha)}{K(\alpha)} = \frac{K'(\beta)}{K(\beta)}, \quad l\frac{K'(\alpha)}{K(\alpha)} = \frac{K'(\gamma)}{K(\gamma)} \quad \textit{and} \quad kl\frac{K'(\alpha)}{K(\alpha)} = \frac{K'(\delta)}{K(\delta)} \tag{1.1}$$

holds for some positive integers k and l. The relation between α, β, γ, and δ induced by the above is called a modular equation of composite degree kl or modular equation of level kl. Also, the relation between α and β induced by (1.1) is called a modular equation of degree k. We define the multipliers m and m' by

$$m = \frac{K'(\alpha)}{K(\beta)} \qquad \textit{and} \qquad m' = \frac{K'(\gamma)}{K(\delta)}.$$

On pages 249 and 250 of his second notebook [12], Ramanujan recorded seven modular equations of composite degree 35. Berndt in [2, pp. 423–425] proved two of these modular equations using theta function identities and the other five by using the theory of modular forms. Recently Vasuki and Sharath [14] proved three of the above mentioned five modular equation of Ramanujan by employing the tools known to Ramanujan, which had been proved by Berndt by using the theory of modular forms. In Section 4 of this paper, we provide an elementary proof of two modular equations that were not considered in [14].

For any complex number a and q,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1.$$

On page 197 of his second notebook [12], Ramanujan defined his general theta function:

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\, b^{n(n-1)/2} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \qquad |ab|<1.$$

Furthermore, he also defined three special cases of $f(a,b)$, namely

$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty,$$

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} = (q;q)_\infty \quad .$$

Besides $\varphi(q)$, $\psi(q)$, $f(-q)$ he further defined

$$\chi(q) = (-q; q^2)_\infty \quad ,$$

which is not a theta function but plays a prominent role in the theory of theta functions. For convenience throughout the paper we set

$$f(-q^n) = f_n.$$

One can easily show that

$$\varphi(-q) = \frac{{f_1}^2}{f_2} \qquad and \qquad \chi(-q) = \frac{f_1}{f_2} \quad . \tag{1.2}$$

Let $P(q)$ and $Q(q)$ be the product or quotient of theta functions f_1, f_k, f_l, and f_{kl}. Then we call the relation between $P(q)$ and $Q(q)$ the P-Q theta function identity of level kl.

Ramanujan recorded twenty-three such P-Q theta function identities in his second notebook [12]. Berndt proved eighteen of them by employing the theory of theta functions in the spirit of Ramanujan, whereas for the remaining five he used the theory of modular forms. In Section 3 of this paper we give a proof of two P-Q theta function identities of level 35, one of which is from those five mentioned above and the other one is from his lost notebook [13]. The proofs are worth noting as we give proofs free of the theory of modular forms and by using theory known to Ramanujan.

2. Preliminary results

In this section, we recall some important results that will be used to prove our main results.

Theorem 2.1 *We have:*
(i) [15], [2, p. 315]

$$If \quad M = \frac{f_1}{q^{1/24} f_2} \quad and \quad N = \frac{f_7}{q^{7/24} f_{14}} \;, \quad then$$
$$(MN)^3 + \frac{8}{(MN)^3} + 7 = \left(\frac{N}{M}\right)^4 + \left(\frac{M}{N}\right)^4 \quad . \tag{2.1}$$

(ii) [3, p. 209]

$$If \quad M = \frac{f_1}{q^{1/4} f_7} \quad and \quad N = \frac{f_2}{q^{1/2} f_{14}} \;, \quad then$$
$$(MN)^2 + \frac{7^2}{(MN)^2} = \left(\frac{N}{M}\right)^6 - 8\left(\frac{N}{M}\right)^2 - 8\left(\frac{M}{N}\right)^2 + \left(\frac{M}{N}\right)^6 \quad . \tag{2.2}$$

(iii) [3, p. 206]

$$If \quad M = \frac{f_1}{q^{1/6} f_5} \quad and \quad N = \frac{f_2}{q^{1/3} f_{10}} \;, \quad then$$
$$MN + \frac{5}{MN} = \left(\frac{N}{M}\right)^3 + \left(\frac{M}{N}\right)^3 \quad . \tag{2.3}$$

Ramanujan recorded identity (2.1) in the form of a modular equation in Entry 19(ix) [2] and Berndt gave a proof of the same using parametrization. For a proof of (2.2) and (2.3), one may refer to [3].

Theorem 2.2 *Let α, β, γ, and δ have degrees 1, 5, 7, and 35, respectively. Then*

$$Q^4 + \frac{1}{Q^4} - \left\{Q^2 + \frac{1}{Q^2}\right\} - 2\left\{P^2 + \frac{1}{P^2}\right\} = 0, \tag{2.4}$$

$$R^4 + \frac{1}{R^4} - \left\{Q^6 + \frac{1}{Q^6}\right\} + 5\left\{Q^4 + \frac{1}{Q^4}\right\} - 10\left\{Q^2 + \frac{1}{Q^2}\right\} + 15 = 0, \tag{2.5}$$

and

$$T^6 + \frac{1}{T^6} + Q^6 + \frac{1}{Q^6} = 4\left\{P^4 + \frac{1}{P^4} - R^4 - \frac{1}{R^4}\right\}, \tag{2.6}$$

where

$$\begin{aligned} P &= \{256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/48}, \\ Q &= \left\{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right\}^{1/48}, \\ R &= \left\{\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right\}^{1/48}, \\ \text{and} \quad T &= \left\{\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right\}^{1/48}. \end{aligned} \tag{2.7}$$

Ramanujan recorded elven Schläfli-type modular equations for composite degrees on pages 86 and 88 of his first notebook [11]. In [4] Berndt proved all the eleven modular equations, one equation by using tools known to Ramanujan and the remaining ten by using the theory of modular forms. Baruah in [1] gave elementary proofs for seven of the ten equations mentioned above. Identities (2.4) and (2.5) are two among them and identity (2.6) is due to Baruah [1]. For the remaining sections we shall use the following notations:

$$u = \frac{qf_1f_{35}}{f_5f_7}, \qquad v = \frac{q^{4/3}f_5f_{35}}{f_1f_7}, \qquad w = \frac{q^{3/2}f_7f_{35}}{f_1f_5},$$

$$u_1 = \frac{q^2f_2f_{70}}{f_{10}f_{14}}, \qquad v_1 = \frac{q^{8/3}f_{10}f_{70}}{f_2f_{14}}, \qquad \text{and} \qquad w_1 = \frac{q^3f_{14}f_{70}}{f_2f_{10}}.$$

3. P-Q theta function identities

Theorem 3.1 *[3, p. 236], [12, p. 303] We have*

$$\frac{1}{w^2} + 49w^2 - 5 = \frac{1}{u^3} - u^3 - 5\left(\frac{1}{u^2} + u^2\right). \tag{3.1}$$

Proof Let

$$\frac{u}{u_1} + \frac{u_1}{u} = t. \tag{3.2}$$

From identity (2.1) of [14], we have

$$\frac{1}{uu_1} + uu_1 = \left(\frac{u_1}{u}\right)^3 + \left(\frac{u}{u_1}\right)^3 + 4\left\{\left(\frac{u_1}{u}\right)^2 + \left(\frac{u}{u_1}\right)^2\right\} + 8\left\{\frac{u_1}{u} + \frac{u}{u_1}\right\} + 12 \ .$$

Making use of (3.2) in the above identity, we obtain

$$\frac{1}{uu_1} + uu_1 = t^3 + 4t^2 + 5t + 4, \tag{3.3}$$

which imples

$$\frac{1}{uu_1} - uu_1 = (t+1)\sqrt{(t+3)(t+2)(t^2+t+2)} \ , \tag{3.4}$$

$$\frac{1}{(uu_1)^{3/2}} + (uu_1)^{3/2} = \left(t^3 + 4t^2 + 5t + 3\right)\sqrt{(t+3)(t^2+t+2)}, \tag{3.5}$$

and

$$\frac{1}{(uu_1)^{3/2}} - (uu_1)^{3/2} = \left(t^3 + 4t^2 + 5t + 5\right)(t+1)\sqrt{t+2} \ . \tag{3.6}$$

We have considered positive sign for (3.4) and (3.6), as $\frac{1}{uu_1} > uu_1$, which follows from their series expansion

$$\frac{1}{uu_1} = q^{-3} + q^{-2} + 3q^{-1} + 4 + 7q + 8q^2 + 12q^3 + 12q^4 + \cdots,$$
$$uu_1 = q^3 - q^4 - 2q^5 + q^6 + q^7 + 3q^8 - 2q^9 - 2q^{10} + \cdots.$$

Similarly, from (3.2) we have

$$\frac{u}{u_1} - \frac{u_1}{u} = \sqrt{t^2-4} \ , \tag{3.7}$$

$$\left(\frac{u}{u_1}\right)^{3/2} + \left(\frac{u_1}{u}\right)^{3/2} = (t-1)\sqrt{t+2}, \tag{3.8}$$

and

$$\left(\frac{u}{u_1}\right)^{3/2} - \left(\frac{u_1}{u}\right)^{3/2} = (t+1)\sqrt{t-2} \ . \tag{3.9}$$

We have considered positive sign for (3.7) and (3.9), as $\frac{u}{u_1} > \frac{u_1}{u}$, which follows from their series expansion

$$\frac{u}{u_1} = q^{-1} - 1 + q^2 + 2q^3 - 2q^5 + q^6 + 2q^8 - 3q^9 + \cdots,$$
$$\frac{u_1}{u} = q + q^2 + q^3 + 2q^4 - q^6 - q^7 - 4q^8 - 5q^9 + \cdots.$$

On multiplying identity (3.5) with (3.9), we obtain

$$u^3 - \frac{1}{u^3} - {u_1}^3 + \frac{1}{{u_1}^3} = \left(t^3 + 4t^2 + 5t + 3\right)(t+1)\sqrt{(t+3)(t-2)(t^2+t+2)} \quad . \tag{3.10}$$

On multiplying identity (3.6) with (3.8), we obtain

$$-u^3 + \frac{1}{u^3} - {u_1}^3 + \frac{1}{{u_1}^3} = \left(t^3 + 4t^2 + 5t + 5\right)(t+1)(t-1)(t+2) \quad . \tag{3.11}$$

Subtracting identity (3.10) from (3.11), we obtain

$$\frac{1}{u^3} - u^3 = \frac{1}{2}\Bigg\{ \left(t^3 + 4t^2 + 5t + 5\right)(t+1)(t-1)(t+2) - \left(t^3 + 4t^2 + 5t + 3\right)(t+1)\sqrt{(t+3)(t-2)(t^2+t+2)} \Bigg\} \quad . \tag{3.12}$$

On multiplying identity (3.2) with identity (3.3), we find

$$u^2 + \frac{1}{u^2} + {u_1}^2 + \frac{1}{{u_1}^2} = t\left(t^3 + 4t^2 + 5t + 4\right) \quad . \tag{3.13}$$

On multiplying identity (3.4) with identity (3.7), we find

$$-u^2 - \frac{1}{u^2} + {u_1}^2 + \frac{1}{{u_1}^2} = (t+1)(t+2)\sqrt{(t+3)(t-2)(t^2+t+2)} \quad . \tag{3.14}$$

Subtracting (3.14) from (3.13), we obtain

$$u^2 + \frac{1}{u^2} = \frac{1}{2}\left\{ t\left(t^3 + 4t^2 + 5t + 4\right) - (t+1)(t+2)\sqrt{(t+3)(t-2)(t^2+t+2)} \right\} \quad . \tag{3.15}$$

Making use of identities (3.12) and (3.15), we deduce

$$\frac{1}{u^3} - u^3 - 5\left(\frac{1}{u^2} + u^2\right) = \frac{1}{2}\Bigg\{ \left(t^3 + 3t + 1\right)\left(t^4 + 3t^3 - 3t^2 - 5t - 10\right) - (t+1)\left(t^3 + 4t^2 - 7\right)\sqrt{(t+3)(t-2)(t^2+t+2)} \Bigg\} \quad . \tag{3.16}$$

Let

$$X = \frac{f_1}{q^{1/24} f_2} \quad , \quad X_1 = \frac{f_5}{q^{5/24} f_{10}} \quad , \quad Y = \frac{f_7}{q^{7/24} f_{14}}, \quad \textit{and} \quad Y_1 = \frac{f_{35}}{q^{35/24} f_{70}}.$$

Multiplying identity (2.1) with the identity obtained replacing q by q^5 in (2.1) and then using (3.2) in it, we deduce

$$\left(\frac{w_1}{w}\right)^4 + \left(\frac{w}{w_1}\right)^4 = \left\{ (XY)^3 + \frac{8}{(XY)^3} \right\}\left\{ (X_1Y_1)^3 + \frac{8}{(X_1Y_1)^3} \right\} + 7\left\{ (XY)^3 + \frac{8}{(XY)^3} + (X_1Y_1)^3 + \frac{8}{(X_1Y_1)^3} \right\} + 51 - \left(t^2 - 2\right)^2 \quad . \tag{3.17}$$

From identity (2.4) we have

$$XYX_1Y_1 + \frac{4}{XYX_1Y_1} = \left(\frac{u}{u_1}\right)^2 + \left(\frac{u_1}{u}\right)^2 + \frac{u}{u_1} + \frac{u_1}{u}.$$

Making use of (3.2) the above identity reduces to

$$XYX_1Y_1 + \frac{4}{XYX_1Y_1} = t^2 + t - 2, ;$$

this implies

$$(XYX_1Y_1)^{3/2} + \frac{8}{(XYX_1Y_1)^{3/2}} = \left(t^2 + t - 4\right)\sqrt{t^2 + t + 2} \tag{3.18}$$

and

$$(XYX_1Y_1)^{3/2} - \frac{8}{(XYX_1Y_1)^{3/2}} = \left(t^2 + t\right)\sqrt{t^2 + t - 6}\ . \tag{3.19}$$

We have considered the positive sign above as $XYX_1Y_1 > \frac{1}{XYX_1Y_1}$, which follows from their series expression

$$XXX_1Y_1 = q^{-2} - q^{-1} + 2q^2 - 2q^3 + 2q^4 - 3q^5 + 3q^6 + \cdots,$$

$$\frac{1}{XYX_1Y_1} = q^2 + q^3 + q^4 + 2q^5 + q^7 + q^8 + q^{10} - q^{11} + \cdots.$$

Using modular equations (2.4) and (2.5) in (2.6) we deduce the modular equation relating T and Q, where T and Q are as in (2.7). Transforming the obtained modular equation into theta functions, we deduce that

$$\left(\frac{X_1Y_1}{XY}\right)^3 + \left(\frac{XY}{X_1Y_1}\right)^3 = t^4 + 7t^3 + 17t^2 + 21t + 16;$$

this implies

$$\left(\frac{X_1Y_1}{XY}\right)^{3/2} + \left(\frac{XY}{X_1Y_1}\right)^{3/2} = (t+3)\sqrt{t^2 + t + 2} \tag{3.20}$$

and

$$\left(\frac{X_1Y_1}{XY}\right)^{3/2} - \left(\frac{XY}{X_1Y_1}\right)^{3/2} = \sqrt{t^4 + 7t^3 + 17t^2 + 21t + 14}\ . \tag{3.21}$$

We have considered the positive sign above as $\frac{X_1Y_1}{XY} > \frac{XY}{X_1Y_1}$, which follows from their series expansion

$$\frac{X_1Y_1}{XY} = q^{-4/3} + q^{-1/3} + q^{2/3} + 2q^{5/3} - q^{11/3} - q^{14/3} - 2q^{17/3} + \cdots,$$

$$\frac{XY}{X_1Y_1} = q^{4/3} - q^{7/3} - q^{13/3} + 2q^{16/3} - 3q^{25/3} + q^{28/3} + q^{31/3} + \cdots.$$

Multiplying identity (3.18) with (3.20) results in

$$(XY)^3+\frac{8}{(XY)^3}+(X_1Y_1)^3+\frac{8}{(X_1Y_1)^3}=(t+3)\left(t^2+t-4\right)\left(t^2+t+2\right)\ .$$

Multiplying identity (3.19) with (3.21) results in

$$-(XY)^3-\frac{8}{(XY)^3}+(X_1Y_1)^3+\frac{8}{(X_1Y_1)^3}$$
$$=\left(t^2+t\right)\sqrt{(t^4+7t^3+17t^2+21t+14)\,(t^2+t-6)}\ .$$

From the above two identities, we deduce the following:

$$(XY)^3+\frac{8}{(XY)^3}=\frac{1}{2}\Big\{(t+3)\left(t^2+t-4\right)\left(t^2+t+2\right)-\left(t^2+t\right)\sqrt{(t^4+7t^3+17t^2+21t+14)\,(t^2+t-6)}\Big\} \tag{3.22}$$

and

$$(X_1Y_1)^3+\frac{8}{(X_1Y_1)^3}=\frac{1}{2}\Big\{(t+3)\left(t^2+t-4\right)\left(t^2+t+2\right)+\left(t^2+t\right)\sqrt{(t^4+7t^3+17t^2+21t+14)\,(t^2+t-6)}\Big\}. \tag{3.23}$$

Substituting (3.22) and (3.23) in (3.17) results in

$$\left(\frac{w}{w_1}\right)^4+\left(\frac{w_1}{w}\right)^4=t^6+10t^5+39t^4+80t^3+99t^2+70t+23.$$

This implies

$$\frac{w}{w_1}+\frac{w_1}{w}=\sqrt{t^3+5t^2+7t+7} \tag{3.24}$$

and

$$\frac{w}{w_1}-\frac{w_1}{w}=(t+1)\sqrt{t+3}. \tag{3.25}$$

We have considered the positive sign above as $\frac{w}{w_1}>\frac{w_1}{w}$, which follows from their series expansion

$$\frac{w}{w_1}=q^{-3/2}+q^{-1/2}+q^{1/2}+2q^{3/2}+2q^{7/2}+q^{9/2}-2q^{11/2}+\cdots,$$
$$\frac{w_1}{w}=q^{3/2}-q^{5/2}-q^{9/2}+2q^{11/2}-2q^{13/2}+2q^{15/2}-q^{17/2}+\cdots.$$

Multiplying identity (2.2) with the identity obtained replacing q by q^5 in (2.2) and then using (3.2) in it, we deduce

$$\frac{1}{ww_1}+7^2ww_1=t\left(t^2+t-5\right)\sqrt{t^3+5t^2+7t+7}. \tag{3.26}$$

This implies

$$\frac{1}{ww_1} - 7^2 ww_1 = \left(t^3+4t^2-7\right)\sqrt{(t-2)\left(t^2+t+2\right)}\ . \tag{3.27}$$

Multiplying identity (3.24) with (3.26) yields

$$\frac{1}{w^2} + 7^2w^2 + \frac{1}{{w_1}^2} + 7^2{w_1}^2 = t\left(t^2+t-5\right)\left(t^3+5t^2+7t+7\right). \tag{3.28}$$

Similarly, multiplication of identity (3.25) with (3.27) yields

$$-\frac{1}{w^2} - 7^2w^2 - \frac{1}{{w_1}^2} + 7^2{w_1}^2 = (t+1)\left(t^3+4t^2-7\right)\sqrt{(t+3)(t-2)\left(t^2+t+2\right)}\ . \tag{3.29}$$

Subtracting identity (3.29) from (3.28) and then subtracting 5 on both sides of the resulting identity yields

$$\begin{aligned}\frac{1}{w^2} + 49w^2 - 5 = \frac{1}{2}\Bigg\{&\left(t^3+3t+1\right)\left(t^4+3t^3-3t^2-5t-10\right)\\ &-(t+1)\left(t^3+4t^2-7\right)\sqrt{(t+3)(t-2)\left(t^2+t+2\right)}\Bigg\}\ .\end{aligned} \tag{3.30}$$

From (3.16) and (3.30) we obtain the required result. □

Theorem 3.2 *[13, p. 55] We have*

$$\frac{1}{v^3} + 125v^3 = \left\{\frac{1}{u^4} - u^4\right\} - \left\{\frac{1}{u^3} + u^3\right\} + 7\left\{\frac{1}{u^2} - u^2\right\} + 14\left\{\frac{1}{u} + u\right\}\ . \tag{3.31}$$

Proof Multiplying identity (2.3) with the identity obtained replacing q by q^7 in (2.3) and then using (3.2) in it, we deduce

$$\frac{1}{vv_1} + 25\ vv_1 = -4t^3 - 20t^2 - 28t - 20 + \left(\frac{v_1}{v}\right)^3 + \left(\frac{v}{v_1}\right)^3\ . \tag{3.32}$$

Using modular equations (2.4) and (2.5) in (2.6) we deduce the modular equation relating T and Q, where T and Q are as in (2.7). Transforming the obtained modular equation into theta functions, we deduce that

$$\left(\frac{v_1}{v}\right)^3 + \left(\frac{v}{v_1}\right)^3 = t^4 + 7t^3 + 17t^2 + 21t + 16\ . \tag{3.33}$$

Substituting (3.33) in (3.32), we get

$$\frac{1}{vv_1} + 25\ vv_1 = t^4 + 3t^3 - 3t^2 - 7t - 4\ .$$

This implies

$$\frac{1}{(vv_1)^{3/2}} + 125\ (vv_1)^{3/2} = (t-1)\left(t^4+3t^3-3t^2-7t-9\right)\sqrt{(t+3)(t+2)} \tag{3.34}$$

and

$$\frac{1}{(vv_1)^{3/2}} - 125\ (vv_1)^{3/2} = \left(t^4 + 3t^3 - 3t^2 - 7t + 1\right) \sqrt{(t-2)\,(t^3 + 5t^2 + 7t + 7)}\ . \tag{3.35}$$

We have considered the positive sign above as $\frac{1}{vv_1} > vv_1$, which follows from their series expression

$$\frac{1}{vv_1} = q^{-4} - q^{-3} - 2q^{-2} + q^{-1} + 1 + 3q - 2q^2 - 4q^3 + \cdots,$$

$$vv_1 = q^4 + q^5 + 3q^6 + 4q^7 + 7q^8 + 8q^9 + 14q^{10} + q^{11} + \cdots.$$

Identity (3.33) implies

$$\left(\frac{v}{v_1}\right)^{3/2} + \left(\frac{v_1}{v}\right)^{3/2} = (t+3)\sqrt{t^2 + t + 2} \tag{3.36}$$

and

$$\left(\frac{v}{v_1}\right)^{3/2} - \left(\frac{v_1}{v}\right)^{3/2} = \sqrt{(t+2)\,(t^3 + 5t^2 + 7t + 7)}\ . \tag{3.37}$$

We have considered the positive sign above as $\frac{v}{v_1} > \frac{v_1}{v}$, which follows from their series expansion

$$\frac{v}{v_1} = q^{-4/3} + q^{-1/3} + q^{2/3} + 2q^{5/3} - q^{11/3} - q^{14/3} - 2q^{17/3} + \cdots,$$

$$\frac{v_1}{v} = q^{4/3} - q^{7/3} - q^{13/3} + 2q^{16/3} - 3q^{25/3} + q^{28/3} + q^{31/3} \cdots.$$

Subtracting the identity obtained multiplying (3.35) with (3.37) from the identity obtained by multiplying (3.34) with (3.36) yields

$$\begin{aligned}\frac{1}{v^3} + 125v^3 =& \frac{1}{2}\Big\{ (t+3)\,(t-1)\left(t^4 + 3t^3 - 3t^2 - 7t - 9\right)\sqrt{(t+2)\,(t+3)\,(t^2+t+2)} \\ &- \left(t^4 + 3t^3 - 3t^2 - 7t + 1\right)\left(t^3 + 5t^2 + 7t + 7\right)\sqrt{t^2 - 4}\Big\}\ .\end{aligned} \tag{3.38}$$

Identity (3.2) implies

$$\left(\frac{u}{u_1}\right)^{1/2} + \left(\frac{u_1}{u}\right)^{1/2} = \sqrt{t+2} \tag{3.39}$$

and

$$\left(\frac{u}{u_1}\right)^{1/2} - \left(\frac{u_1}{u}\right)^{1/2} = \sqrt{t-2}\ . \tag{3.40}$$

We have considered the positive sign above, as $\frac{u}{u_1} > \frac{u_1}{u}$, which follows from their series expansion as explained in (3.7).

Identity (3.3) implies

$$\frac{1}{(uu_1)^{1/2}} + (uu_1)^{1/2} = \sqrt{(t+3)(t^2+t+2)} \tag{3.41}$$

and

$$\frac{1}{(uu_1)^{1/2}} - (uu_1)^{1/2} = (t+1)\sqrt{t+2}\ . \tag{3.42}$$

We have considered the positive sign above as $\frac{1}{uu_1} > uu_1$, which follows from their series expansion as explained in (3.4). Subtracting the identity obtained by multiplying (3.40) with (3.42) from the identity obtained by multiplying (3.39) with (3.41) yields

$$u + \frac{1}{u} = \frac{1}{2}\left\{\sqrt{(t+2)(t+3)(t^2+t+2)} - (t+1)\sqrt{(t^2-4)}\right\}\ . \tag{3.43}$$

In the same way, subtracting the identity obtained by multiplying (3.2) with (3.4) from the identity obtained by multiplying (3.3) with (3.7) yields

$$u^2 - \frac{1}{u^2} = \frac{1}{2}\left\{\left(t^3+4t^2+5t+4\right)\sqrt{t^2-4} - t(t+1)\sqrt{(t+2)(t+3)(t^2+t+2)}\right\}\ . \tag{3.44}$$

Similarly, subtracting the identity obtained by multiplying (3.4) with (3.7) from the identity obtained by multiplying (3.2) with (3.3) yields

$$u^2 + \frac{1}{u^2} = \frac{1}{2}\left\{t\left(t^3+4t^2+5t+4\right) - (t+1)(t+2)\sqrt{(t+3)(t-2)(t^2+t+2)}\right\}\ . \tag{3.45}$$

On the same lines, subtracting the identity obtained by multiplying (3.6) with (3.9) from the identity obtained by multiplying (3.5) with (3.8), we obtain

$$\begin{aligned} u^3 + \frac{1}{u^3} = \frac{1}{2}\Bigg\{ &(t-1)\left(t^3+4t^2+5t+3\right)\sqrt{(t+3)(t+2)(t^2+t+2)} \\ &- (t+1)^2\left(t^3+4t^2+5t+5\right)\sqrt{t^2-4}\Bigg\}\ . \end{aligned} \tag{3.46}$$

Making use of identities (3.43), (3.44), (3.45), and(3.46), we deduce

$$\begin{aligned} &\left\{\frac{1}{u^4} - u^4\right\} - \left\{\frac{1}{u^3} + u^3\right\} + 7\left\{\frac{1}{u^2} - u^2\right\} + 14\left\{\frac{1}{u} + u\right\} \\ &= \frac{1}{2}\Bigg\{(t+3)(t-1)\left(t^4+3t^3-3t^2-7t-9\right)\sqrt{(t+2)(t+3)(t^2+t+2)} \\ &\qquad - \left(t^4+3t^3-3t^2-7t+1\right)\left(t^3+5t^2+7t+7\right)\sqrt{t^2-4}\Bigg\}. \end{aligned} \tag{3.47}$$

From (3.38) and (3.47), we obtain the required result.

□

4. Modular equations

Theorem 4.1 *[2, p. 423] Let β, γ, and δ have degrees 5, 7, and 35, respectively. Let m and m' denote the multipliers connecting α, β, and γ, δ, respectively. Then*

$$\begin{aligned}(\alpha\delta)^{1/4}+\{(1-\alpha)(1-\delta)\}^{1/4}+2^{4/3}\{\alpha\delta(1-\alpha)(1-\delta)\}^{1/12}\\+(\beta\gamma)^{1/4}+\{(1-\beta)(1-\gamma)\}^{1/4}+2^{4/3}\{\beta\gamma(1-\beta)(1-\gamma)\}^{1/12}\\=1+\{1+2^{4/3}\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/24}\}^2\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\{(\alpha\delta)^{1/4}+\{(1-\alpha)(1-\delta)\}^{1/4}+2^{4/3}\{\alpha\delta(1-\alpha)(1-\delta)\}^{1/12}\}\\\times\{(\beta\gamma)^{1/4}+\{(1-\beta)(1-\gamma)\}^{1/4}+2^{4/3}\{\beta\gamma(1-\beta)(1-\gamma)\}^{1/12}\}\\=1-2^{7/3}\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/24}\\\times\{(\alpha\beta\gamma\delta)^{1/8}+\{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/8}\}\quad.\end{aligned} \tag{4.2}$$

Proof of (4.1) From Entry 42 [4, p. 379] we have

$$\begin{aligned}2\{256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/24}\\+\frac{2}{\{256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/24}}\\=\left\{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right\}^{1/12}+\left\{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right\}^{1/12}\\-\left\{\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right\}^{1/24}-\left\{\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right\}^{1/24}\quad.\end{aligned}$$

From Entry 12(v) [2, p. 124] the above identity can be written as

$$\begin{aligned}\frac{4q^2}{\chi(q)\chi(q^5)\chi(q^7)\chi(q^{35})}+\frac{\chi(q)\chi(q^5)\chi(q^7)\chi(q^{35})}{q^2}=\frac{q^2\chi^2(q^5)\chi^2(q^7)}{\chi^2(q)\chi^2(q^{35})}\\+\frac{\chi^2(q)\chi^2(q^{35})}{q^2\chi^2(q^5)\chi^2(q^7)}-\frac{q\chi(q^5)\chi(q^7)}{\chi(q)\chi(q^{35})}-\frac{\chi(q)\chi(q^{35})}{q\chi(q^5)\chi(q^7)}\quad.\end{aligned} \tag{4.3}$$

Replacing q by $-q$ in (4.3) we obtain

$$\begin{aligned}\frac{4q^2}{\chi(-q)\chi(-q^5)\chi(-q^7)\chi(-q^{35})}+\frac{\chi(-q)\chi(-q^5)\chi(-q^7)\chi(-q^{35})}{q^2}\\=\frac{q^2\chi^2(-q^5)\chi^2(-q^7)}{\chi^2(-q)\chi^2(-q^{35})}+\frac{\chi^2(-q)\chi^2(-q^{35})}{q^2\chi^2(-q^5)\chi^2(-q^7)}\\+\frac{q\chi(-q^5)\chi(-q^7)}{\chi(-q)\chi(-q^{35})}+\frac{\chi(-q)\chi(-q^{35})}{q\chi(-q^5)\chi(-q^7)}\quad.\end{aligned} \tag{4.4}$$

Let

$$X=\frac{1}{q^2}\chi(-q)\chi(-q^5)\chi(-q^7)\chi(-q^{35})\quad. \tag{4.5}$$

Making use of (1.2) and (4.5) in (4.4), we obtain

$$\frac{4}{X}+X=\left(\frac{u_1}{u}\right)^2+\left(\frac{u}{u_1}\right)^2+\frac{u_1}{u}+\frac{u}{u_1} \quad .$$

The above identity can be simplified as

$$\left(\frac{2}{X}\right)^2-y\left(\frac{X}{2}\right)+1=0, \tag{4.6}$$

where

$$y=\frac{1}{2}\left\{\left(\frac{u_1}{u}\right)^2+\left(\frac{u}{u_1}\right)^2+\frac{u_1}{u}+\frac{u}{u_1}\right\}.$$

Solving (4.6), we obtain

$$\frac{4}{X}=y-\sqrt{y^2-4} \quad . \tag{4.7}$$

By the definition of X, we can see that $X > 2$ when q approaches 0. Hence, we select the negative sign above. Consider

$$y^2-4=\frac{1}{4}\left\{\left(\frac{u_1}{u}\right)^4+\left(\frac{u}{u_1}\right)^4+2\left(\frac{u_1}{u}\right)^3+2\left(\frac{u}{u_1}\right)^3+\left(\frac{u_1}{u}\right)^2+\left(\frac{u}{u_1}\right)^2+2\frac{u_1}{u}+2\frac{u}{u_1}\right\} \quad .$$

y^2-4 can be written as the sum of two terms:

$$y^2-4=A\left(u,u_1\right)+B\left(u_1,u\right),$$

where

$$\begin{aligned} A\left(u,u_1\right)=u^{-3}{u_1}^{-3}&\left(u^2{u_1}^2+u^3+2u^2u_1+2u{u_1}^2+{u_1}^3-uu_1\right)\\ &\times\left(u^3u_1+u^3+2u^2u_1+2u{u_1}^2+{u_1}^3-{u_1}^2\right)\end{aligned}$$

and

$$B\left(u_1,u\right)=\frac{1}{4}\left\{\left(\frac{u_1}{u}\right)^2+\left(\frac{u}{u_1}\right)^2+3\frac{u_1}{u}+3\frac{u}{u_1}+4+2u-\frac{2}{u}\right\}^2 \quad .$$

Identity (2.14) of [14] shows that $A\left(P,Q\right)$ is zero. Hence, we have

$$y^2-4=\frac{1}{4}\left\{\left(\frac{u_1}{u}\right)^2+\left(\frac{u}{u_1}\right)^2+3\frac{u_1}{u}+3\frac{u}{u_1}+4+2u-\frac{2}{u}\right\}^2 \quad . \tag{4.8}$$

Substituting (4.8) in (4.7) and simplification results in

$$\frac{4}{X}+2+\frac{u_1}{u}\left(1+\frac{u^2}{u_1}\right)+\frac{u}{u_1}\left(1-\frac{u_1}{u^2}\right)=0 \quad . \tag{4.9}$$

Making use of (1.2) and (4.5), we write (4.9) as

$$\frac{4q^2}{\chi(-q)\chi(-q^5)\chi(-q^7)\chi(-q^{35})}+\frac{q\chi(-q^5)\chi(-q^7)}{\chi(-q)\chi(-q^{35})}\left\{1+\frac{\varphi(-q)\varphi(-q^{35})}{\varphi(-q^5)\varphi(-q^7)}\right\}$$
$$+2+\frac{\chi(-q)\chi(-q^{35})}{q\chi(-q^5)\chi(-q^7)}\left\{1-\frac{\varphi(-q^5)\varphi(-q^7)}{\varphi(-q)\varphi(-q^{35})}\right\}=0. \tag{4.10}$$

Multiplying (4.10) with $\frac{q^2}{\chi(-q)\chi(-q^5)\chi(-q^7)\chi(-q^{35})}$ and then replacing q by $-q$ in the resulting identity yields us

$$\frac{4q^4}{\chi^2(q)\chi^2(q^5)\chi^2(q^7)\chi^2(q^{35})}+\frac{2q^2}{\chi(q)\chi(q^5)\chi(q^7)\chi(q^{35})}=$$
$$\frac{q^3}{\chi^2(q)\chi^2(q^{35})}\left\{1+\frac{\varphi(q)\varphi(q^{35})}{\varphi(q^5)\varphi(q^7)}\right\}+\frac{q}{q\chi^2(q^5)\chi^2(q^7)}\left\{1-\frac{\varphi(q^5)\varphi(q^7)}{\varphi(q)\varphi(q^{35})}\right\}.$$

Using Entry 10(i)[2, p.122] and Entry 12(v)[2, p. 124], the above can be written as

$$2^{8/3}\{\alpha\beta\gamma\delta\,(1-\alpha)\,(1-\beta)\,(1-\gamma)\,(1-\delta)\}^{1/12}$$
$$+2^{7/3}\{\alpha\beta\gamma\delta\,(1-\alpha)\,(1-\beta)\,(1-\gamma)\,(1-\delta)\}^{1/24}$$
$$=2^{4/3}\{\alpha\delta\,(1-\alpha)\,(1-\delta)\}^{1/12}\left\{1+\sqrt{\frac{m}{m'}}\right\}$$
$$+2^{4/3}\{\beta\gamma\,(1-\beta)\,(1-\gamma)\}^{1/12}\left\{1+\sqrt{\frac{m'}{m}}\right\}.$$

Making use of elementary algebra, the above can be rewritten as

$$1+\left\{1+2^{4/3}\{\alpha\beta\gamma\delta\,(1-\alpha)\,(1-\beta)\,(1-\gamma)\}^{1/24}\right\}^2-$$
$$2^{4/3}\{\alpha\delta\,(1-\alpha)\,(1-\delta)\}^{1/12}-2^{4/3}\{\beta\gamma\,(1-\beta)\,(1-\gamma)\}^{1/12}=2+$$
$$2^{4/3}\{\alpha\delta\,(1-\alpha)\,(1-\delta)\}^{1/12}\sqrt{\frac{m}{m'}}-2^{4/3}\{\beta\gamma\,(1-\beta)\,(1-\gamma)\}^{1/12}\sqrt{\frac{m'}{m}}. \tag{4.11}$$

Addition of Entry 18(iii) and (iv) [2, p. 423] results in

$$(\alpha\delta)^{1/4}+\{(1-\alpha)\,(1-\delta)\}^{1/4}+(\beta\gamma)^{1/4}+\{(1-\beta)\,(1-\gamma)\}^{1/4}=2$$
$$+2^{4/3}\{\alpha\delta\,(1-\alpha)\,(1-\delta)\}^{1/12}\sqrt{\frac{m}{m'}}-2^{4/3}\{\beta\gamma\,(1-\beta)\,(1-\gamma)\}^{1/12}\sqrt{\frac{m'}{m}}. \tag{4.12}$$

Comparing (4.11) and (4.12), we obtain the required result. □

Proof of (4.2) Let

$$A=(\alpha\delta)^{1/24}\quad,\quad B=(\beta\gamma)^{1/24}\quad,\quad M=\sqrt{\frac{m}{m'}},$$
$$A'=\{(1-\alpha)\,(1-\delta)\}^{1/24},\quad \textit{and}\quad B'=\{(1-\beta)\,(1-\gamma)\}^{1/24}. \tag{4.13}$$

Using (4.13) in (4.2), we obtain

$$\begin{aligned}(AB)^6+(AB')^6+(A'B)^6+(A'B')^6+2^{4/3}(AA')^2B^6+2^{4/3}(AA')^2B'^6\\+2^{4/3}(BB')^2A^6+2^{4/3}(BB')^2A'^6+2^{4/3}(AA'BB')^2\\=1-2^{7/3}AA'BB'\left\{(AB)^3+(A'B')^3\right\}\quad.\end{aligned}\tag{4.14}$$

With multiplication of Entry 18(iii) with Entry 18(iv) and then using (4.13), we obtain

$$\begin{aligned}(AB)^6+(AB')^6+(A'B)^6+(A'B')^6-A^6-A'^6-B^6-B'^6+1\\=-2^{8/3}(AA'BB')^2\quad.\end{aligned}\tag{4.15}$$

Rearranging the terms of (4.1) gives us

$$\begin{aligned}-A^6-A'^6-B^6-B'^6\quad&=\quad 2^{4/3}(AA')^2+2^{4/3}(BB')^2-2\\&-2^{8/3}(AA'BB')^2-2^{7/3}(AA'BB')\quad.\end{aligned}$$

Substituting the above in (4.15) yields

$$\begin{aligned}(AB)^6+(AB')^6+(A'B)^6+(A'B')^6=&1+2^{7/3}AA'BB'\\&-2^{4/3}(AA')^2-2^{4/3}(BB')^2\quad.\end{aligned}\tag{4.16}$$

Substituting (4.16) in (4.14), we find

$$\begin{aligned}2^{4/3}(AA')^2\left(B^6+B'^6-1\right)+2^{4/3}(BB')^2\left(A^6+A'^6-1\right)\\+2^{8/3}(AA'BB')^2+2^{7/3}AA'BB'\left\{1+(AB)^3+(A'B')^3\right\}=0\quad.\end{aligned}$$

Using Entry 18 (iii), (iv), and (v) [2, p. 423], the above identity can be written as

$$\begin{aligned}2^{1/3}(AA')^4\left\{\frac{2^{1/2}(BB')^3-2^{1/6}AA'}{2^{1/2}(AA')^3+2^{1/6}AA'}\right\}-2^{1/3}(BB')^4\left\{\frac{2^{1/6}BB'-2^{1/6}(AA')^3}{2^{1/2}BB'+2^{1/6}(BB')^3}\right\}\\+2^{1/3}(AA'BB')^2+AA'BB'\left\{1+(AB)^3+(A'B')^3\right\}=0\quad.\end{aligned}\tag{4.17}$$

We get from (2.7) and (4.13)

$$P^2=2^{1/3}AA'BB'\qquad \textit{and}\qquad Q^2=\frac{AA'}{BB'}\quad.\tag{4.18}$$

Simplifying (4.17) using elementary algebra and then substituting (4.18) results in

$$\begin{aligned}\frac{2P^2+(P^4-P^2)\left(Q^2+\frac{1}{Q^2}\right)-\left(Q^4+\frac{1}{Q^4}\right)}{P^2+\frac{1}{P^2}+Q^2+\frac{1}{Q^2}}+P^2+1+(AB)^3\\+(A'B')^3=0\quad.\end{aligned}\tag{4.19}$$

From Entry 19(i) [2, p. 314], we have

$$(\alpha\gamma)^{1/8}+\{(1-\alpha)(1-\gamma)\}^{1/8}=1 \quad , \tag{4.20}$$

$$(\beta\delta)^{1/8}+\{(1-\beta)(1-\delta)\}^{1/8}=1 \quad . \tag{4.21}$$

Identities (4.20) and (4.21) respectively imply

$$(\alpha\gamma)^{1/8}-\{(1-\alpha)(1-\gamma)\}^{1/8}=\sqrt{1-4\{\alpha\gamma(1-\alpha)(1-\gamma)\}^{1/8}} \quad , \tag{4.22}$$

$$(\beta\delta)^{1/8}-\{(1-\beta)(1-\delta)\}^{1/8}=\sqrt{1-4\{\beta\delta(1-\beta)(1-\delta)\}^{1/8}} \quad . \tag{4.23}$$

Adding the identity obtained by multiplying (4.20) with (4.21) with the identity obtained by multiplying (4.22) with (4.23), we have

$$(AB)^3+(A'B')^3=\frac{1}{2}\left\{1+\sqrt{1-2^{3/2}P^3\left(T^3+\frac{1}{T^3}\right)+8P^6}\right\} \quad . \tag{4.24}$$

Substituting (4.24) in (4.19) results in

$$2P^2\left\{\frac{P^4Q^6-P^2Q^6-Q^8+P^4Q^2-P^2Q^4-P^2Q^2-1}{Q^2\left(P^2+Q^2\right)\left(P^2Q^2+1\right)}\right\}+2P^3+3$$
$$=-\sqrt{1-2^{3/2}P^3\left(T^3+\frac{1}{T^3}\right)+8P^6} \quad .$$

Squaring on both sides of the above identity and then rearranging the terms yields

$$\left[2P^2\left\{\frac{P^4Q^6-P^2Q^6-Q^8+P^4Q^2-P^2Q^4-P^2Q^2-1}{Q^2\left(P^2+Q^2\right)\left(P^2Q^2+1\right)}\right\}+2P^3+3\right]^2$$
$$-1-8P^6+2P^3\sqrt{2}(T^3+\frac{1}{T^3})=0 \quad . \tag{4.25}$$

Making use of (2.4), (2.5), and (2.6), we obtain T^3+T^{-3} in terms of P and Q

$$T^3+\frac{1}{T^3}=\frac{P\left(Q^{12}-4Q^{10}+8Q^8-14Q^6+8Q^4-4Q^2+1\right)}{Q^6\left(P^2+1\right)} \quad . \tag{4.26}$$

Substituting (4.26) in (4.25) results in

$$M\left(P,Q\right)\ N\left(P,Q\right)=0, \tag{4.27}$$

where

$$M\left(P,Q\right)=-P^2Q^8+2P^4Q^4+P^2Q^6+P^2Q^2+2Q^4-P^2$$

and

$$\begin{aligned}N(P,Q) &= 2P^{12}Q^{6} + P^{10}Q^{8} + 2P^{8}Q^{10} + P^{6}Q^{12} - 8P^{8}Q^{8} - 3P^{6}Q^{10} \\ &+ P^{10}Q^{4} + 2P^{8}Q^{6} + 5P^{6}Q^{8} + 4P^{4}Q^{10} - 8P^{8}Q^{4} - 24P^{6}Q^{6} \\ &- 8P^{4}Q^{8} + 2P^{2}Q^{10} + 2P^{8}Q^{2} + 5P^{6}Q^{4} - 6P^{4}Q^{6} - 3P^{2}Q^{8} + P^{6} \\ &- 3P^{6}Q^{2} - 8P^{4}Q^{4} - 5P^{2}Q^{6} + 4P^{4}Q^{2} - 3P^{2}Q^{4} - 2Q^{6} + 2P^{2}Q^{2}.\end{aligned}$$

By (2.4), it follows that $M(P,Q) = 0$. This implies that (4.27) holds and equivalently (4.2) holds. □

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