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Construction of the holonomy invariant foliated cocycles on the tangent bundle via formal integrability

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Abstract: This paper is dedicated to exhaustive structural analysis of the holonomy invariant foliated cocycles on the tangent bundle of an arbitrary $(m+n)$-dimensional manifold. For this purpose, by applying Spencer theory of formal integrability, sufficient conditions for the metric associated with the semispray S are determined to extend to a transverse metric for the lifted foliated cocycle on TM. Accordingly, this geometric structure converts to a holonomy invariant foliated cocycle on the tangent space, which is totally adapted to the Helmholtz conditions.

Key words: Foliated cocycle, holonomy group, metrizability, formal integrability, transverse metric

1. Introduction

Differential geometry of the total space of a manifold's tangent bundle has its origins in diverse fields of study such as calculus of variations, differential equations, theoretical physics, and mechanics. In recent years, it can be regarded as a distinguished domain of differential geometry and has noteworthy applications in specific problems of mathematical biology and mainly in the theory of physical fields [3–5, 25–27]. This significance provides a constructive setting for the development of novel notions and geometric structures such as systems of second-order differential equations (SODEs), metric structures, semisprays, and nonlinear connections. Accordingly, analysis of the above-mentioned concepts can be considered as a powerful tool for the thorough investigation of the geometric properties of a tangent bundle.

From a historical point of view, principled investigation of the differential geometry of tangent bundles started with Dombrowski [16], Kobayashi and Nomiza [20], and Yano and Ishihara [36] in the 1960s and 1970s. Specifically, Crampin [11] and Grifone [17] considerably contributed to the geometry of the tangent bundle by introducing the notion of the nonlinear connection on the tangent bundle of a system of SODEs. In [25] Miron introduced and investigated the concept of generalized Lagrange spaces. Moreover, regarding covariant derivatives and geometric objects that can be associated to a system of SODEs, comprehensive research was undertaken in [2, 13, 21, 23, 34] (refer to [9] for more details).

In the last decades increasing numbers of studies have been dedicated to the qualitative investigation of the solutions of systems of (non-)autonomous second (higher)-order ordinary (partial) differential equation fields via some corresponding geometric structures. The notable fact regarding these investigations is the significant demand for a unifying geometric setting for a differential equation field considering the associated

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geometric structures and invariants. The inverse problem of the calculus of variations is fundamentally based on the following notable question: what are the conditions under which the solutions of a typical SODE on an m-dimensional manifold as the configuration space with the local coordinates x^i

$$\frac{d^2x^i}{dt^2} + 2G^i(x,\dot{x}) = 0, \qquad i \in \{1,2,\cdots,m\}, \tag{1.1}$$

can be regarded as the solutions of the associated Euler-Lagrange equations for some Lagrangian function L

$$\frac{d}{dt}\left(\frac{\partial \mathrm{L}}{\partial x^i}\right) - \frac{\partial \mathrm{L}}{\partial x^i} = 0, \qquad i \in \{1,2,\cdots,m\}. \tag{1.2}$$

In addition, system (1.1) can be totally characterized via a second-order vector field on the tangent bundle TM denoted by semispray. Moreover, TM is considered as the velocity space with local coordinates x^i, $y^i := \dot{x}^i$.

$$\mathcal{S} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}. \tag{1.3}$$

Helmholtz conditions can be regarded as one of the significant points of view to the inverse problem of the calculus of variations and are fundamentally based on the necessary and sufficient conditions for the existence of a multiplier matrix $g_{ij}(x,\dot{x})$ such that for some Lagrangian L the following identity holds:

$$g_{ij}(x,\dot{x})\left(\frac{d^2x^j}{dt^2} + 2G^i(x,\dot{x})\right) = \frac{d}{dt}\left(\frac{\partial \mathrm{L}}{\partial \dot{x}^i}\right) - \frac{\partial \mathrm{L}}{\partial x^i}. \tag{1.4}$$

Note that in this case the semispray $\mathcal{S}$ is denoted by variational or a Lagrangian vector field. Furthermore, the system (1.4) can be thoroughly reformulated as follows:

$$\mathcal{L}_{\mathcal{S}}\left(\frac{\partial \mathrm{L}}{\partial \dot{x}^i} dx^i\right) = d\mathrm{L}, \tag{1.5}$$

where $\mathcal{L}_{\mathcal{S}}$ is the Lie derivative with respect to semispray $\mathcal{S}$. Meanwhile, for the multiplier matrix g_{ij}, the Helmholtz conditions are illustrated by:

$$g_{ij} = g_{ji}, \qquad \frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^j}, \tag{1.6}$$

$$\nabla g_{ij} = 0, \qquad g_{ik}R^k_j = g_{jk}R^k_i. \tag{1.7}$$

It is worth mentioning that conditions (1.6) can be regarded as the necessary and sufficient conditions for the existence of a Lagrange function that is defined locally and has as its Hessian the multiplier matrix g_{ij}. Likewise, conditions (1.7) demonstrate the compatibility between the given SODE structure and the multiplier matrix via some related induced geometric structures such as the Jacobi endomorphism R^i_j and the dynamical covariant derivative ∇.

The problem of metrizability has been investigated from several aspects in recent years. Indeed, a semispray is called metrizable if the paths of the semispray are just the geodesics of some metric space. The problem of compatibility between a system of SODEs and a metric structure on a tangent bundle has been studied by many authors and it is known as one of the Helmholtz conditions from the inverse problem of Lagrangian mechanics [1, 8, 12, 14, 15, 18, 22, 34]. The noticeable fact is that Helmholtz conditions can be totally reformulated in terms of some regular and linear partial differential operators by applying Frölicher–Nijenhuis theory as a powerful tool [10]. As a consequence, the formal integrability of the declared differential operators can be exhaustively addressed via Spencer theory (refer to [10] for more complete details). In this paper, taking into account [10], sufficient conditions for the metric associated with the semispray S are determined to extend to a transverse metric for the lifted foliated cocycle on TM.

In mathematics, foliation theory can be regarded as a powerful geometric device that is applied in order to study manifolds, consisting of an integrable subbundle of the tangent bundle. In other words, a foliation locally looks like a decomposition of the manifold as a union of parallel submanifolds of smaller dimension. Such foliations of manifolds occur naturally in various geometric fields, such as solutions of differential equations and integrable systems or in differential topology. In 1959 Reinhart introduced a particular type of foliations constructed via a particular geometric structure called a metric foliated cocycle [32]. When for a given foliation there exists a metric g on M that is transverse (or bundle-like) for $\mathcal{F}$, we say that $(M, g, \mathcal{F})$ is a metric foliated cocycle. This notion is quite intuitive. In other words, the existence of this geometric structure leads to the creation of a particular metric for which the leaves of the foliation remain locally at constant distance from each other. Indeed, from another point of view, the theory of foliations has a close relationship to that of differential equations. For example, a nowhere vanishing vector field X on a manifold M is totally equivalent to an oriented one-dimensional foliation. Therefore, solutions of a system of ordinary differential equations are the integral curves of X. According to the Frobenius theorem, foliations of higher dimensions associate with systems of partial differential equations, so the foliation is said to be metric whenever the solutions (or leaves) are locally equidistant. If, in addition, the space of leaves $\mathbf{L}$ is well behaved, the map $M \longrightarrow \mathbf{L}$ that projects a point of the manifold M to the leaf on which it lies is denoted by a metric fibration.

Consequently, a metric foliation $\mathcal{F}$ of M can be considered as a decomposition of M into connected subsets, known as leaves, which are locally equidistant; namely, for any $p \in M$ there exist neighborhoods $\mathcal{V} \subset \mathcal{U}$ of p such that the following holds: given two arbitrary leaves $\mathbf{L}_i$ and connected components $\mathcal{B}_i$ of $\mathbf{L}_i \cap \mathcal{U}, \quad i = 1, 2$, the distance function $q \longrightarrow d(q, \mathcal{B}_1)$ is constant on $\mathcal{B}_2 \cap \mathcal{V}$. This class of foliations can be regarded as a good candidate and a significant device for modeling situations drawn from mechanics and physics and particularly plays a fundamental role in generalizing Riemannian or Finsler geometry to foliated manifolds. Consequently, metric foliated cocycles form a natural class of foliations that is worth investigating from different aspects [6, 19, 28, 29, 35].

In the theory of foliations the notion of holonomy is closely related to the existence of a transverse metric structure on the foliation. For a given point x on a manifold equipped with a foliation of codimension n, one may consider how the leaves near that point intersect a small n-dimensional disk that is transversal to the leaves and includes the mentioned point. The ways in which these leaves depart from this disk and return to it are identified in a group, denoted by the holonomy group at x. In the case of metric foliations, assume that $\pi : M \longrightarrow \mathbf{L}$ is the corresponding metric fibration and $\gamma : [0, 1] \longrightarrow \mathbf{L}$ is a piecewise smooth curve in the base space. Then the map $\mathbb{H}_\gamma : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(1))$ among the fibers over the endpoints of the curve γ is denoted by the holonomy diffeomorphism associated to γ. It is worth noticing that the map declared above projects a point p in the first fiber to the endpoint of the horizontal lift of γ that originates at point p.

In [30, 31] Popescu et al. defined the notion of the Lagrangian adopted to the lifted foliation and in [24] Riemannian foliations that are compatible with SODE structure were discussed. In this paper, we have comprehensively analyzed the structure of the holonomy invariant foliated cocycles on the tangent bundle of an arbitrary manifold via the notion of formal integrability, which was first introduced by Bucataru and Muzsnay in [10] and is fundamentally based on Frölicher–Nijenhuis formalism and is extensively fruitful since it provides a noteworthy setting to apply Spencer theory in order to investigate the formal integrability of Helmholtz conditions. This reformulation of the inverse problem of the calculus of variations enables us to apply Spencer theory in order to construct a transverse metric on the tangent bundle, which leads to the creation of the holonomy invariant foliated cocycles. The structure of the present paper is as follows: in Section 2, according to [10], a brief discussion regarding the reformulation of the Helmholtz conditions in terms of a formal integrability of a partial differential operator is presented. Section 3 is devoted to geometric investigation of metric foliated cocycles via the concept of holonomy groups. In Section 4, a thorough analysis of the holonomy invariant foliated cocycles via the concept of formal integrability is presented. Some concluding remarks are mentioned at the end of the paper.

2. Reformulation of the Helmholtz conditions via formal integrability

Let M be an m-dimensional manifold and (TM, π, M) denote its tangent bundle with local coordinates (x^i, y^i) and VTM the corresponding vertical subbundle. The tangent structure $\mathcal{J}$ is locally expressed by $\mathcal{J} = \dfrac{\partial}{\partial y^i} \otimes dx^i$ and the vector field $\mathbb{C} \in \Gamma(TTM)$ defined by $\mathbb{C} = y^i \dfrac{\partial}{\partial y^i}$ is called the Liouville vector field. In addition, a k-form ω is called semibasic if $\omega(X_1, X_2, \cdots, X_k) = 0$ whenever one of the vector fields X_i is vertical for $i \in \{1, \cdots, k\}$. Moreover, the module of semibasic k-forms is denoted by $\mathrm{Sec}(\Lambda^k T_V^*)$. Also, a vector valued k-form A on $TM\backslash\{0\}$ is said to be semi-basic if it takes values in the vertical bundle and specifically when one of the vectors X_i, $i \in \{1, \cdots, k\}$ is vertical the following relation holds: $A(X_1, X_2, \cdots, X_k) = 0$.

Hence, according to Frölicher–Nijenhuis theory a semispray (spray) on M is a vector field $\mathcal{S} \in \Gamma(TTM\backslash\{0\})$ such that $\mathcal{J}\mathcal{S} = \mathbb{C}$ (and $[\mathbb{C}, \mathcal{S}] = \mathcal{S}$). Now consider the almost tangent structure $\mathbb{P} = -\mathcal{L}_{\mathcal{S}}\mathcal{J} = h - v$ where h and v are the horizontal and vertical projectors induced by $\mathcal{S}$, respectively. Then the Jacobi endomorphism (or Douglas tensor) Φ is defined as the following $(1,1)$-type tensor field:

$$\Phi = v \circ \mathcal{L}_{\mathcal{S}} h = -v \circ \mathcal{L}_{\mathcal{S}} v = R^i_j \frac{\partial}{\partial y^i} \otimes dx^j. \tag{2.1}$$

The dynamical covariant derivative ∇ is defined by:

$$\begin{aligned} &\nabla X = h[\mathcal{S}, hX] + v[\mathcal{S}, vX], \quad \forall\ X \in \Gamma(TTM\backslash\{0\}) \\ &\nabla = \mathcal{L}_{\mathcal{S}} + h \circ \mathcal{L}_{\mathcal{S}} h + v \circ \mathcal{L}_{\mathcal{S}} v = \mathcal{L}_{\mathcal{S}} + \Psi. \end{aligned} \tag{2.2}$$

Taking into account that ∇ is a zero-degree derivation on $\Lambda^k(TM\backslash\{0\})$, it can be uniquely decomposed into the sum of a Lie derivation $\mathcal{L}_{\mathcal{S}}$ and an algebraic derivation i_Ψ as follows: $\nabla = \mathcal{L}_{\mathcal{S}} - i_\Psi$. According to [10] the following significant relations hold:

$$\begin{aligned} &(a): \ \nabla\mathcal{S} = 0, \quad \nabla\mathbb{C} = 0, \quad \nabla i_{\mathcal{S}} = i_{\mathcal{S}}\nabla, \quad \nabla i_{\mathbb{C}} = i_{\mathbb{C}}\nabla; \\ &(b): \ \nabla h = 0, \quad \nabla v = 0, \quad \nabla\mathcal{J} = 0, \quad \nabla\mathbb{F} = 0; \\ &(c): \ d\nabla - \nabla d = d_\Psi, \quad \nabla i_h = i_h\nabla = 0, \quad \nabla i_{\mathcal{J}} - i_{\mathcal{J}}\nabla = 0. \end{aligned} \tag{2.3}$$

According to (2.3), a semispray $\mathcal{S}$ on M is called a Lagrangian vector field if there exist $\mathrm{L} \in C^\infty(TM\backslash\{0\})$ such that $\mathcal{L}_{\mathcal{S}} d_{\mathcal{J}} \mathrm{L} = d\mathrm{L}$. Mainly due to [10] we have the following:

Theorem 2.1 *A semispray $\mathcal{S}$ is a Lagrangian vector field if and only if there exists a semibasic 1-form Θ on $TM\backslash\{0\}$ that satisfies the following reformulations of the Helmholtz conditions:*

$$d_h\Theta = 0, \quad d_{\mathcal{J}}\Theta = 0, \quad \nabla d\Theta = 0, \quad d_{\Phi}\Theta = 0. \tag{2.4}$$

Consequently, the semispray $\mathcal{S}$ is a Lagrangian vector field if and only if the following partial differential operator is formally integrable:

$$\mathcal{P}_{\mathrm{L}} = \big(d_{\mathcal{J}}, d_h, d_{\Phi}, \nabla d\big): \ Sec(T_v^*) \longrightarrow Sec\big(\oplus^{(4)} \Lambda^2 T_v^*\big). \tag{2.5}$$

Overall, considering the above discussion, a spray $\mathcal{S}$ is projectively metrizable if there exists a 1-homogeneous function $F \in C^{\infty}(TM\backslash\{0\})$ such that $\mathcal{L}_{\mathcal{S}} d_{\mathcal{J}} F = dF$; moreover, a spray $\mathcal{S}$ is Finsler metrizable if there exists a 2-homogeneous function $\mathrm{L} \in C^{\infty}(TM)\backslash\{0\}$ such that $\mathcal{L}_{\mathcal{S}} d_{\mathcal{J}} \mathrm{L} = \mathrm{L}$. Equivalently, related to the notion of projective metrizability from the Frölicher–Nijenhuis theory approach, we have [10]:

Proposition 2.2 *A spray $\mathcal{S}$ is projectively metrizable if and only if there exists a semibasic 1-form Θ on $TM\backslash\{0\}$ such that the following identities satisfied:*

$$\mathcal{L}_{\mathbb{C}}\Theta = 0, \quad d_{\mathcal{J}}\Theta = 0, \quad d_h\Theta = 0. \tag{2.6}$$

It is noticeable that in order to address the formal integrability of the partial differential system (2.6) the following first-order partial differential operator should be reckoned:

$$\mathcal{P}_1 = \big(\mathcal{L}_{\mathbb{C}}, d_{\mathcal{J}}, d_h\big): \ \mathrm{Sec}(T_v^*) \longrightarrow \big(T_v^* \oplus \Lambda^2 T_v^* \oplus \Lambda^2 T_v^*\big). \tag{2.7}$$

Furthermore, $\mathcal{P}_1$ induces the following morphism of vector bundles defined by:

$$p_0(\mathcal{P}_1): \ J_1 T_v^* \longrightarrow F := T_v^* \oplus \Lambda^2 T_v^* \oplus \Lambda^2 T_v^*. \tag{2.8}$$

Subsequently, the lth-order jet prolongation is expressed by: $p_l(\mathcal{P}_1) : \ J_{l+1} T_v^* \longrightarrow J_l F$. Meanwhile, for $\vartheta \in TM$, $R_{l+1,\vartheta} = \mathrm{Ker}(P_1(\mathcal{P}_1)_{\vartheta}) \subset J_{l+1,\vartheta} T_v^*$ can be regarded as the space of solutions of order $(l+1)$ of $\mathcal{P}_1$ at ϑ. In addition $\mathcal{P}_1$ is formally integrable at ϑ if R_l is a vector bundle and for all $l \geq 1$, $\bar{\pi}_{l,\vartheta} : \ R_{l+1,\vartheta} \longrightarrow R_{l,\vartheta}$ is onto. Consequently, in the analytic case the concept of formal integrability implies the existence of a locally defined section in R_1 and therefore a solution of (2.6). Overall, according to the Cartan–Kähler theorem, if $\bar{\pi}_1 : \ R_2 \longrightarrow R_1$ is onto and the symbol $\varsigma_2(\mathcal{P}_1)$ is involutive then $\mathcal{P}_1$ can be regarded as a formally integrable partial differential operator. Note the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & g_2(\mathcal{P}_1) \arrow[r, "i"] \arrow[d] & \mathcal{S}^2 T^* \otimes T_v^* \arrow[r, "\varsigma_2"] \arrow[d] & T^* \otimes F \arrow[r, "\tau"] \arrow[d] & \mathcal{K} \arrow[r] & 0 \\
0 \arrow[r] & R_2 \arrow[r, "i"] \arrow[d, "\bar{\pi}_1"] & \mathcal{J}_2 T_v^* \arrow[r, "p_1(\mathcal{P}_1)"] \arrow[d, "\pi_1"] & \mathcal{J}_1 F \arrow[d, "\pi"] & & \\
0 \arrow[r] & R_1 \arrow[r, "i"] & \mathcal{J}_1 T_v^* \arrow[r, "p_0(\mathcal{P}_1)"] \arrow[d] & F \arrow[d] & & \\
 & & 0 & 0 & &
\end{tikzcd}$$

In order to determine the surjectivity of $\bar{\pi}_1 : \ R_2 \longrightarrow R_1$ it is required to investigate the following map: $\tau : \ T^* \otimes \big(T_v^* \oplus \Lambda^2 T_v^* \oplus \Lambda^2 T_v^*\big) \longrightarrow \mathcal{K}$, where $\mathcal{K} = \mathrm{Coker}(\varsigma_2(\mathcal{P}_1))$. Since $\dim \mathcal{K} = \dfrac{m^2(m-1)}{2}$, it is deduced that $\mathcal{K} \cong \oplus^{(2)} \Lambda^2 T_v^* \oplus^{(3)} \Lambda^3 T_v^*$. The five main components of the map τ are given by:

$$\begin{aligned} &\tau_1\big(A, B_1, B_2\big) = \tau_{\mathcal{J}} A - i_{\mathbb{C}} B_1; \quad \tau_2\big(A, B_1, B_2\big) = \tau_{\mathcal{J}} A - i_{\mathbb{C}} B_2;\\ &\tau_3\big(A, B_1, B_2\big) = \tau_{\mathcal{J}} B_1; \quad \tau_4\big(A, B_1, B_2\big) = \tau_h B_2;\\ &\tau_5\big(A, B_1, B_2\big) = \tau_h B_1 + \tau_{\mathcal{J}} B_2. \end{aligned} \tag{2.9}$$

Additionally, $\varsigma_2(\mathcal{P}_1)$ is involutive; namely, it admits a quasiregular basis that specifically satisfies Cartan's test if and only if all the groups of the Spencer cohomology vanish (refer to [10] for more complete explanations).

Consequently, a first-order formal solution $\Theta \in \Lambda^1 T_v^*$ of system (2.6) can be lifted into a second-order solution (in other words, $\bar{\pi} : \ R_2 \longrightarrow R_1$ is onto) if and only if $d_R \Theta = 0$ where R is the curvature tensor defined by $R = \dfrac{1}{2}[h, h] = -\dfrac{1}{3}[\mathcal{J}, \Phi]$.

Overall, the following three specific cases can be regarded as the significant cases when system (2.6) is formally integrable and subsequently the corresponding spray $\mathcal{S}$ is projectively metrizable [10]: (i) Flat case $R = 0$; (ii) Isotropic case $R = \Omega \wedge \mathcal{J}$ for Ω, a semibasic 1-form; (iii) Any spray on a two-dimensional manifold is projectively metrizable.

3. Holonomy groups and metric foliated cocycles

Let M be a manifold of dimension $m+n$. Then a foliation $\mathcal{F}$ of codimension n is defined via an open cover $\mathcal{U} = \{\mathcal{U}_i\}_{i \in I}$, and for each i, a local diffeomorphism $\Psi : \ \mathbb{R}^{m+n} \longrightarrow \mathcal{U}_i$ such that on each nonempty $\mathcal{U}_i \cap \mathcal{U}_j$ the following change of coordinates occurs:

$$\Psi_j^{-1} \circ \Psi_i : \ (x, y) \in \Psi_i^{-1}(\mathcal{U}_i \cap \mathcal{U}_j) \longrightarrow (\tilde{x}, \tilde{y}) \in \Psi_j^{-1}(\mathcal{U}_i \cap \mathcal{U}_j),$$

where $\tilde{x} = \Psi_{ij}(x, y)$ and $\tilde{y} = \zeta_{ij}(y)$. Consequently, the manifold M is decomposed into m-dimensional connected submanifolds, which are denoted by a leaf of $\mathcal{F}$. Moreover, $\mathcal{V} \subset M$ is called saturated for the foliation $\mathcal{F}$ if it can be regarded as the union of leaves. In other words, if $x \in \mathcal{V}$ then the leaf passing through x is contained in $\mathcal{V}$. Now assume that $\mathcal{F}$ is a foliation of codimension n on the manifold M and $\pi : \ \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the second projection. Then the map $h_i = \pi \circ \Psi_i^{-1} : \ \mathcal{U}_i \longrightarrow \mathbb{R}^n$ is a submersion and on $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$ the following identity holds: $h_j = \zeta_{ij} \circ h_i$. Furthermore, the fibers of submersion h_i are considered as the $\mathcal{F}$-plaques of $(\mathcal{U}_i, \mathcal{F})$ and the foliation $\mathcal{F}$ is thoroughly characterized via the submersions h_i and the local diffeomorphisms ζ_{ij} of $\mathbb{R}^n$. Overall, a foliation $\mathcal{F}$ of codimension n on M is totally identified via an open cover $\{\mathcal{U}_i\}_{i \in I}$ and submersions $h_i : \mathcal{U}_i \longrightarrow \mathcal{T}$ over an n-dimensional manifold $\mathcal{T}$ and a diffeomorphism $\zeta_{ij} : \ h_i(\mathcal{U}_i \cap \mathcal{U}_j) \longrightarrow h_j(\mathcal{U}_i \cap \mathcal{U}_j)$ such that $h_j = \zeta_{ij} \circ h_i$ for $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$. Then $\mathcal{X} = \big\{\mathcal{U}_i, h_i, \mathcal{T}, \zeta_{ij}\big\}$ is denoted by a *foliated cocycle* characterizing the foliation $\mathcal{F}$. According to the above definition, the n-manifold $\mathcal{T}^* = \bigsqcup \mathcal{T}_i^*$, $\mathcal{T}_i^* = h_i(\mathcal{U}_i)$, is denoted as the transverse manifold corresponding to the cocycle $\mathcal{X}$ and the pseudogroup $\mathcal{H}$ of local diffeomorphisms of $\mathcal{T}^*$ generated by ψ_{ij}, the holonomy pseudogroup representative on $\mathcal{T}^*$, which is associated to the cocycle $\mathcal{X}$. It is noticeable that $\mathcal{T}^*$ is a complete transverse manifold and the equivalence class of $\mathcal{H}$ is called the holonomy pseudogroup of $(M, \mathcal{F})$. In this paper, in what follows, it is assumed that

$\mathcal{F}$ is constructed via a cocycle $\mathcal{X}$ and the transverse manifold and holonomy pseudogroup associated to $\mathcal{X}$ are denoted by $\mathcal{T}^*$ and $\mathcal{H}$, respectively. The foliation $\mathcal{F}$ is called (transversally) metric if it is constructed via a cocycle $\mathcal{X} = \{\mathcal{U}_i, h_i, \mathcal{T}, \zeta_{ij}\}$ modeled on a manifold $(\mathcal{T}_0^*, g_0)$ and such that the transformations ζ_{ij} are local isometries of the metric structure g_0.

A symmetric C^∞-bilinear form $h : \Gamma(TM)\times\Gamma(TM) \to C^\infty$ is said to be positive if it satisfies $h(X,X) \geq 0$ for any $X \in \Gamma(TM)$. Such a form induces a positive bilinear form h_x on the tangent space T_xM at any point $x \in M$. The kernel $ker(h_x)$ is the linear subspace: $\{v \in T_x(M) : h_x(v, T_x(M)) = 0\}$ of T_xM. The Lie derivative $\mathcal{L}_X h$ of h in the direction of a vector field $X \in \Gamma(TM)$ is the symmetric C^∞-bilinear form on $\Gamma(TM)$ given by:

$$\mathcal{L}_X h(Y,Z) = X(h(Y,Z)) - h([X,Y],Z) - h(Y,[X,Z]).$$

A transverse metric on $(M,\mathcal{F})$ can be regarded as a positive bilinear form h on $\Gamma(TM)$ such that:
(1) $ker(h_x) = T_x(\mathcal{F})$ for any $x \in M$ and
(2) $\mathcal{L}_X h = 0$ for any vector field X on M tangent to $\mathcal{F}$.
Taking into account [28], a foliation together with a transverse metric h on $(M,\mathcal{F})$ is denoted by a metric foliation of M.

The notion of holonomy has a close relationship to existence of a transverse metric structure on the foliation. For an arbitrary point x on a manifold equipped with a foliation of codimension n, one of the main problems is regarding how the leaves near that point intersect a small n-dimensional disk that is transversal to the leaves and includes the mentioned point. In this situation, the significant concept of holonomy is introduced. In other words, the holonomy group at x exhaustively demonstrates the ways in which these leaves depart from this disk and return to it. This group is a quotient group of the fundamental group of leaves through x and is of particular significance because it includes much information about the structure of the foliation around the leaf through x.

The concept of holonomy is totally characterized via the notion of a germ of a locally defined diffeomorphism. A germ of a map from x to y is an equivalence class of maps $f : \mathcal{V} \to \mathcal{U}$ from an open neighborhood $\mathcal{V}$ of x to an open neighborhood $\mathcal{U}$ of y with $y = f(x)$. The germs of diffeomorphisms $(M,x) \longrightarrow (M,x)$ form a group denoted by: $Diff_x(M)$.
Let $(M,\mathcal{F})$ be a foliated manifold of codimension n, and let $\mathbf{L}_\alpha$ be a leaf of $\mathcal{F}$. Let $x, y \in \mathbf{L}_\alpha$ be two points on this leaf, and let A and B be transversal sections at x and y, i.e. submanifolds of M transversal to leaves of $\mathcal{F}$ with $x \in A$ and $y \in B$. To any path γ from x to y in $\mathbf{L}_\alpha$ we will associate a germ of a diffeomorphism

$$hol(\gamma) = hol^{A,B}(\gamma) : (A,x) \to (B,y),$$

denoted by the holonomy of the path γ in $\mathbf{L}_\alpha$ with respect to transversal sections A and B, as follows:
Let $\mathcal{U}$ be a foliated chart such that $\gamma([0,1]) \subset \mathcal{U}$ and let x and y lie on the same plaque in $\mathcal{U}$. Then we can find a small open neighborhood $\mathcal{N}$ of x in A with $\mathcal{N} \subset \mathcal{U}$ on which a smooth map $f : \mathcal{N} \to B$ can be defined, which satisfies $f(x) = y$, and for any $z \in \mathcal{N}$ the point $f(z)$ lies on the same plaque in $\mathcal{U}$ as z. We can choose $\mathcal{N}$ small enough such that f is a diffeomorphism onto its image. Then we define the following:

$$hol^{A,B}(\gamma) = germ_x(f).$$

The above definition is completely independent of the choice of $\mathcal{U}$ and f. If γ and $\tilde{\gamma}$ are homotopic paths in $\mathbf{L}_\alpha$ (with fixed end-points) from x to y and if A and B are transversal sections respectively at x and y, then

$hol^{A,B}(\gamma) = hol^{A,B}(\tilde{\gamma})$. Hence, we can regard $hol^{A,B}$ as being defined on the homotopy classes of paths in $\mathbf{L}_\alpha$ from x to y. Particularly, for a transversal section A at $x \in L_\alpha$, we obtain the following map:

$$hol^A = hol^{A,A} : \pi_1(\mathbf{L}_\alpha, x) \to Diff_x(A),$$

which is a group homomorphism from the fundamental group of the leaf $\mathbf{L}_\alpha$ at x to the group of germs at x of local diffeomorphisms of A with respect to the point x. We will say that hol^A is the holonomy representation of the leaf $\mathbf{L}_\alpha$ at x. Its image is the holonomy group of $\mathbf{L}_\alpha$ at x (refer to [28] for more details).

It is worth mentioning that any manifold can be equipped with a metric structure, but the specific property is to equip a foliated manifold with a metric for which the length of curves or tangent vectors that are transversal to the leaves remain invariant under the action of the corresponding holonomy group. In this case, the holonomy group can be explicated as a group of isometries for an arbitrary transversal n-dimensional disk.

The metric structure g_Q on the transverse bundle $Q = \frac{TM}{T(\mathcal{F})}$ of a foliation $\mathcal{F}$ is holonomy invariant if $\mathcal{L}_X g_Q = 0$ for any $X \in \Gamma\big(T(\mathcal{F})\big)$.

Theorem 3.1 *Let M be a manifold of dimension $m+n$ that is equipped with a foliation $\mathcal{F}$ of codimension n. Then $\mathcal{F} = \{U_i, h_i, \mathcal{T}, \zeta_{ij}, g\}$ is a metric foliated cocycle if and only if the induced metric on the transverse bundle is holonomy invariant.*

Proof: Let $\mathcal{F}$ be a foliation of dimension m and codimension n on manifold M. Then consider the local coordinates $(x^i) = (x^a, x^\alpha)$ where $a, b, ... \in \{1, ..., m\}$ and $\alpha, \beta, ... \in \{1, ..., n\}$. Now, suppose that $\mathbf{L}_t$ is a leaf of $\mathcal{F}$ and $\{(\mathcal{U}, \varphi) : (x^1, ..., x^m, x^{m+1}, ..., x^{m+n})\}$ is a foliated chart on the $(m+n)$-foliated manifold $(M, \mathcal{F})$. This means that each plaque P_c^t of $\mathcal{F}$ in $\mathcal{U}$ is described by equations of the form

$$x^{m+1} = c^{m+1}, ..., x^{m+n} = c^{m+n},$$

where $c = (c^{m+1}, ..., c^{m+n})$ is a point of $\mathbb{R}^n$. Hence, $\{\frac{\partial}{\partial x^a}\}, a \in \{1, ..., m\}$ are vector fields on $\mathcal{U}$ that are tangent to each n-dimensional submanifold P_c^t of $\mathcal{U}$. Let $\{(\widetilde{\mathcal{U}}, \widetilde{\varphi}) : (\tilde{x}^1, ..., \tilde{x}^m, \tilde{x}^{m+1}, ..., \tilde{x}^{m+n})\}$ be another foliated chart in such a way that $\mathcal{U} \cap \widetilde{\mathcal{U}} \neq \emptyset$. Assume that $\mathcal{P}_c^t$ and $\mathcal{P}_{\tilde{c}}^t$ are two plaques in $\mathcal{U}$ and $\widetilde{\mathcal{U}}$, respectively, in such a way that $\mathcal{P}_c^t \cap \mathcal{P}_{\tilde{c}}^t \neq \emptyset$. As $\mathcal{P}_c^t$ and $\mathcal{P}_{\tilde{c}}^t$ are the domains of some local charts on the leaf $\mathbf{L}_t$, which is a m-dimensional submanifold of M, on $\mathcal{P}_c^t \cap \mathcal{P}_{\tilde{c}}^t$ we have:

$$\frac{\partial}{\partial x^a} = \frac{\partial \tilde{x}^b}{\partial x^a} \frac{\partial}{\partial \tilde{x}^b} \quad , \quad a, b, ... \in \{1, ..., m\}. \tag{3.1}$$

Since $\mathcal{U} \cap \widetilde{\mathcal{U}}$ is covered by the intersections of plaques of $\mathcal{F}$, it can be deduced that (3.1) is satisfied on the whole $\mathcal{U} \cap \widetilde{\mathcal{U}}$. On $\mathcal{U} \cap \widetilde{\mathcal{U}}$ the following is generally assumed:

$$\frac{\partial}{\partial x^a} = \frac{\partial \tilde{x}^b}{\partial x^a} \frac{\partial}{\partial \tilde{x}^b} + \frac{\partial \tilde{x}^\alpha}{\partial x^a} \frac{\partial}{\partial \tilde{x}^\alpha}. \tag{3.2}$$

Thus, according to (3.1), it can be inferred that:

$$\frac{\partial \tilde{x}^\alpha}{\partial x^a} = 0 \quad , \quad \forall\ \alpha \in \{m+1, ..., m+n\}\ ,\ a \in \{1, ..., m\}. \tag{3.3}$$

Hence, the coordinate transformations on the $(m+n)$-foliated manifold $(M,\mathcal{F})$ have the following special form:

$$(a):\tilde{x}^a=\tilde{x}^a(x^b,x^\beta)\quad,\quad(b):\tilde{x}^\alpha=\tilde{x}^\alpha(x^\beta).\tag{3.4}$$

As $\{\frac{\partial}{\partial x^a}\}$, $a\in\{1,...,m\}$, are tangent to leaves of $\mathcal{F}$, $\{\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^\alpha}\}$ is called an $\mathcal{F}$-natural frame field on $(M,\mathcal{F})$ (refer to [6] for more details). Then the transformations of $\mathcal{F}$-natural frame fields on $(M,\mathcal{F})$ are given by relation (3.1) and

$$\frac{\partial}{\partial x^\alpha}=\frac{\partial\tilde{x}^a}{\partial x^\alpha}\frac{\partial}{\partial\tilde{x}^a}+\frac{\partial\tilde{x}^\beta}{\partial x^\alpha}\frac{\partial}{\partial\tilde{x}^\beta}.\tag{3.5}$$

Let $\mathbf{L}_\alpha$ be a leaf of $\mathcal{F}$, Υ a path in $\mathbf{L}_\alpha$, and E and K transversal sections of $\mathcal{F}$ with $\Upsilon(0)\in E$ and $\Upsilon(1)\in K$. Then we must prove that

$$\mathrm{Hol}^{E,K}(\Upsilon):(E,\Upsilon(0))\longrightarrow(K,\Upsilon(1))$$

is the germ of an isometry, or in other words that $\mathcal{H}=\mathrm{Hol}^{E,K}(\Upsilon)$ preserves the metric. According to the definition of holonomy, we can assume that Υ is inside a surjective chart, $\Omega=(x^1,...,x^m,x^{m+1},...,x^{m+n}):\mathcal{U}\longrightarrow\mathbb{R}^m\times\mathbb{R}^n$ of $\mathcal{F}$ and $E,K\subset\mathcal{U}$. Without loss of generality, assume that $\Omega(t)\subset\{0\}\times\mathbb{R}^n$, so that the vector fields $\frac{\partial}{\partial x^\alpha}|_E$ form a frame for the tangent bundle of E. Furthermore, assume that the holonomy diffeomorphism $\mathcal{H}:E\longrightarrow K$ is defined on all of E. By definition of H we have $x^\alpha\circ\mathcal{H}=x^\alpha|_E,\quad for\quad\alpha=1,...,n$. Therefore, $\frac{\partial(x^\alpha\circ h)}{\partial x^\beta}(p)=\delta_{\alpha\beta},\quad for\quad\alpha,\beta=1,...,n$, so:

$$\mathcal{H}_\star(\frac{\partial}{\partial x^\alpha}(p))\in\frac{\partial}{\partial x^\alpha}(\mathcal{H}(p))+T_{\mathcal{H}(p)}(\mathcal{F}),\quad\forall p\in E.$$

Here we view $T_{\mathcal{H}(p)}(\mathcal{F})$ as a subspace of $T_{\mathcal{H}(p)}(M)$. Particularly, we have [28]:

$$\begin{aligned}&g|_E(\mathcal{H}_{\star p}(\frac{\partial}{\partial x^\alpha}(p)),\mathcal{H}_{\star p}(\frac{\partial}{\partial x^\beta}(p)))\\&=g(\frac{\partial}{\partial x^\alpha}(\mathcal{H}(p)),\frac{\partial}{\partial x^\beta}(\mathcal{H}(p)))\\&=g_{\alpha\beta}(\mathcal{H}(p))=g_{\alpha\beta}(p)=g|_E(\frac{\partial}{\partial x^\alpha}(p),\frac{\partial}{\partial x^\beta}(p)).\end{aligned}$$

Consequently, for an arbitrary transversal section E at $x\in\mathbf{L}_\alpha$ we obtain the following map:

$$\mathrm{Hol}^E=\mathrm{Hol}^{E,E}:\pi_1(\mathbf{L}_\alpha,x)\to Diff_x(E),$$

which is a group homomorphism from the fundamental group of the leaf $\mathbf{L}_\alpha$ at x to the group of germs at x of local diffeomorphisms of E with respect to the point x. We will say that Hol^E is the holonomy representation of the leaf $\mathbf{L}_\alpha$ at x. Its image is the holonomy group of $\mathbf{L}_\alpha$ at x.

Theorem 3.2 *Let M be a manifold of dimension $m+n$ that is equipped with a foliation $\mathcal{F}$ of codimension n. Then $\mathcal{F}=\{U_i,h_i,\mathcal{T},\zeta_{ij},g\}$ is a holonomy invariant foliated cocycle if and only if for any $X,Y,Z\in\Gamma(TM)$ the following relation satisfied:*

$$\pi X(g(\tilde{\pi}Y,\tilde{\pi}Z))-g([\pi X,\tilde{\pi}Y],\tilde{\pi}Z)-g([\pi X,\tilde{\pi}Z],\tilde{\pi}Y)=0.\tag{3.6}$$

Proof: According to Theorem (3.1), it is deduced that $\mathcal{F} = \{U_i, h_i, \mathcal{T}, \zeta_{ij}, g\}$ is a metric foliated cocycle. Suppose that $T(\mathcal{F})$ is the tangent distribution to the foliation and $T(\mathcal{F})^{\perp}$ denotes the complementary orthogonal distribution to $T(\mathcal{F})$ in TM, which we consider as the transversal distribution corresponding to $\mathcal{F}$. Here we denote by the same symbol g the metrics induced by g on $T(\mathcal{F})$ and $T(\mathcal{F})^{\perp}$. The projection morphisms of TM on $T(\mathcal{F})$ and $T(\mathcal{F})^{\perp}$ with respect to the decomposition $TM = T(\mathcal{F}) \oplus T(\mathcal{F})^{\perp}$ are denoted by π and $\widetilde{\pi}$, respectively. According to Theorem (5.1) of [6], there exists a unique connection $\nabla^{\mathcal{I}}$ (resp. $\nabla^{\mathcal{I}^{\perp}}$) with respect to the above decomposition. We call $\nabla^{\mathcal{I}}$ and $\nabla^{\mathcal{I}^{\perp}}$ the intrinsic connections on $T(\mathcal{F})$ and $T(\mathcal{F})^{\perp}$, respectively. According to [6], we have:

$$\begin{aligned} {\nabla^{\mathcal{I}}}_{\widetilde{\pi}X}\pi Y &= \pi[\widetilde{\pi}X, \pi Y], \\ {\nabla^{\mathcal{I}^{\perp}}}_{\pi X}\widetilde{\pi}Y &= \widetilde{\pi}[\pi X, \widetilde{\pi}Y]. \end{aligned} \tag{3.7}$$

Now, according to metric isomorphism $T(\mathcal{F})^{\perp} \approx \frac{TM}{T(\mathcal{F})}$, the proof is completed.

4. Structural analysis of the holonomy invariant foliated cocycles on the tangent bundle through formal integrability

Let M be a smooth $(m+n)$-dimensional manifold and ∇ be a linear connection on M. Recall that the tangent bundle TM possesses a natural $(m+n)$-foliation, which is defined by fibers. The distribution VTM, which is tangent to this foliation, is denoted by the vertical foliation on TM. From the geometric point of view, the linear connection ∇ assigns an $(m+n)$-distribution HTM on TM that is complementary to VTM as follows: let (x^i, y^i) be a coordinate system on TM, where (x^i), $i \in \{1, \cdots, m+n\}$ are local coordinates on M. Then we define the following:

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \Gamma^k_{ij}(x) \frac{\partial}{\partial x^k}, \qquad H^k_j(x, y) = y^i \Gamma^k_{ij}(x). \tag{4.1}$$

Considering the fact that $\{\Gamma^k_{ij}(x)\}$ are the local coefficients of a linear connection on M, the distribution HTM can be locally defined as follows:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - H^j_i(x, y) \frac{\partial}{\partial y^j}, \qquad i \in \{1, \cdots, m+n\}. \tag{4.2}$$

The distribution HTM is denoted by the horizontal distribution on TM, which is induced by ∇.

A path $\vartheta^* : [0,1] \longrightarrow TM$ is called horizontal if for all $t \in [0,1]$ we have $\frac{d\vartheta^*}{dt} \in HTM_{\vartheta^*(t)}$. Now consider the case that $\vartheta : [0,1] \longrightarrow M$ is a smooth piecewise path in M that connects the point $x = \vartheta(0)$ to $y = \vartheta(1)$ in M. As a consequence, for each $u \in T_xM$ there exists only a unique horizontal lift $\vartheta^* : [0,1] \longrightarrow TM$ with $\vartheta^*(0) = u$. Hence, it is inferred that ϑ^* is horizontal. Moreover, we have $\pi(\vartheta^*(t)) = \vartheta(t)$ where the map $\pi : TM \longrightarrow M$ is the natural projection. Thus, it is deduced that

$$\xi_{\vartheta(t)} : T_xM \longrightarrow T_{\vartheta(t)}M, \qquad T_{\vartheta(t)}(u) = \vartheta^*(t), \qquad \forall \;\; t \in [0,1] \tag{4.3}$$

can be regarded as an isomorphism of vector spaces. The map $\xi_{\vartheta(t)}$ is denoted by the parallel transport or parallel displacement along path ϑ. Specifically, whenever the manifold M is equipped with a metric structure

g and ∇ is the Levi-Civita connection associated to g, then the map $\xi_{\vartheta(t)}$ can be considered as a linear isometry. Conversely, for a given distribution HTM that is complementary to VTM, the covariant differentiation can be defined via the notion of the parallel transport. This can be fulfilled as follows: consider X and Y as two vector fields on M, and for any arbitrary point $x \in M$, the integral curve $\vartheta : [0,1] \longrightarrow M$ of the vector field X through the point x is selected. This means that $\vartheta(0) = x$ and $\acute{\vartheta}(t) = X(\vartheta(t))$. Consequently, the covariant derivative $\nabla_X Y$ of Y with respect to X is the vector field defined by:

$$(\nabla_X Y)(x) = \lim \frac{1}{t}\big(\xi^{-1}_{\vartheta(t)} Y(\vartheta(t) - Y(x)\big). \tag{4.4}$$

Now, if $x \in M$ and $\vartheta : [0,1] \longrightarrow M$ is a loop at point x, namely $\vartheta(0) = \vartheta(1) = x$, then the parallel transport $\xi_{\vartheta(t)}$ can be regarded as an automorphism of T_xM. It is worth noticing that all such automorphisms construct a group Hol_x, which is denoted by the holonomy group of the connection ∇ at point x. Meanwhile, taking into account the fact that manifold M is assumed to be connected, it can be deduced that at different points the corresponding holonomy groups are isomorphic to each other. Thus, we can consider Hol as the holonomy group of the connection ∇.

In addition, let ∇ be a linear connection on an $(m+n)$-dimensional manifold M with $m, n > 0$. Then an m-dimensional distribution D on M is called parallel with respect to ∇ if and only if the distribution D is invariant under parallel transports. In other words, for all $x, y \in M$ and all smooth piecewise paths ϑ from x to y, the following condition is satisfied: $\xi_\vartheta(D_x) = D_y$.

Theorem 4.1 *Let M be a connected smooth manifold of dimension $m+n$ with $m, n > 0$ and ∇ be a linear connection on M. Suppose that Hol_x denotes the holonomy group of the connection ∇ at $x \in M$. If the action of the group Hol_x on T_xM leaves a nontrivial m-dimensional subspace D_x of T_xM invariant, namely Hol is m-reducible and ∇ is torsion free, then there exists canonically a foliated cocycle $\mathcal{F}^{\mathbb{T}} = \{\tilde{U}_i, \tilde{h}_i, \tilde{\mathcal{T}}, \tilde{\zeta}_{ij}\}$ on the tangent space TM.*

Proof Since the holonomy group Hol of the connection ∇ is m-reducible, then for an arbitrary point $x \in M$ we select D_x as the subspace that is invariant under the action of the group Hol_x. Now we construct a distribution D on the manifold M as follows: for any other point $y \in M$ we consider D_y as the image of D_x under any parallel transport ξ_ϑ from T_xM to T_yM. Now we select any other path ϱ connecting the point x to y in order to illustrate that D_y does not depend on the choice of the path ϱ. Subsequently, $\varrho^{-1} \circ \vartheta$ can be regarded as a loop at point x and ultimately it is demonstrated that D_x is invariant under the parallel transport $\xi_{\varrho^{-1}\circ\vartheta} = \xi_\varrho^{-1} \circ \xi_\vartheta$. Consequently, it is displayed that $\xi_\varrho^{-1} \circ \xi_\vartheta(D_x) = D_x$ and thus we have proved that $\xi_\vartheta(D_x) = \xi_\varrho(D_x)$. Therefore, D is a well-defined distribution on manifold M and, from its construction procedure, the parallelism and smoothness of D directly results. Overall, for a given connection ∇ on manifold M, the existence problem for a distribution D that is parallel with respect to ∇ was comprehensively discussed above. In addition, from relation (4.4) it directly results that the distribution D is parallel with respect to the linear connection ∇ if and only if ∇ is an adapted connection to D. Thus, considering the fact that ∇ is torsion-free, it is deduced that:

$$[X, Y] = \nabla_X Y - \nabla_Y X, \quad \forall\, X, Y \in \Gamma(TM). \tag{4.5}$$

Hence, it is indicated that $[X, Y] \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$. Consequently, it is proved that the distribution D is involutive. Thus, for any $x \in M$ there exists a local foliated chart $\{(U, \Psi) : (x^a, x^\alpha)\}$ on M in such a way that all the submanifolds of U given by $x^\alpha = c^\alpha$, $\alpha \in \{m+1, \cdots, m+n\}$ are integral manifolds of distribution D. Indeed, these integral manifolds can be considered as plaques of D in M and so a new topology can be constructed on manifold M, whose basis comprises all plaques of D in M and is denoted by $\tau(D)$. Taking into account the fact that manifold M is totally covered via the set of all plaques of distribution D, it is inferred that $(M, \tau(D))$ can be reckoned as an m-dimensional integral manifold of D. Furthermore, a connected component of $(M, \tau(D))$ passing through the point $x \in M$ is considered as the leaf $\mathbf{L}_t$ of D through x and any other m-dimensional manifold of D is an open submanifold of $(M, \tau(D))$. Consequently, it is illustrated that a disjoint partition is induced on M, which is structurally constructed via the leaves of D. Thus, M admits an m-dimensional foliated atlas that includes the local charts covered by plaques of the distribution D. Accordingly, the integrable m-dimensional distribution D defines a foliated cocycle $\mathcal{F} = \{U_i, h_i, \mathcal{T}, \zeta_{ij}\}$ of codimension n on M. Now, for any $x \in M$, we consider the leaf $\mathbf{L}_t$ of $\mathcal{F}$, which passes through x, and we define $D_x = T_x\mathbf{L}_t$. This distribution is known as the tangent distribution to the foliation $\mathcal{F}$ and is denoted by $D(\mathcal{F})$. Thus, if $\{(U, \Psi) : (x^a, x^\alpha)\}$ is a foliated chart on $(M, \mathcal{F})$, then $\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^m}\}$ are tangent to $\mathbf{L}_t \cap U$ and consequently in local coordinates we have: $D(\mathcal{F}) = \mathrm{span}\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^m}\}$. A chart can be induced: $(\bar{U}, x^a, y^b, x^\alpha, y^\beta)$ on TM where (x^α, y^β) are the transverse coordinates. Let $(\widetilde{U}, \tilde{x}^a, \tilde{y}^b, \tilde{x}^\alpha, \tilde{y}^\beta)$ be another coordinate system on TM. Then the theorem follows directly from the transformation rule:

$$\tilde{x}^a = \tilde{x}^a(x^b, x^\beta) \quad , \quad \tilde{x}^\alpha = \tilde{x}^\alpha(x^\beta) \quad , \quad \tilde{y}^a = \frac{\partial \tilde{x}^a}{\partial x^b} y^b + \frac{\partial \tilde{x}^a}{\partial x^\beta} y^\beta \quad , \quad \tilde{y}^\alpha = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta} y^\beta.$$

Taking into account the coordinate transformations, it is inferred that the two foliated cocycles mentioned above can be regarded as the natural lift of $\mathcal{F}$ to the tangent space TM. These two foliated cocycles are locally spanned by $\{\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y^a}\}$ and $\{\frac{\partial}{\partial y^a}\}$. □

The semispray $\mathcal{S} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$ determines a nonlinear connection N with local coefficients $G^i_j = \frac{\partial G^i}{\partial y^j}$. The nonlinear connection N has local components as follows: $(G^i_j) = \begin{pmatrix} G^a_b & G^\alpha_b \\ G^a_\beta & G^\alpha_\beta \end{pmatrix}$. Each of the local components $G^a_b, G^\alpha_b, G^a_\beta, G^\alpha_\beta$ has $x^a, x^\alpha, y^b, y^\beta$ as variables.

The nonlinear connection N defines a local base of its horizontal vector fields given by:

$$\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - G^b_a \frac{\partial}{\partial y^b} - G^\beta_a \frac{\partial}{\partial y^\beta}, \tag{4.6}$$

$$\frac{\delta}{\delta x^\alpha} = \frac{\partial}{\partial x^\alpha} - G^b_\alpha \frac{\partial}{\partial y^b} - G^\beta_\alpha \frac{\partial}{\partial y^\beta}. \tag{4.7}$$

In [30, 31] Popescu et al. defined the notion of the Lagrangian adopted to the lifted foliation and in [24] Riemannian foliations compatible with SODE structure were discussed. In this section, by imposing the following four significant conditions we will provide an appropriate setting in order to construct holonomy invariant foliated cocycles on the tangent space TM. As will be demonstrated, the concept of formal integrability is applied as a fundamental tool.

Definition 4.2 *Let $\mathcal{F} = \{U_i, h_i, \mathcal{T}, \zeta_{ij}\}$ be a foliated cocycle of codimension n on M and $\mathcal{F}^{\mathbb{T}} = \{\tilde{U}_i, \tilde{h}_i, \tilde{\mathcal{T}}, \tilde{\zeta}_{ij}\}$ be the foliated cocycle on the tangent space TM. Let $\mathcal{S}$ be a semispray that is locally represented as $\mathcal{S} =$*

$y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$. *Then* $\mathcal{S}$ *is called the adopted foliated semispray (AFS) and the metric* g *is called the adopted transverse metric (ATM) if the following four conditions are satisfied:*

(1) The following partial differential operator

$$\mathcal{P}_{\mathrm{L}} = \big(d_{\mathcal{J}}, d_h, d_\Phi, \nabla d\big) : \; Sec(T_v^*) \longrightarrow Sec\big(\oplus^{(4)} \Lambda^2 T_v^*\big) \tag{4.8}$$

is formally integrable.

(2) $g_{b\beta} = g(\frac{\partial}{\partial y^b}, \frac{\partial}{\partial y^\beta}) = 0$.

(3) The local functions $(g_{\alpha\beta})$ *and* $(g^{\alpha\beta})$ *are basic functions, i.e. they do not depend on the tangent variables* (x^a, y^a).

(4) The semispray $\mathcal{S}$ *is foliated; namely, the following identities hold:*

$$\begin{cases} (i) : G^\alpha_b = \dfrac{\partial G^\alpha}{\partial y^b} = 0\ ,\ (ii) : \dfrac{\partial G^\alpha}{\partial x^b} = 0; \quad or, \\ (i') : G^b_\alpha = \dfrac{\partial G^b}{\partial y^\alpha} = 0\ ,\ (ii') : \dfrac{\partial G^b}{\partial x^\alpha} = 0. \end{cases} \tag{4.9}$$

Theorem 4.3 *Let* M *be an* $(m+n)$*-dimensional manifold and* $\mathcal{F} = \{U_i, h_i, \mathcal{T}, \zeta_{ij}\}$ *be a foliated cocycle of codimension* n *on* M. *Suppose that* $\mathcal{S} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ *is an AFS and* g *is an ATM. Then there exists a metric* $g^{\mathbb{T}}$ *on* TM *such that* $\mathcal{F}^{\mathbb{T}} = \{\tilde{U}_i, \tilde{h}_i, \tilde{\mathcal{T}}, \tilde{\zeta}_{ij}, g^{\mathbb{T}}\}$ *is a holonomy invariant foliated cocycle on* TM.

Proof A coordinate system in M is defined by $\{(\mathcal{V},\psi) : x^1, \cdots, x^{m+n}\}$ or briefly $\{(\mathcal{V},\psi) : x^i\}$, where $\mathcal{V}$ is an open subset of M, $\psi : \mathcal{V} \longrightarrow \mathbb{R}^{m+n}$ is a diffeomorphism of $\mathcal{V}$ onto $\psi(\mathcal{V})$, and $(x^1, \cdots, x^{m+n}) = \psi(x)$ for any $x \in \mathcal{V}$. The canonical projection of TM on M and by T_xM the fiber at $x \in M$ is denoted by π, i.e. $T_xM = \pi^{-1}(x)$. The coordinate system $\{(\mathcal{V},\psi) : x^i\}$ in M defines a coordinate system $\{(\mathcal{V}^*, \Psi) : x^1, \cdots, x^{m+n}, y^1, \cdots, y^{m+n}\} = \{(\mathcal{V}^*, \Psi) : x^i, y^i\}$ in TM, where $\mathcal{V}^* = \pi^{-1}(\mathcal{V})$ and $\Psi : \mathcal{V}^* \longrightarrow \mathbb{R}^{2(m+n)}$ is a diffeomorphism of $\mathcal{V}^*$ on $\psi(\mathcal{V}) \times \mathbb{R}^{m+n}$, and $(x^1, \cdots, x^{m+n}, y^1, \cdots, y^{m+n}) = \Psi(y_x)$, for any $x \in \mathcal{V}$ and $y_x \in T_xM$. For short we denote by (x,y) the coordinates of y_x. Next, consider another coordinate system $\{(\tilde{\mathcal{V}}, \tilde{\psi}); x^i\}$ in M such that $\mathcal{V} \cap \tilde{\mathcal{V}} \neq \emptyset$. Then the local coordinates (x,y) and $(\tilde{x},\tilde{y})$ on TM are related by:

$$\tilde{x}^i = \tilde{x}^i(x^1, x^2, \cdots, x^{m+n}), \qquad \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial \tilde{x}^j} y^j. \tag{4.10}$$

Consequently, from (4.10) the local frame fields $\{\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial y^i}\}$ and $\{\dfrac{\partial}{\partial \tilde{x}^i}, \dfrac{\partial}{\partial \tilde{y}^i}\}$ are related as follows:

$$\begin{cases} \dfrac{\partial}{\partial x^i} = \dfrac{\partial \tilde{x}^j}{\partial x^i}\dfrac{\partial}{\partial \tilde{x}^j} + \dfrac{\partial^2 \tilde{x}^j}{\partial x^i \partial x^k} y^k \dfrac{\partial}{\partial \tilde{y}^j}, \\ \dfrac{\partial}{\partial y^i} = \dfrac{\partial \tilde{x}^j}{\partial x^i}\dfrac{\partial}{\partial \tilde{y}^j}. \end{cases} \tag{4.11}$$

Analogously, it is inferred that the local coframe fields $\{dx^i, dy^i\}$ and $\{d\tilde{x}^i, d\tilde{y}^i\}$ on the cotangent bundle T^*M of M are related by

$$d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j}\, dx^j, \qquad d\tilde{y}^i = \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} y^j\, dx^k + \frac{\partial \tilde{x}^i}{\partial x^j}\, dy^j. \tag{4.12}$$

A complementary distribution HTM to VTM in TTM is called a nonlinear connection or a horizontal distribution on TM. This connection is of special geometric significance and its existence leads to the following decomposition:

$$TTM = HTM \oplus VTM. \tag{4.13}$$

Then we take a local frame field $\{X_i, \frac{\partial}{\partial y^i}\}$ on $\acute{\mathcal{V}} \subset TM$ adapted to (4.13), i.e. $X_i \in \Gamma(HTM)$ and $\frac{\partial}{\partial y^i} \in \Gamma(VTM)$. Thus, we have

$$\frac{\partial}{\partial x^i} = E_i^j(x,y)X_j + G_i^j(x,y)\frac{\partial}{\partial y^j}, \tag{4.14}$$

where E_i^j and G_i^j are smooth functions that are locally defined on TM. Hence, the transition matrix from the local frame field $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}$ to $\{X_j, \frac{\partial}{\partial y^j}\}$ is

$$\Delta = \begin{pmatrix} E_i^j(x,y) & 0 \\ & \\ G_i^k(x,y) & \delta_h^k \end{pmatrix}. \tag{4.15}$$

As Δ is a nonsingular matrix it follows that the $(m+n)\times(m+n)$ matrix $[E_i^j(x,y)]$ is also a nonsingular matrix. Thus, the set of local vector fields $\{\frac{\delta}{\delta x^1}, \cdots, \frac{\delta}{\delta x^{m+n}}\}$ given by $\frac{\delta}{\delta x^i} = E_i^j(x,y)X_j$ is a basis in $\Gamma(HTM|_{\acute{\mathcal{U}}})$. In this way (4.14) becomes

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_i^j(x,y)\frac{\partial}{\partial y^j}. \tag{4.16}$$

Denote by $\tilde{G}_h^k$ the functions in (4.16) given with respect to another coordinate system $\{(\tilde{\acute{\mathcal{V}}}, \tilde{\acute{\Phi}}); \tilde{x}^i, \tilde{y}^i\}$ on TM such that $\acute{\mathcal{V}} \cap \tilde{\acute{\mathcal{V}}} \neq \emptyset$. Then, by using (4.11) and (4.16), we obtain

$$\begin{cases} G_i^j(x,y)\dfrac{\partial \tilde{x}^k}{\partial x^j} = \widetilde{G}_h^k(\tilde{x},\tilde{y})\dfrac{\partial \tilde{x}^h}{\partial x^i} + \dfrac{\partial^2 \tilde{x}^k}{\partial x^i \partial x^h} y^h, \\ \dfrac{\delta}{\delta x^i} = \dfrac{\partial \widetilde{x}^j}{\partial x^i}\dfrac{\delta}{\delta \widetilde{x}^j}. \end{cases} \tag{4.17}$$

A nonlinear connection HTM enables us to define an almost product structure on TM as follows. Consider a vector field X on TM. Then locally we have

$$X = X^i\frac{\delta}{\delta x^i} + \dot{X}^i\frac{\partial}{\partial y^i}, \tag{4.18}$$

where X^i and $\dot{X}^i$ satisfy

$$\tilde{X}^j = \frac{\partial \tilde{x}^j}{\partial x^i} X^i \quad , \quad \dot{\tilde{X}}^j = \frac{\partial \tilde{x}^j}{\partial x^i} \dot{X}^i \tag{4.19}$$

with respect to (4.10). Then we define

$$\begin{cases} Q : \Gamma(TTM) \longrightarrow \Gamma(TTM) \\ QX = \dot{X}^i \dfrac{\delta}{\delta x^i} + X^i \dfrac{\partial}{\partial y^i}. \end{cases} \tag{4.20}$$

By (4.11), (4.17), and (4.19) it follows from (4.20) that QX does not depend on the local chart on TTM. Moreover, $Q^2 = I$ and therefore Q is an almost product structure on TM. We call Q the almost product structure corresponding to the nonlinear connection HTM [7]. Furthermore, denote by h and v the projection morphisms of TTM to HTM and VTM, respectively. Then we have

$$(a): \ Q \circ h = v \circ Q \quad \text{and} \quad (b): \ Q \circ v = h \circ Q. \tag{4.21}$$

By means of Q and v we define the following $\mathcal{F}(TM)$-bilinear mapping:

$$\begin{cases} \mathcal{R}_1 : \Gamma(VTM) \times \Gamma(VTM) \longrightarrow \Gamma(VTM) \\ \mathcal{R}_1(X, Y) = -v[QX, QY], \quad \forall \ X, Y \in \Gamma(VTM). \end{cases} \tag{4.22}$$

According to [7] we see that $\mathcal{R}_1$ is a tensor field of Type $(1,2)$. By direct calculations using (4.22), (4.20), and (4.16), the following significant relation is deduced:

$$\mathcal{R}_1(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}) = [\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}] = R^k_{ij} \frac{\partial}{\partial y^k}, \tag{4.23}$$

where we have set

$$R^k_{ij} = \frac{\delta G^k_i}{\delta x^j} - \frac{\delta G^k_j}{\delta x^i}. \tag{4.24}$$

From (4.22) it follows that $\mathcal{R}_1 = (R^k_{ij})$ is the obstruction to the integrability of the horizontal distribution. More precisely, we have the following result:
The horizontal distribution HTM is involutive if and only if $\mathcal{R}_1 = 0$ on TM, or equivalently, on each coordinate neighborhood $\acute{\mathcal{V}} \subset TM$ we have $R^k_{ij} = 0$. Ultimately, by using (4.16) and direct calculations, we obtain:

$$[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}] = \frac{\partial G^k_i}{\partial y^j} \frac{\partial}{\partial y^k}. \tag{4.25}$$

Since the semispray $\mathcal{S}$ is foliated, $G^b_\alpha = 0$. Thus, from (4.6), it can be deduced that $\dfrac{\delta}{\delta x^\alpha} = \dfrac{\partial}{\partial x^\alpha} - G^\beta_\alpha \dfrac{\partial}{\partial y^\beta}$. Also, by definition (4.2), the following relations can be deduced: $\dfrac{\partial G^\beta_\alpha}{\partial x^a} = \dfrac{\partial^2 G^\beta}{\partial x^a \partial y^\alpha} = 0$ and $\dfrac{\partial G^\beta_\alpha}{\partial y^a} = \dfrac{\partial^2 G^\beta}{\partial y^a \partial y^\alpha} = 0$.

Thus, G_α^β does not depend on tangent variables (x^a, y^a). Let g be the ATM (Definition 4.2). By applying g, a metric $g^{\mathbb{T}}$ on TM can be defined as follows:

$$
\begin{aligned}
&g_{IJ}^{\mathbb{T}}(x,y) = \begin{pmatrix} g_{ij}(x,y) & 0 \\ & \\ 0 & g_{ij}(x,y) \end{pmatrix}, \\
&I, J \in \{1, \cdots, 2(m+n)\}, \ i, j \in \{1, \cdots, m+n\}.
\end{aligned}
\tag{4.26}
$$

This means that with respect to the frame field $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$, which is locally defined on TM, the following can be stated:

$$
g^{\mathbb{T}}(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = g^{\mathbb{T}}(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = g_{ij}, \qquad g^{\mathbb{T}}(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) = 0. \tag{4.27}
$$

As demonstrated above, $\{\dfrac{\delta}{\delta x^\alpha}, \dfrac{\partial}{\partial y^\beta}\}$, $\alpha, \beta \in \{m+1, ..., m+n\}$, is a local base of foliated vector fields for the foliation $\mathcal{F}^{\mathbb{T}}$. Furthermore:

$$
g^{\mathbb{T}}(\frac{\delta}{\delta x^\alpha}, \frac{\delta}{\delta x^\beta}) = g^{\mathbb{T}}(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}) = g_{\alpha\beta}, \qquad g^{\mathbb{T}}(\frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial y^\beta}) = 0. \tag{4.28}
$$

Since $g_{\alpha\beta}$ is a basic function, by definition it can be inferred that $g^{\mathbb{T}}$ is a transverse metric for the foliation $\mathcal{F}^{\mathbb{T}}$. Hence, $\mathcal{F}^{\mathbb{T}} = \{\tilde{U}_i, \tilde{h}_i, \tilde{\mathcal{T}}, \tilde{\zeta}_{ij}, g^{\mathbb{T}}\}$ is a holonomy invariant foliated cocycle on TM. $\square$

Corollary 4.4 *Let $\mathcal{F} = \{U_i, h_i, \mathcal{T}, \zeta_{ij}\}$ be a foliated cocycle of codimension n on M and $\mathcal{F}^{\mathbb{T}} = \{\tilde{U}_i, \tilde{h}_i, \tilde{\mathcal{T}}, \tilde{\zeta}_{ij}\}$ be the foliated cocycle on the tangent space TM. Suppose that $\mathcal{S} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ is an AFS. Then $V\mathcal{F}^{\mathbb{T}}$ and $H\mathcal{F}^{\mathbb{T}}$ induce the nonlinear connection (G_b^a) $a, b \in \{1, ..., m\}$, on the leaves of $\mathcal{F}^{\mathbb{T}}$.*

Proof Since the semispray $\mathcal{S}$ is an AFS, according to relation (4.9) we have $G_a^\beta = 0$. Thus, from relation (4.6), it can be inferred that:

$$
\frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - G_a^b \frac{\partial}{\partial y^b},
$$

which means that $H\mathcal{F}^{\mathbb{T}}$ is locally spanned by the vector fields $\{\frac{\delta}{\delta x^a}\}$ $a \in \{1, ..., m\}$. Since $\{\frac{\partial}{\partial y^a}\}$ is a local base in $\Gamma(V\mathcal{F}^{\mathbb{T}})$, we have: $T(\mathcal{F}^{\mathbb{T}}) = V\mathcal{F}^{\mathbb{T}} \oplus H\mathcal{F}^{\mathbb{T}}$, where $H\mathcal{F}^{\mathbb{T}} = T(\mathcal{F}^{\mathbb{T}}) \cap HTM$ and $V\mathcal{F}^{\mathbb{T}} = T(\mathcal{F}^{\mathbb{T}}) \cap VTM$. $\square$

Theorem 4.5 *Let M be an $(m+n)$-dimensional smooth manifold with $m, n > 0$ and $\mathcal{F} = \{U_i, h_i, \mathcal{T}, \zeta_{ij}\}$ be a parallel nondegenerate foliated cocycle of codimension n on M. Suppose that $\mathcal{S} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ is an AFS and let $\tilde{g}$ be the metric associated to semispray S. Then for any $(x,y) \in TM$ there exists a neighborhood $\Omega_{\mathbb{T}}^* \subset TM$ and two submanifolds $\Omega_{\mathbb{T}}$ and $\Omega_{\mathbb{T}}^{\perp}$ of dimensions $2m$ and $2n$ admitting the metric structures g and $g^{\perp}$ such that $(\Omega_{\mathbb{T}}^*, \tilde{g})$ is the product of $(\Omega_{\mathbb{T}}, g)$ and $(\Omega_{\mathbb{T}}^{\perp}, g^{\perp})$. Furthermore, there exist two complementary orthogonal totally geodesic and holonomy invariant foliated cocycles on TM.*

Proof Assume that $\mathcal{F}$ is a parallel nondegenerate m-foliation on an $(m+n)$-dimensional manifold $(M,\tilde{g})$. Hence, the corresponding tangent distribution D to $\mathcal{F}$ is nondegenerate and parallel with respect to the Levi-Civita connection $\tilde{\tilde{\nabla}}$ on $(M,\tilde{g})$. Consequently, the distribution $D^{\perp}$ is parallel, nondegenerate, and complementary orthogonal to D. This yields the second parallel n-foliation $\mathcal{F}^{\perp}$. Hence, $(M,D,D^{\perp})$ can be reckoned as an almost product manifold and the pair $(\mathcal{F},\mathcal{F}^{\perp})$ is a $\tilde{\nabla}$-grid. Now consider $\mathbf{L}$ and $\acute{\mathbf{L}}$ to be the leaves through x^* of $\mathcal{F}$ and $\mathcal{F}^{\perp}$, respectively. Then there is a foliated chart $(\mathcal{V},\sigma)$ about x^* with local coordinates $(x^1,\cdots,x^m,x^{m+1},\cdots,x^{m+n})$ such that each plaque of $\mathcal{F}$ is given by the following equations:

$$x^{m+1}=c^{m+1},\cdots,x^{m+n}=c^{m+n}. \tag{4.29}$$

Moreover, since x^* is the origin of the coordinate system, we may take $(x^1,\cdots,x^m,0,\cdots,0)$ as local coordinates on $\mathcal{V}\cap\mathbf{L}$. Similarly, we take another foliated chart $(\mathcal{W},\theta)$ about x^* with respect to $\mathcal{F}^{\perp}$ such that $(0,\cdots,0,x^{m+1},\cdots,x^{m+n})$ are local coordinates on $\mathcal{W}\cap\acute{\mathbf{L}}$. Then we choose the open neighborhoods Ω and $\Omega^{\perp}$ of x^* in $\mathbf{L}$ and $\acute{\mathbf{L}}$ such that $\Omega\times\Omega^{\perp}\subset\mathcal{V}\cap\mathcal{W}$. Thus, $\Omega^*=\Omega\times\Omega^{\perp}$ is the required neighborhood of x^* in M. It is worth mentioning that we can take $(x^1,\cdots,x^m,x^{m+1},\cdots,x^{m+n})$ as a coordinate system on Ω^* compatible with both foliations $\mathcal{F}$ and $\mathcal{F}^{\perp}$. That is to say:

$$D=\operatorname{span}\{\frac{\partial}{\partial x^1},\cdots,\frac{\partial}{\partial x^m}\},\qquad D^{\perp}=\operatorname{span}\{\frac{\partial}{\partial x^{m+1}},\cdots,\frac{\partial}{\partial x^{m+n}}\} \tag{4.30}$$

on Ω^*. The coordinate system $\{(\mathcal{V}\times\mathcal{W},\sigma\times\theta):(x^a,x^\alpha)\}$ on $\Omega^*=\Omega\times\Omega^{\perp}$ defines a coordinate system $\{(\mathcal{V}^*\times\mathcal{W}^*,\Sigma\times\Theta):(x^a,x^\alpha;y^a,y^\alpha)\}$ on $T\Omega^*\simeq T\Omega\oplus T\Omega^{\perp}$, where for any $x\in\mathcal{V}$ and $y_x\in T_x\Omega$, we have $\mathcal{V}^*=\pi_1^{-1}(\mathcal{V})$ and $\Sigma:\mathcal{V}^*\to\mathbf{R}^{2m}$ is a diffeomorphism of $\mathcal{V}^*$ on $\sigma(\mathcal{V})\times\mathbf{R}^m$, and $(x^1,\ldots,x^m;y^1,\ldots,y^m)=\Sigma(y_x)$. Correspondingly, for any $v\in\mathcal{W}$ and $v_u\in T_uM_2$, we have $\mathcal{W}^*=\pi_2^{-1}(\mathcal{W})$ and $\Theta:\mathcal{W}^*\to\mathbf{R}^{2n}$ is a diffeomorphism of $\mathcal{W}^*$ on $\theta(\mathcal{W})\times\mathbf{R}^n$ and $(x^{m+1},\ldots,x^{m+n};y^{m+1},\ldots,y^{m+n})=\Theta(v_u)$. For simplicity, we denote by $(x,y)=(x^1,\ldots,x^m;y^1,\ldots,y^m)$ the coordinate of y_x (likewise $(u,v)=(x^{m+1},\ldots,x^{m+n};y^{m+1},\ldots,y^{m+n})$ the coordinate of v_u). Therefore, we have $(x,u,y,v)\in T_x\Omega\oplus T_u\Omega^{\perp}$. Now we consider another coordinate system, $\{(\tilde{\mathcal{V}},\tilde{\mathcal{W}},\tilde{\sigma}\times\tilde{\theta}):(\tilde{x}^a,\tilde{x}^\alpha)\}$ on $\Omega^*=\Omega\times\Omega^{\perp}$, such that $\mathcal{V}\cap\tilde{\mathcal{V}}\neq\emptyset$ and $\mathcal{W}\cap\tilde{\mathcal{W}}\neq\emptyset$. Then the local coordinates (x,u,y,v) and $(\tilde{x},\tilde{u},\tilde{y},\tilde{v})$ on $T\Omega^*=T\Omega\oplus T\Omega^{\perp}$ are related by [7]:

$$\begin{cases}\tilde{x}^a=\tilde{x}^a(x^1,\ldots,x^m),\quad \tilde{x}^\alpha=\tilde{x}^\alpha(x^{m+1},\ldots,x^{m+n}),\\ \tilde{y}^a=\dfrac{\partial\tilde{x}^a}{\partial x^b}y^b,\quad \tilde{y}^\alpha=\dfrac{\partial\tilde{x}^\alpha}{\partial x^\beta}y^\beta.\end{cases} \tag{4.31}$$

In the following, we will denote $\Omega^*_{\mathbb{T}}=T\Omega^*$, $\Omega_{\mathbb{T}}=T\Omega$, and $\Omega^{\perp}_{\mathbb{T}}=T\Omega^{\perp}$. If $\{\frac{\partial}{\partial x^a},\frac{\partial}{\partial y^a}\}$ and $\{\frac{\partial}{\partial x^\alpha},\frac{\partial}{\partial y^\alpha}\}$ are the basis on $\Omega_{\mathbb{T}}$ and $\Omega^{\perp}_{\mathbb{T}}$, resp., then we shall suppose that $\{\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^\alpha},\frac{\partial}{\partial y^a},\frac{\partial}{\partial y^\alpha}\}$ are their lifts to $\Omega_{\mathbb{T}}\times\Omega^{\perp}_{\mathbb{T}}$, respectively. As a consequence of (4.31) the local frame fields $\{\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^\alpha},\frac{\partial}{\partial y^a},\frac{\partial}{\partial y^\alpha}\}$ and $\{\frac{\partial}{\partial\tilde{x}^b},\frac{\partial}{\partial\tilde{x}^\beta},\frac{\partial}{\partial\tilde{y}^b},\frac{\partial}{\partial\tilde{y}^\beta}\}$

satisfy the following [33]:

$$\begin{cases} \dfrac{\partial}{\partial x^a} = \dfrac{\partial \tilde{x}^b}{\partial x^a}\dfrac{\partial}{\partial \tilde{x}^b} + \dfrac{\partial \tilde{y}^b}{\partial x^a}\dfrac{\partial}{\partial \tilde{y}^b}, \quad \dfrac{\partial}{\partial x^\alpha} = \dfrac{\partial \tilde{x}^\beta}{\partial x^\alpha}\dfrac{\partial}{\partial \tilde{x}^\beta} + \dfrac{\partial \tilde{y}^\beta}{\partial x^\alpha}\dfrac{\partial}{\partial \tilde{y}^\beta} \\ \dfrac{\partial}{\partial y^a} = \dfrac{\partial \tilde{x}^b}{\partial x^a}\dfrac{\partial}{\partial \tilde{y}^b}, \quad \dfrac{\partial}{\partial y^\alpha} = \dfrac{\partial \tilde{x}^\beta}{\partial x^\alpha}\dfrac{\partial}{\partial \tilde{y}^\beta} \\ rank \begin{bmatrix} \dfrac{\partial \tilde{x}^i}{\partial x^j} & 0 \\ 0 & \dfrac{\partial \tilde{x}^\alpha}{\partial x^\beta} \end{bmatrix} = m+n. \end{cases} \tag{4.32}$$

It is worth noticing that the $(m+n)$-dimensional product manifold $\Omega^* = \Omega \times \Omega^\perp$ can be considered as the configuration space of a dynamical system, which is governed by the following system of second-order ordinary differential equations:

$$\begin{cases} \dfrac{\mathrm{d}^2 x^a}{\mathrm{d}t^2} + 2G^a(x, u, \dfrac{\mathrm{d}x}{\mathrm{d}t}, \dfrac{\mathrm{d}u}{\mathrm{d}t}) = 0 \qquad 1 \le a \le m, \\ \dfrac{\mathrm{d}^2 x^\alpha}{\mathrm{d}t^2} + 2G^\alpha(x, u, \dfrac{\mathrm{d}x}{\mathrm{d}t}, \dfrac{\mathrm{d}u}{\mathrm{d}t}) = 0 \qquad m+1 \le \alpha \le m+n, \end{cases} \tag{4.33}$$

where system (4.33) is defined over a local chart on $\Omega^*_{\mathbb{T}} \simeq \Omega_{\mathbb{T}} \oplus \Omega^\perp_{\mathbb{T}}$. The functions $G^a(x,u,y,v)$ and $G^\alpha(x,u,y,v)$ are of class C^∞ on $\Omega^*_{\mathbb{T}} - \{0\}$ and only continuous on the null section. Hence, we have a collection of systems (4.33) on every induced local chart on $\Omega^*_{\mathbb{T}}$ that are compatible on the intersection of the induced local chart. This is equivalent to the fact that under the change (4.31) of local induced coordinates on $\Omega^*_{\mathbb{T}}$, the functions $G^a(x,u,y,v)$ and $G^\alpha(x,u,y,v)$ transform as follows [33]:

$$2\tilde{\mathbf{G}}^b = \frac{\partial \tilde{x}^b}{\partial x^a} 2G^a - \frac{\partial \tilde{y}^b}{\partial x^a} y^a, \qquad 2\tilde{\mathbf{G}}^\beta = \frac{\partial \tilde{x}^\beta}{\partial x^a} 2G^a - \frac{\partial \tilde{y}^\beta}{\partial x^\alpha} y^\alpha. \tag{4.34}$$

Proof Now, taking into account the change of local coordinates (4.31) on TM and by considering (4.32), it is deduced that

$$y^a \tfrac{\partial}{\partial x^a} - 2\mathbf{G}^a(x,u,y,v)\frac{\partial}{\partial y^a} + y^\alpha \frac{\partial}{\partial x^\alpha} - 2\mathbf{G}^\alpha(x,u,y,v)\frac{\partial}{\partial y^\alpha}$$
$$= \tilde{y}^a \frac{\partial}{\partial \tilde{x}^a} - 2\tilde{\mathbf{G}}^a(\tilde{x},\tilde{u},\tilde{y},\tilde{v})\partial \tilde{y}^a + \tilde{y}^\alpha \frac{\partial}{\partial \tilde{x}^\alpha} - 2\tilde{\mathbf{G}}^\alpha(\tilde{x},\tilde{u},\tilde{y},\tilde{v})\frac{\partial}{\partial \tilde{y}^\alpha}$$

if and only if the functions $\mathbf{G}^a$, $\tilde{\mathbf{G}}^a$, $\mathbf{G}^\alpha$, and $\tilde{\mathbf{G}}^a$ are related by (4.34). □

Thus, a vector field $S \in \Gamma(TTM)$ is called a semispray or a second-order vector field if on every domain of local charts of $\Omega^*_{\mathbb{T}}$ we have a collection of the functions $\{\mathbf{G}^a, \mathbf{G}^\alpha\}$ such that:

$$S = y^a \frac{\partial}{\partial x^a} - 2\mathbf{G}^a(x,u,y,v)\frac{\partial}{\partial y^a} + y^\alpha \frac{\partial}{\partial x^\alpha} - 2\mathbf{G}^\alpha(x,u,y,v)\frac{\partial}{\partial y^\alpha}. \tag{4.35}$$

The functions $\{\mathbf{G}^a, \mathbf{G}^\alpha\}$ are called the local coefficients of the semispray. Suppose that S is a semispray to form (4.35) with local coefficients $\{\mathbf{G}^a, \mathbf{G}^\alpha\}$. We put:

$$\mathbf{N} = (\mathbf{G}^i_j) = \begin{pmatrix} \mathbf{G}^a_b := \dfrac{\partial \mathbf{G}^a}{\partial y^b} & \mathbf{G}^\alpha_b := \dfrac{\partial \mathbf{G}^\alpha}{\partial y^b} \\ \mathbf{G}^a_\beta := \dfrac{\partial \mathbf{G}^a}{\partial y^\beta} & \mathbf{G}^\alpha_\beta := \dfrac{\partial \mathbf{G}^\alpha}{\partial y^\beta} \end{pmatrix}. \tag{4.36}$$

Consequently, $\mathbf{N}$ is a nonlinear connection on $\Omega^*_{\mathbb{T}} = \Omega_{\mathbb{T}} \oplus \Omega^{\perp}_{\mathbb{T}}$. In local coordinates the semispray induced by the nonlinear connection $\mathbf{N} = (\mathbf{G}^i_j)$ with coefficients (4.36) is defined by:

$$S = y^a \frac{\partial}{\partial x^a} - (\mathbf{G}^a_b y^b + \mathbf{G}^a_\beta y^\beta)\frac{\partial}{\partial y^a} + y^\alpha \frac{\partial}{\partial x^\alpha} - (\mathbf{G}^\alpha_b y^b + \mathbf{G}^\alpha_\beta y^\beta)\frac{\partial}{\partial y^\alpha}. \tag{4.37}$$

This means that coefficients of the induced semispray are identified as follows:

$$2\mathbf{G}^a(x,u,y,v) = \mathbf{G}^a_b y^b + \mathbf{G}^a_\beta y^\beta, \quad 2\mathbf{G}^\alpha(x,u,y,v) = \mathbf{G}^\alpha_b y^b + \mathbf{G}^\alpha_\beta y^\beta.$$

By applying (4.36), we define:

$$\begin{cases} \dfrac{\delta^*}{\delta x^a} = \dfrac{\partial}{\partial x^a} - \mathbf{G}^b_a \dfrac{\partial}{\partial y^b} - \mathbf{G}^\beta_a \dfrac{\partial}{\partial y^\beta} \\ \dfrac{\delta^*}{\delta x^\alpha} = \dfrac{\partial}{\partial x^\alpha} - \mathbf{G}^b_\alpha \dfrac{\partial}{\partial y^b} - \mathbf{G}^\beta_\alpha \dfrac{\partial}{\partial y^\beta}. \end{cases} \tag{4.38}$$

Therefore, if we put $\mathcal{V}(TM) := span\{\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^\alpha}\}$, $\mathcal{H}(TM) := span\{\frac{\delta^*}{\delta x^a}, \frac{\delta^*}{\delta x^\alpha}\}$, then we can write $TTM = \mathcal{V}(TM) \oplus \mathcal{H}(TM)$. It is worth mentioning that we can take the coordinate system $\{(\mathcal{V}^* \times \mathcal{W}^*, \Sigma \times \Theta) : (x^a, x^\alpha; y^a, y^\alpha)\}$ as a coordinate system on $\Omega^*_{\mathbb{T}}$ compatible with both foliations $\mathcal{F}_{\mathbb{T}}$ and $\mathcal{F}^{\perp}_{\mathbb{T}}$. That is to say:

$$T(\mathcal{F}_{\mathbb{T}}) = \mathrm{span}\{\frac{\partial}{\partial y^a}, \frac{\delta^*}{\delta x^a}\}, \qquad T(\mathcal{F}^{\perp}_{\mathbb{T}}) = \mathrm{span}\{\frac{\partial}{\partial y^\alpha}, \frac{\delta^*}{\delta x^\alpha}\}. \tag{4.39}$$

It is clear that with respect to the above coordinate system the matrix of the local components of $\widetilde{g}$ has the form

$$[\widetilde{g}_{ij}(x,y)] = \begin{pmatrix} g_{ab}(x,y) & 0 \\ 0 & g_{\alpha\beta}(x,y), \end{pmatrix}, \quad i,j \in \{1,...,m+n\}. \tag{4.40}$$

This means that with respect to the frame field $\{\dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^i}\}$, which is locally defined on TM, the following can be stated:

$$\begin{cases} g = g_{ab}(x,y) = \widetilde{g}(\dfrac{\delta^*}{\delta x^a}, \dfrac{\delta^*}{\delta x^b}) = \widetilde{g}(\dfrac{\partial}{\partial y^a}, \dfrac{\partial}{\partial y^b}), \\ \widetilde{g}(\dfrac{\delta}{\delta x^a}, \dfrac{\partial}{\partial y^b}) = 0, \quad a,b \in \{1,...,m\}, \quad \text{and} \\ g^{\perp} = g_{\alpha\beta}(x,y) = \widetilde{g}(\dfrac{\delta^*}{\delta x^\alpha}, \dfrac{\delta^*}{\delta x^\beta}) = \widetilde{g}(\dfrac{\partial}{\partial y^\alpha}, \dfrac{\partial}{\partial y^\beta}), \\ \widetilde{g}(\dfrac{\delta}{\delta x^\alpha}, \dfrac{\partial}{\partial y^\beta}) = 0, \quad \alpha,\beta \in \{1,...,n\}. \end{cases} \tag{4.41}$$

Now, by applying the metric $\widetilde{g}$, the following decomposition can be reached:

$$TT(M) = T(\mathcal{F}_{\mathbb{T}}) \oplus T(\mathcal{F}_{\mathbb{T}}^{\perp}). \tag{4.42}$$

By π and $\widetilde{\pi}$ the projection morphism can be denoted on $T(\mathcal{F}_{\mathbb{T}})$ and $T(\mathcal{F}_{\mathbb{T}}^{\perp})$, respectively. As stated previously in Theorem (3.2), according to [6] with respect to the above decomposition there exist unique linear connections $\nabla^{\mathcal{I}}$ and $\nabla^{\mathcal{I}^{\perp}}$ on $T(\mathcal{F}_{\mathbb{T}})$ and $T(\mathcal{F}_{\mathbb{T}}^{\perp})$, respectively, denoted by intrinsic connections. Now, according to [6], by considering the Levi-Civita connection $\widetilde{\nabla}$ on $(TM, \widetilde{g})$ the following can be stated:

$$\begin{cases} (a) : \nabla^{\mathcal{I}}_X \pi Y = \pi \widetilde{\nabla}_{\pi X} \pi Y + \pi[\widetilde{\pi} X, \pi Y], \\ (b) : \nabla^{\mathcal{I}^{\perp}}_X \widetilde{\pi} Y = \widetilde{\pi} \widetilde{\nabla}_{\widetilde{\pi} X} \widetilde{\pi} Y + \widetilde{\pi}[\pi X, \widetilde{\pi} Y]. \end{cases} \tag{4.43}$$

We call $\mathbb{H} : \Gamma(T(\mathcal{F}_{\mathbb{T}})) \times \Gamma(T(\mathcal{F}_{\mathbb{T}})) \to \Gamma(T(\mathcal{F}_{\mathbb{T}}^{\perp}))$ and $\widetilde{\mathbb{H}} : \Gamma(T(\mathcal{F}_{\mathbb{T}}^{\perp})) \times \Gamma(T(\mathcal{F}_{\mathbb{T}}^{\perp})) \to \Gamma(T(\mathcal{F}_{\mathbb{T}}))$, given by

$$\begin{cases} \mathbb{H}(\pi X, \pi Y) = \widetilde{\pi} \widetilde{\nabla}_{\pi X} \pi Y \\ \widetilde{\mathbb{H}}(\widetilde{\pi} X, \widetilde{\pi} Y) = \pi \widetilde{\nabla}_{\widetilde{\pi} X} \widetilde{\pi} Y, \end{cases} \tag{4.44}$$

the second fundamental forms of $T(\mathcal{F}_{\mathbb{T}})$ and $T(\mathcal{F}_{\mathbb{T}}^{\perp})$, respectively. Since $\mathcal{F}_{\mathbb{T}}$ is a parallel nondegenerate foliation on TM, then for any $X, Y \in \Gamma(T(\mathcal{F}_{\mathbb{T}}))$ we have $\widetilde{\nabla}_X Y \in \Gamma(T(\mathcal{F}_{\mathbb{T}}))$. Thus, by (4.44) it follows that the second fundamental form $\mathbb{H}$ of $\mathcal{F}_{\mathbb{T}}$ vanishes identically on TM. Hence, $\mathcal{F}_{\mathbb{T}}$ defines a totally geodesic foliated cocycle on TM. Moreover, considering the fact that $\widetilde{g}$ is parallel with respect to $\widetilde{\nabla}$, it is inferred that

$$\begin{aligned} &\widetilde{g}(\widetilde{\nabla}_x Y, Z) + \widetilde{g}(Y, \widetilde{\nabla}_x Z) = 0, \\ &\quad \forall\ X \in \Gamma(TTM),\ Y \in \Gamma\big(T(\mathcal{F}_{\mathbb{T}})\big),\ Z \in \Gamma\big(T(\mathcal{F}_{\mathbb{T}}^{\perp})\big). \end{aligned} \tag{4.45}$$

As $\widetilde{\nabla}_x Y \in \Gamma\big(T(\mathcal{F}_{\mathbb{T}})\big)$, we have $\widetilde{g}(\widetilde{\nabla}_x Y, Z) = 0$. Hence, $\widetilde{g}(Y, \widetilde{\nabla}_x Z) = 0$, which implies that $\widetilde{\nabla}_x Z \in \Gamma\big(T(\mathcal{F}_{\mathbb{T}}^{\perp})\big)$. Thus, $T(\mathcal{F}_{\mathbb{T}}^{\perp})$ is also parallel with respect to $\widetilde{\nabla}$ and therefore integrable. In a similar way as above, for any $X, Y \in \Gamma\big(T(\mathcal{F}_{\mathbb{T}}^{\perp})\big)$ we have $\widetilde{\nabla}_x Y \in \Gamma\big(T(\mathcal{F}_{\mathbb{T}}^{\perp})\big)$. Thus, according to (4.44) it follows that the second fundamental form $\widetilde{\mathbb{H}}$ of $\mathcal{F}_{\mathbb{T}}^{\perp}$ vanishes identically on TM. Hence, $\mathcal{F}_{\mathbb{T}}^{\perp}$ is also totally geodesic. Finally, by definition (4.2) we obtain that $\big[g_{ab}\big]$ and $\big[g_{\alpha\beta}\big]$ represent the matrices of two ATMs on $\Omega_{\mathbb{T}}$ and $\Omega_{\mathbb{T}}^{\perp}$, respectively. Consequently, it is demonstrated that both above foliations are holonomy invariant and the proof is completed. □

5. Conclusions

Geometric analysis of the tangent bundle (TM, π, M) over a smooth manifold M is one of the most significant fields of modern differential geometry and has outstanding applications in various problems, specifically in the theory of physical fields. This paper is devoted to the thorough investigation of the holonomy invariant foliated cocycles on the tangent space of an arbitrary $(m+n)$-dimensional manifold via the concept of formal integrability as a powerful device and a significant reformulation of the inverse problem of the calculus of variations. For this purpose, first of all a brief discussion regrading the expression of Helmholtz conditions in terms of Frölicher–Nijenhuis formalism and partial differential operators is presented. Mainly, it is illustrated that via this approach by applying Spencer theory a noteworthy setting is provided in order to construct the holonomy invariant foliated cocycles on the tangent bundle. Finally, by imposing four significant conditions on the foliated cocycle $\mathcal{F} = \{U_i, h_i, \mathcal{T}, \zeta_{ij}\}$ of codimension n on M, the holonomy invariant foliated cocycle $\mathcal{F}^{\mathbb{T}} = \{\tilde{U}_i, \tilde{h}_i, \tilde{\mathcal{T}}, \tilde{\zeta}_{ij}, g^{\mathbb{T}}\}$ on the tangent space TM is constructed. Accordingly, an important geometric structure on the tangent space is created, which is totally adapted to the Helmholtz conditions.

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