Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 728 – 750
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doi:10.3906/mat-1702-10

Compactness of the commutators of intrinsic square functions on weighted Lebesgue spaces

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Received: 04.02.2017 • **Accepted/Published Online:** 26.02.2019 • **Final Version:** 27.03.2019

Abstract: The aim of this paper is to study the compactness for the commutators of intrinsic square functions, including the intrinsic g_λ^*-function and the intrinsic Littlewood–Paley g-function. Using a weighted version of the Frechét–Kolmogorov–Riesz theorem, the compactness for their commutators generated with the CMO functions is obtained on the weighted Lebesgue spaces.

Key words: Intrinsic square functions, intrinsic g_λ^*-function, commutators, weighted Lebesgue spaces, CMO functions

1. Introduction

For $0<\alpha\leq 1$, let $\mathcal{C}_\alpha$ be the family of functions $\phi:\mathbb{R}^n\mapsto\mathbb{R}$ such that ϕ's support is contained in $\{x:|x|\leq 1\}$, $\int\phi dx=0$, and for $x,x'\in\mathbb{R}^n$,

$$|\phi(x)-\phi(x')|\leq|x-x'|^\alpha.$$

For $(y,t)\in\mathbb{R}_+^{n+1}$ and $f\in L^1_{loc}(\mathbb{R}^n)$, set

$$A_\alpha f(t,y)\equiv\sup_{\phi\in\mathcal{C}_\alpha}|f*\phi_t(y)|,$$

where $\phi_t(y)=t^{-n}\phi(\frac{y}{t})$. Then we define the varying-aperture intrinsic square (intrinsic Lusin) function of f by the formula

$$G_{\alpha,\beta}(f)(x)=\left(\iint_{\Gamma_\beta(x)}(A_\alpha f(t,y))^2\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}},$$

where $\Gamma_\beta(x)=\{(y,t)\in\mathbb{R}_+^{n+1}:|x-y|<\beta t\}$. Denote $G_{\alpha,1}(f)=G_\alpha(f)$.

The intrinsic square functions were first introduced by Wilson in order to answer a conjecture proposed by Fefferman and Stein on the boundedness of the Lusin area function S on the weighted L^2 Lebesgue space [19, 20]. The intrinsic square function has several interesting features. First, it is independent of any particular kernel, such as the Poisson kernel. It dominates pointwise the classical square function (Lusin area integral) and its real-variable generalizations. Second, although the function $G_{\alpha,\beta}(f)$ is defined by the kernels with uniform compact support, there is a pointwise relation between $G_{\alpha,\beta}(f)$ with different β:

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2010 *AMS Mathematics Subject Classification:* 42B20, 42B25, 47B07

$$G_{\alpha,\beta}(f)(x) \leq \beta^{\frac{3n}{2}+\alpha} G_\alpha(f)(x).$$

We point out that for the classical square functions such a pointwise relation is not available. We can see details in [19].

The intrinsic Littlewood–Paley g-function and the intrinsic g_λ^*-function are defined respectively by

$$g_\alpha f(x) = \left(\int_0^\infty (A_\alpha f(t,y))^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

$$g_{\lambda,\alpha}^* f(x) = \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} (A_\alpha f(t,y))^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}},$$

where $A_\alpha f(t,y)$ is defined as above. In [19], Wilson proved the following result.

Theorem A([19]) Let $1 < p < \infty$, $0 < \alpha \leq 1$, $\omega \in A_p$, and then G_α is bounded from $L^p(\omega)$ to itself.

After that, many papers focused on the boundedness of these operators. Among them, we list [4, 11–13, 17, 18] and references therein. Let b be a locally integrable function on $\mathbb{R}^n$. The commutators generated with BMO functions are defined by

$$[b, G_\alpha] f(x) = \left(\iint_{\Gamma(x)} \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \phi_t(y-z) f(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}},$$

$$[b, g_\alpha] f(x) = \left(\int_0^\infty \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \phi_t(y-z) f(z) dz \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

and

$$[b, g_{\lambda,\alpha}^*] f(x) = \left(\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \phi_t(y-z) f(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}.$$

As usual, a function $f \in L^1_{loc}(\mathbb{R}^n)$ is said to be in BMO if

$$\|f\|_{BMO} = \sup_{x \in \mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy < \infty,$$

where $f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy$ and $B(x,r)$ denotes the ball centered at x with radius r. Moreover, CMO denotes the closure of C_c^∞ in the BMO topology. C_c^∞ is the set of C^∞ functions with compact supports in $\mathbb{R}^n$.

A weight is a function $\omega \in L^1_{loc}(\mathbb{R}^n)$ such that $\omega(x) > 0$ almost everywhere. A weight ω is said to belong to the Muckenhoupt class $A_p, 1 < p < \infty$, if

$$\sup \left(\frac{1}{|Q|} \int_Q \omega(x) dx \right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{-\frac{p'}{p}} dx \right)^{\frac{p}{p'}} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, and $\frac{1}{p} + \frac{1}{p'} = 1$.

In [17], Wang obtained the boundedness of these commutators on weighted Lebesgue space. We state it as follows.

Theorem B([17]) Let $1 < p < \infty$, $0 < \alpha \leq 1$, $\omega \in A_p$. Then the commutators $[b, G_\alpha]$ and $[b, g^*_{\lambda,\alpha}]$ are bounded from $L^p(\omega)$ to themselves whenever $b \in BMO$.

In this paper, we are interested in the compactness for the commutators of G_α, g_α, and $g^*_{\lambda,\alpha}$ on the weighted Lebesgue spaces. We briefly summarize some classical and recent works in the literature, which lead to the motivation in this paper. In the linear setting, the first paper on the compactness of commutators was written by Uchiyama [15]. He improved the boundedness result of the commutators of singular integral operators to compactness when the symbol is in CMO. Since then, many authors have studied the compactness of various operators on different function spaces, such as [1–3, 5–9, 14, 16] and the references therein. In the sublinear setting, Chen and Ding [7] first studied the compactness of the classical Littlewood–Paley operators on the Lebesgue spaces. The compactness of the commutator of singular integral operators on weighted spaces was not known until the work of Clop and Cruz [9]. Here, we will show that the commutators of the intrinsic square functions are compact on the weighted Lebesgue spaces when the symbol is in CMO. We will also study the commutators of g_α and $g^*_{\lambda,\alpha}$. It is worth pointing out that, unlike the singular integral operators in [9], it is easy to see that these three operators are sublinear. Thus, in this paper, we will use some new ideas to overcome the nonlinear difficulty in the weighted setting.

Our main results in this paper are stated as follows.

Theorem 1.1 *Let $1 < p < \infty$, $0 < \alpha \leq 1$, $\omega \in A_p$. If $b \in CMO$, then $[b, G_\alpha]$ is a compact operator on $L^p(\omega)$.*

Theorem 1.2 *Let $1 < p < \infty$, $0 < \alpha \leq 1$, $\omega \in A_p$. If $b \in CMO$, then for $\lambda > 3 + \frac{2\alpha}{n}$, we have that $[b, g^*_{\lambda,\alpha}]$ is a compact operator on $L^p(\omega)$.*

In [19], the author proved that the functions $G_\alpha f$ and $g_\alpha f$ are pointwise comparable. Thus, as a consequence of the proof of Theorem 1.1, we have the following results.

Corollary 1.3 Let $1 < p < \infty$, $0 < \alpha \leq 1$, $\omega \in A_p$. If b $\in CMO$, then $[b, g_\alpha]$ is a compact operator on $L^p(\omega)$.

Throughout this paper, we use the notation $A \preceq B$ to mean that there is a positive constant C independent of all essential variables such that $A \leq CB$. Moreover, the constant C may be different from place to place.

2. Definitions and lemmas

Lemma 2.1 *([10]) Let $\omega \in A_p$ with $1 < p < \infty$. Then:*

1) for any ball B and $k > 1$, there exists an absolutely constant C such that

$$\omega(kB) \leq C k^{np} \omega(B);$$

2) there is a $q < p$ such that $\omega \in A_q$.

Lemma 2.2 *Let $b_1, b_2 \in BMO, \lambda > 0$, and $0 < \alpha \leq 1$. The commutators of G_α and $g^*_{\lambda,\alpha}$ are defined in Section 1. Then we have:*

1) $|[b_1, G_\alpha]f(x) - [b_2, G_\alpha]f(x)| \le |[b_1 - b_2, G_\alpha]f(x)|$ *;*

2) $|[b_1, g^*_{\lambda,\alpha}]f(x) - [b_2, g^*_{\lambda,\alpha}]f(x)| \le |[b_1 - b_2, g^*_{\lambda,\alpha}]f(x)|$ *.*

Proof 1)

$$
\begin{aligned}
|[b_1, G_\alpha]f(x) - [b_2, G_\alpha]f(x)| =& \left| \left(\iint_{\Gamma(x)} \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} (b_1(x) - b_1(z))\phi_t(y-z)f(z)dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \right. \\
& \left. - \left(\iint_{\Gamma(x)} \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} (b_2(x) - b_2(z))\phi_t(y-z)f(z)dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \right| \\
\le& \left(\iint_{\Gamma(x)} \sup_{\phi \in \mathcal{C}_\alpha} \left(\left| \int_{\mathbb{R}^n} (b_1(x) - b_1(z))\phi_t(y-z)f(z)dz \right| \right. \right. \\
& \left. \left. - \left| \int_{\mathbb{R}^n} (b_2(x) - b_2(z))\phi_t(y-z)f(z)dz \right| \right)^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \\
\le& \left| \left(\iint_{\Gamma(x)} \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} [(b_1(x) - b_2(x)) - (b_1(z) - b_2(z))]\phi_t(y-z)f(z)dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \right. \\
=& |[b_1 - b_2, G_\alpha]f(x)|.
\end{aligned}
$$

The proof of conclusion 2) is similar to the proof of conclusion 1). We omit the details. □

Definition 2.1 $T : V \to Y$ *is said to be a compact operator if* T *is continuous and maps bounded subsets into strongly precompact subsets.*

A criterion for compactness in weighted L^p spaces is provided by the following weighted version of the Frechét–Kolmogorov–Riesz theorem.

Lemma 2.3 *([9]) Let* $p \in (1, \infty), \omega \in A_p$*, and let* $F \subset L^p(\omega)$*. Then* F *is a strongly precompact subset if it satisfies the following conditions:*

(C1) F *is uniformly bounded, i.e.* $\sup_{f \in F} \|f\|_{L^p(\omega)} < \infty$*;*

(C2) F *is uniformly equicontinuous, i.e.* $\sup_{f \in F} \|f(\cdot + h) - f(\cdot)\|_{L^p(\omega)} \overset{h \to 0}{\longrightarrow} 0$*;*

(C3) F *uniformly vanishes at infinity, i.e.* $\lim_{R \to \infty} \int_{|x|>R} |f(x)|^p \omega(x)dx = 0$*.*

3. Proof of Theorem 1.1

Since $b \in CMO \subset BMO$, by Theorem B, we have that the commutator $[b, G_\alpha]$ is continuous in $L^p(\omega)$. Hence, for any bounded set $F \subset L^p(\omega)$, where $f \in F$ must satisfy $\|f\|_{L^p(\omega)} \preceq 1$, it suffices to prove that $\{[b, G_\alpha]f : f \in F, b \in CMO\}$ is a strongly precompact subset. According to a density argument, if $b \in CMO$, then there exists a sequence of functions $b^\epsilon \in C_c^\infty(\mathbb{R}^n)$, such that

$$\|b^\epsilon - b\|_{BMO} < \epsilon.$$

Thus, by Lemma 2.2 and Theorem B, we get

$$\|[b, G_\alpha] - [b^\epsilon, G_\alpha]\|_{L^p(\omega)\to L^p(\omega)} \le \|[b - b^\epsilon, G_\alpha]\|_{L^p(\omega)\to L^p(\omega)} \preceq \epsilon.$$

Therefore, it will be enough to prove that $G = \{[b, G_\alpha]f : f \in F, b \in C_c^\infty(\mathbb{R}^n)\}$ is strongly precompact. Then, by Lemma 2.3, we need to show that conditions (C1)–(C3) hold uniformly for set G. First, by Theorem B, we have

$$\sup_{f\in F} \|[b, G_\alpha]f\|_{L^p(\omega)} \preceq \|b\|_{BMO} \sup_{f\in F} \|f\|_{L^p(\omega)} < \infty.$$

Thus, condition (C1) holds for G. Next, we verify condition (C3); that is,

$$\lim_{R\to\infty} \left(\int_{|x|>R} |[b, G_\alpha]f(x)|^p \,\omega(x)dx \right)^{\frac{1}{p}} = 0. \tag{3.1}$$

Suppose supp $b \subset \{z : |z| < \eta\}$. Let us choose $R > 2\eta$. For $|x| > R > 2\eta$, we have $b(x) = 0$ and $|x| > 2|z|$. Thus, $|x| \le |x - z| + |z| \le |x - z| + \frac{1}{2}|x|$, and we get $|x| \le 2|x - z|$. Therefore, by the Minkowski inequality, we have

$$\begin{aligned}
[b, G_\alpha]f(x) &= \left(\int_0^\infty \int_{|x-y|<t} \sup_{\phi\in\mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} (b(x) - b(z))\phi_t(y - z)f(z)dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \\
&= \left(\int_0^\infty \int_{|x-y|<t} \sup_{\phi\in\mathcal{C}_\alpha} \left| \int_{|z|<\eta} b(z)\phi_t(y - z)f(z)dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \\
&\preceq \left(\int_0^\infty \int_{|x-y|<t} \left| \int_{\substack{|y-z|\le t \\ |z|<\eta}} |b(z)f(z)|dz \right|^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} \\
&\le \int_{|z|<\eta} \left(\int_0^\infty \int_{\substack{|x-y|\le t \\ |y-z|\le t}} |b(z)f(z)|^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} dz \\
&\preceq \int_{|z|<\eta} \left(\int_{\frac{|x-z|}{2}}^\infty \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}} |b(z)f(z)|\, dz \\
&\preceq \|b\|_\infty \int_{|z|<\eta} \frac{|f(z)|}{|x - z|^n} dz \preceq \frac{1}{|x|^n} \int_{|z|<\eta} |f(z)|dz.
\end{aligned}$$

Then, by the Hölder inequality, it follows that

$$\begin{aligned}\left(\int_{|x|>R} |[b,G_\alpha]f(x)|^p\,\omega(x)dx\right)^{\frac{1}{p}} &\preceq \left(\int_{|x|>R} \frac{1}{|x|^{np}}\left(\int_{|z|<\eta}|f(z)|dz\right)^p \omega(x)dx\right)^{\frac{1}{p}}\\ &= \int_{|z|<\eta}|f(z)|dz\left(\int_{|x|>R}\frac{1}{|x|^{np}}\omega(x)dx\right)^{\frac{1}{p}}\\ &\preceq \left(\int_{|x|>R}\frac{1}{|x|^{np}}\omega(x)dx\right)^{\frac{1}{p}}\int_{|z|<\eta}|f(z)|\omega(z)^{\frac{1}{p}}\omega(z)^{-\frac{1}{p}}dz\\ &\leq \left(\int_{|x|>R}\frac{1}{|x|^{np}}\omega(x)dx\right)^{\frac{1}{p}}\left(\int_{|z|<\eta}|f(z)|^p\omega(z)dz\right)^{\frac{1}{p}}\left(\int_{|z|<\eta}\omega(z)^{-\frac{p'}{p}}dz\right)^{\frac{1}{p'}}\\ &\preceq \left(\int_{|x|>R}\frac{1}{|x|^{np}}\omega(x)dx\right)^{\frac{1}{p}}\|f\|_{L^p(\omega)}.\end{aligned}$$

By Lemma 2.1, one can choose a $q<p$ such that $\omega\in A_q$, and we claim that

$$\begin{aligned}\int_{|x|>R}\frac{\omega(x)}{|x|^{np}}dx &= \sum_{j=1}^{\infty}\int_{2^{j-1}R<|x|\leq 2^jR}\frac{\omega(x)}{|x|^{np}}dx\\ &\leq \sum_{j=1}^{\infty}(2^{j-1}R)^{-np}\int_{|x|\leq 2^jR}\omega(x)dx\\ &= \sum_{j=1}^{\infty}(2^{j-1}R)^{-np}\omega(B(0,2^jR))\\ &\preceq \sum_{j=1}^{\infty}2^{-nj(p-q)}R^{-n(p-q)}\omega(B(0,1))\\ &\preceq R^{-n(p-q)}.\end{aligned}$$

Therefore, letting $R\to\infty$, we obtain (3.1). At last, we estimate

$$\sup_{f\in F}\|[b,G_\alpha]f(\cdot+h)-[b,G_\alpha]f(\cdot)\|_{L^p(\omega)}.$$

Similar to Lemma 2.2, we obtain that

$$|[b,G_\alpha]f(x+h)-[b,G_\alpha]f(x)| \leq \left(\int_0^\infty\int_{|x-y|<t}\sup_{\phi\in\mathcal{C}_\alpha}|I(x,y,t)|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}, \tag{3.2}$$

where

$$I(x,y,t)=\int\phi(\frac{y+h-z}{t})[b(x+h)-b(z)]f(z)dz-\int\phi(\frac{y-z}{t})[b(x)-b(z)]f(z)dz.$$

For every x, choose $0<\epsilon<\frac{1}{2}$, and let us write

$$\begin{aligned}I(x,y,t)=&\int_{|x-z|<2^{1/\epsilon}|h|}\phi(\frac{y-z}{t})[b(z)-b(x)]f(z)dz\\&+\int_{|x-z|<2^{1/\epsilon}|h|}\phi(\frac{y+h-z}{t})[b(x+h)-b(z)]f(z)dz\\&+\int_{|x-z|\geq 2^{1/\epsilon}|h|}\left[\phi(\frac{y+h-z}{t})-\phi(\frac{y-z}{t})\right][b(x+h)-b(z)]f(z)dz\\&+\int_{|x-z|\geq 2^{1/\epsilon}|h|}\phi(\frac{y-z}{t})[b(x+h)-b(x)]f(z)dz\\=:&\ I+II+III+IV.\end{aligned}\tag{3.3}$$

First, we estimate $\left(\int_0^\infty\int_{|x-y|<t}\sup_{\phi\in\mathcal{C}_\alpha}|I|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}$. By the Minkowski inequality, we obtain

$$\begin{aligned}\left(\int_0^\infty\int_{|x-y|<t}\sup_{\phi\in\mathcal{C}_\alpha}|I|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}&=\left(\int_0^\infty\int_{|x-y|<t}\sup_{\phi\in\mathcal{C}_\alpha}\left|\int_{\substack{|x-z|<2^{1/\epsilon}|h|\\|y-z|\leq t}}\phi(\frac{y-z}{t})[b(z)-b(x)]f(z)dz\right|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}\\&\preceq\|\nabla b\|_\infty\int_{|x-z|<2^{1/\epsilon}|h|}\left(\int_{\frac{|x-z|}{2}}^\infty\int_{|y-z|\leq t}\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}|x-z||f(z)|dz\\&\preceq\|\nabla b\|_\infty\int_{|x-z|<2^{1/\epsilon}|h|}\frac{|f(z)|}{|x-z|^{n-1}}dz\\&=\|\nabla b\|_\infty\sum_{j=1}^\infty\int_{2^{-j+1/\epsilon}|h|\leq|x-z|<2^{-j+1+1/\epsilon}|h|}\frac{|f(z)|}{|x-z|^{n-1}}dz\\&\preceq\sum_{j=1}^\infty\frac{1}{(2^{-j+1/\epsilon}|h|)^{n-1}}\int_{|x-z|\leq 2^{-j+1+1/\epsilon}|h|}|f(z)|dz\\&\preceq 2^{n+1/\epsilon}|h|\sum_{j=1}^\infty 2^{-j}Mf(x)\\&\preceq 2^{1/\epsilon}|h|Mf(x),\end{aligned}$$

where M is the Hardy–Littlewood maximal function defined as

$$Mf(x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(z)|dz.$$

Using the boundedness of M on $L^p(\omega)$ (see [10] Corollary 9.2.7), we obtain that

$$\left(\int_{\mathbb{R}^n}\left(\int_0^\infty\int_{|x-y|<t}\sup_{\phi\in\mathcal{C}_\alpha}|I|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{p}{2}}\omega(x)dx\right)^{\frac{1}{p}}\preceq 2^{1/\epsilon}|h|\|f\|_{L^p(\omega)}.\tag{3.4}$$

Then we estimate $\left(\int_0^\infty\int_{|x-y|<t} \sup_{\phi\in\mathcal{C}_\alpha} |II|^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}$. By the Minkowski inequality, we have that

$$\begin{aligned}
&\left(\int_0^\infty\int_{|x-y|<t} \sup_{\phi\in\mathcal{C}_\alpha} |II|^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}\\
&= \left(\int_0^\infty\int_{|x-y|<t} \sup_{\phi\in\mathcal{C}_\alpha} \left|\int_{\substack{|x-z|<2^{1/\epsilon}|h|\\ |y+h-z|\le t}} \phi(\frac{y+h-z}{t})[b(z)-b(x+h)]f(z)dz\right|^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}\\
&\preceq \|\nabla b\|_\infty \int_{|x-z|<2^{1/\epsilon}|h|} \left(\int_{\frac{|x+h-z|}{2}}^\infty \int_{|y+h-z|\le t} \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}} |x+h-z||f(z)|dz\\
&\preceq \|\nabla b\|_\infty \int_{|x+h-z|<2^{1/\epsilon+1}|h|} \frac{|f(z)|}{|x+h-z|^{n-1}}dz\\
&= \|\nabla b\|_\infty \sum_{j=1}^\infty \int_{2^{-j+1+1/\epsilon}|h|\le|x+h-z|<2^{-j+2+1/\epsilon}|h|} \frac{|f(z)|}{|x+h-z|^{n-1}}dz\\
&\preceq \sum_{j=1}^\infty \frac{1}{(2^{-j+1+1/\epsilon}|h|)^{n-1}} \int_{|x+h-z|\le 2^{-j+2+1/\epsilon}|h|} |f(z)|dz\\
&\preceq 2^{n+1/\epsilon}|h| \sum_{j=1}^\infty 2^{-j} Mf(x)\\
&\preceq 2^{1/\epsilon}|h|Mf(x).
\end{aligned}$$

Using the boundedness of M on $L^p(\omega)$ again, we have that

$$\left(\int_{\mathbb{R}^n}\left(\int_0^\infty\int_{|x-y|<t} \sup_{\phi\in\mathcal{C}_\alpha} |II|^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{p}{2}} \omega(x)dx\right)^{\frac{1}{p}} \preceq 2^{1/\epsilon}|h|\|f\|_{L^p(\omega)}. \tag{3.5}$$

Now we estimate III. We will divide the integral into three parts. Denote

$$A_1 = \{z\in\mathbb{R}^n : |y-z|\le t, |y-z+h|\le t\},$$

$$A_2 = \{z\in\mathbb{R}^n : |y-z|\le t, |y-z+h|> t\},$$

$$A_3 = \{z\in\mathbb{R}^n : |y-z|> t, |y-z+h|\le t\}.$$

Clearly, if $z\in\mathbb{R}^n\setminus A_1\cup A_2\cup A_3$, we have III=0. Denote

$$III_{A_i} = \int_{A_i\cap\{z||x-z|\ge 2^{1/\epsilon}|h|\}} \left[\phi(\frac{y+h-z}{t}) - \phi(\frac{y-z}{t})\right][b(x+h)-b(z)]f(z)dz,$$

where $i = 1, 2, 3$. We first estimate the integral over A_1. By the property of function ϕ and the Minkowski inequality, we get

$$
\begin{aligned}
&\left(\int_0^\infty \int_{|x-y|<t} \sup_{\phi\in\mathcal{C}_\alpha} |III_{A_1}|^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}} \\
&\preceq \|b\|_\infty \left(\int_0^\infty \int_{|x-y|<t} \left|\int_{A_1\cap\{z||x-z|\geq 2^{1/\epsilon}|h|} \left|\frac{h}{t}\right|^\alpha |f(z)|dz\right|^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}} \\
&\preceq |h|^\alpha \int_{|x-z|\geq 2^{1/\epsilon}|h|} \left(\int_{t>\frac{|x-z|}{2}} \int_{|y-z|\leq t} \frac{1}{t^{3n+2\alpha+1}} dydt\right)^{\frac{1}{2}} |f(z)|dz \\
&\preceq |h|^\alpha \int_{|x-z|\geq 2^{1/\epsilon}|h|} \frac{|f(z)|}{|x-z|^{n+\alpha}} dz \\
&\preceq \sum_{j=0}^\infty |h|^\alpha \int_{2^{j+1/\epsilon}|h|\leq|x-z|\leq 2^{j+1+1/\epsilon}|h|} \frac{|f(z)|}{|x-z|^{n+\alpha}} dz \\
&\preceq 2^{-\frac{\alpha}{\epsilon}} \sum_{j=0}^\infty 2^{-j\alpha} Mf(x) \preceq 2^{-\frac{\alpha}{\epsilon}} Mf(x).
\end{aligned}
$$

Using the boundedness of M on $L^p(\omega)$ again, we have that

$$
\left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{|x-y|<t} \sup_{\phi\in\mathcal{C}_\alpha} |III_{A_1}|^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{p}{2}} \omega(x)dx\right)^{\frac{1}{p}} \preceq 2^{-\frac{\alpha}{\epsilon}} \|f\|_{L^p(\omega)}. \tag{3.6}
$$

The integrals over A_2 and A_3 are symmetric, so we only give the details over A_2. Noticing that $|y-z| \leq t < |y-z+h|$ on A_2 and $|x-y| < t$, if $|y-z| \leq \frac{1}{3}|x-z|$, then $|y-z| + |h| \geq |y-z+h| > t > |x-y| = |x-z+z-y| \geq |x-z| - |y-z| \geq \frac{2}{3}|x-z| \Rightarrow |h| \geq \frac{2}{3}|x-z| - |y-z| \geq \frac{1}{3}|x-z|$; that is, $|x-z| \leq 3|h|$. This is a contradiction, so $|y-z| > \frac{1}{3}|x-z| \geq \frac{1}{3}2^{1/\epsilon}|h| \geq \frac{4}{3}|h|$. Then, by the Minkowski inequality, it follows that

$$
\begin{aligned}
&\left(\int_0^\infty \int_{|x-y|<t} \sup_{\phi\in\mathcal{C}_\alpha} \left|\int_{A_2\cap\{z||x-z|\geq 2^{1/\epsilon}|h|\}} \phi(\frac{y-z}{t})[b(x+h)-b(z)]f(z)dz\right|^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}} \\
&\preceq \|b\|_\infty \left(\int_0^\infty \int_{|x-y|<t} \left|\int_{\substack{|x-z|\geq 2^{1/\epsilon}|h| \\ |y-z|\leq t<|y-z+h|}} f(z)dz\right|^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}} \\
&\preceq \int_{|x-z|\geq 2^{1/\epsilon}|h|} \left(\int_{|y-z|>\frac{1}{3}|x-z|} \int_{|y-z|\leq t<|y-z+h|} \frac{dtdy}{t^{3n+1}}\right)^{\frac{1}{2}} |f(z)|dz
\end{aligned}
$$

$$\begin{aligned}
&\preceq \int_{|x-z|\geq 2^{1/\epsilon}|h|}\left(\int_{|y-z|>\frac{1}{3}|x-z|}\left|\frac{1}{|y-z|^{3n}}-\frac{1}{|y+h-z|^{3n}}\right|dy\right)^{\frac{1}{2}}|f(z)|dz\\
&\preceq |h|^{1/2}\int_{|x-z|\geq 2^{1/\epsilon}|h|}\left(\int_{|y-z|>\frac{1}{3}|x-z|}\frac{1}{|y-z|^{3n+1}}dy\right)^{\frac{1}{2}}|f(z)|dz\\
&\preceq |h|^{1/2}\int_{|x-z|\geq 2^{1/\epsilon}|h|}\frac{1}{|x-z|^{n+1/2}}|f(z)|dz\\
&\preceq |h|^{1/2}\sum_{j=0}^{\infty}\int_{2^{j+1/\epsilon}|h|\leq|x-z|\leq 2^{j+1+1/\epsilon}|h|}\frac{1}{|x-z|^{n+1/2}}|f(z)|dz\\
&\preceq 2^{-\frac{1}{2\epsilon}}\sum_{j=0}^{\infty}2^{-\frac{j}{2}}Mf(x)\\
&\preceq 2^{-\frac{1}{2\epsilon}}Mf(x).
\end{aligned}$$

Using the boundedness of M on $L^p(\omega)$ again, we have that

$$\left(\int_{\mathbb{R}^n}\left(\int_0^\infty\int_{|x-y|<t}\sup_{\phi\in\mathcal{C}_\alpha}|III_{A_2}|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{p}{2}}\omega(x)dx\right)^{\frac{1}{p}}\preceq 2^{-\frac{1}{2\epsilon}}\|f\|_{L^p(\omega)}. \tag{3.7}$$

For the last part,

$$\begin{aligned}
\left(\int_0^\infty\int_{|x-y|<t}\sup_{\phi\in\mathcal{C}_\alpha}|IV|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}} &= \left(\int_0^\infty\int_{|x-y|<t}\sup_{\phi\in\mathcal{C}_\alpha}\left|\int_{\substack{|x-z|\geq 2^{1/\epsilon}|h|\\ |y-z|\leq t}}\phi(\frac{y-z}{t})[b(x)-b(x+h)]f(z)dz\right|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}\\
&= |b(x)-b(x+h)|\left(\int_0^\infty\int_{|x-y|<t}\sup_{\phi\in\mathcal{C}_\alpha}\left|\int_{\substack{|x-z|\geq 2^{1/\epsilon}|h|\\ |y-z|\leq t}}\phi(\frac{y-z}{t})f(z)dz\right|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}\\
&:= |b(x)-b(x+h)|G_{\alpha,\epsilon,h}f(x).
\end{aligned}$$

We claim that

$$G_{\alpha,\epsilon,h}f(x)\preceq M(G_\alpha f)(x)+(M(|f|^q)(x))^{\frac{1}{q}}+Mf(x), \tag{3.8}$$

where $1<q<p$. Let Q denote the cube centered at x and with diameter $r=2^{1/\epsilon}|h|/4$. Furthermore, let $f_1(x)=f\chi_{4Q}(x)$ and $f_2(x)=f(x)-f_1(x)$. Then we get

$$\begin{aligned}
G_{\alpha,\epsilon,h}f(x) &= \frac{1}{|Q|}\int_Q|G_{\alpha,\epsilon,h}f(x)|d\xi\\
&\leq \frac{1}{|Q|}\int_Q|G_\alpha f(\xi)|d\xi+\frac{1}{|Q|}\int_Q|G_\alpha f_1(\xi)|d\xi\\
&\quad +\frac{1}{|Q|}\int_Q|G_\alpha f_2(\xi)-G_{\alpha,\epsilon,h}f(x)|d\xi\\
&\leq M(G_\alpha f)(x)+I_1+I_2.
\end{aligned}$$

By Theorem A (let $\omega = 1$) and Hölder's inequality, for any $1 < q < p$, we get that

$$I_1 \leq \frac{1}{|Q|^{1/q}} \left(\int_Q |G_\alpha f_1(\xi)|^q d\xi \right)^{\frac{1}{q}} \preceq \frac{1}{|Q|^{1/q}} \left(\int_Q |f_1(\xi)|^q d\xi \right)^{\frac{1}{q}} \preceq (M(|f|^q)(x))^{\frac{1}{q}}.$$

Now, let us estimate I_2. Similar to Lemma 2.2, we have

$$\begin{aligned}
&|G_\alpha f_2(\xi) - G_{\alpha,\epsilon,h} f(x)| \\
&= \Bigg| \left(\int_0^\infty \int_{|x-y|<t} \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{|\xi - x + y - z| \leq t} \phi(\frac{\xi - x + y - z}{t}) f_2(z) dz \right|^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} \\
&- \left(\int_0^\infty \int_{|x-y|<t} \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\substack{|x-z| \geq 2^{1/\epsilon}|h| \\ |y-z| \leq t}} \phi(\frac{y-z}{t}) f(z) dz \right|^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} \Bigg| \\
&\leq \Bigg(\int_0^\infty \int_{|x-y|<t} \sup_{\phi \in \mathcal{C}_\alpha} \Bigg| \int_{\substack{|x-z| \geq 2^{1/\epsilon}|h| \\ |\xi - x + y - z| \leq t}} \phi(\frac{\xi - x + y - z}{t}) f(z) dz \\
&\qquad - \int_{\substack{|x-z| \geq 2^{1/\epsilon}|h| \\ |y-z| \leq t}} \phi(\frac{y-z}{t}) f(z) dz \Bigg|^2 \frac{dydt}{t^{3n+1}} \Bigg)^{\frac{1}{2}} \\
&\leq \left(\int_0^\infty \int_{|x-y|<t} \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\substack{|x-z| \geq 2^{1/\epsilon}|h| \\ |y-z| \leq t \\ |\xi - x + y - z| > t}} \phi(\frac{y-z}{t}) f(z) dz \right|^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} \\
&+ \left(\int_0^\infty \int_{|x-y|<t} \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\substack{|x-z| \geq 2^{1/\epsilon}|h| \\ |y-z| > t \\ |\xi - x + y - z| \leq t}} \phi(\frac{\xi - x + y - z}{t}) f(z) dz \right|^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} \\
&+ \left(\int_0^\infty \int_{|x-y|<t} \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\substack{|x-z| \geq 2^{1/\epsilon}|h| \\ |y-z| \leq t \\ |\xi - x + y - z| \leq t}} \left[\phi(\frac{\xi - x + y - z}{t}) - \phi(\frac{y-z}{t}) \right] f(z) dz \right|^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} \\
&:= I_{21} + I_{22} + I_{23}.
\end{aligned}$$

Noticing $\xi \in Q$, similar to the proof of III_{A_2}, we get

$$\begin{aligned}
I_{21} &\preceq \int_{|x-z| \geq 2^{1/\epsilon}|h|} \left(\int_{|y-z| > \frac{1}{3}|x-z|} \int_{|y-z| \leq t < |\xi - x + y - z|} \frac{dtdy}{t^{3n+1}} \right)^{\frac{1}{2}} |f(z)| dz \\
&\preceq \int_{|x-z| \geq 2^{1/\epsilon}|h|} \frac{|\xi - x|^{\frac{1}{2}}}{|x-z|^{n+\frac{1}{2}}} |f(z)| dz \\
&\preceq (2^{1/\epsilon}|h|)^{\frac{1}{2}} \int_{|x-z| \geq 2^{1/\epsilon}|h|} \frac{1}{|x-z|^{n+\frac{1}{2}}} |f(z)| dz \\
&\preceq Mf(x).
\end{aligned}$$

Similarly, we obtain

$$I_{22} \preceq Mf(x).$$

For I_{23}, similar to III_{A_1}, we have

$$\begin{aligned}
I_{23} &\preceq \left(\int_0^\infty \int_{|x-y|<t} \left| \int_{\substack{|x-z|\geq 2^{1/\epsilon}|h| \\ |y-z|\leq t \\ |\xi - x + y - z| \leq t}} \left|\frac{\xi - x}{t}\right|^\alpha |f(z)|dz \right|^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}} \\
&\preceq (2^{1/\epsilon}|h|)^\alpha \int_{|x-z|\geq 2^{1/\epsilon}|h|} \left(\int_{t>\frac{|x-z|}{2}} \int_{|y-z|\leq t} \frac{dydt}{t^{3n+2\alpha+1}}\right)^{\frac{1}{2}} |f(z)|dz \\
&\preceq (2^{1/\epsilon}|h|)^\alpha \int_{|x-z|\geq 2^{1/\epsilon}|h|} \frac{|f(z)|}{|x-z|^{n+\alpha}} dz \\
&\preceq Mf(x).
\end{aligned}$$

Combining by the estimates of I_{21}, I_{22} and I_{23}, we obtain $I_2 \preceq Mf(x)$. Combining I_1 and I_2, we have claim (3.8). Now we are ready to give the estimates of IV. Noticing that $b \in C_c^\infty(\mathbb{R}^n)$, we have $|b(x) - b(x+h)| \leq C|h|$. Then, applying the claim (3.8) for $1 < q < p$, Theorem A, and the $L^p(\omega)(p>1)$ boundedness of M and M^q, we have

$$\left\| \left(\int_0^\infty \int_{|x-y|<t} \sup_{\phi \in \mathcal{C}_\alpha} |IV|^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}} \right\|_{L^p(\omega)} \preceq |h| \|f\|_{L^p(\omega)}. \tag{3.9}$$

By (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), and (3.9), we obtain

$$\sup_{f\in F} \|[b, G_\alpha]f(\cdot + h) - [b, G_\alpha]f(\cdot)\|_{L^p(\omega)} \preceq (2^{1/\epsilon}|h| + 2^{-\frac{\alpha}{\epsilon}} + 2^{-\frac{1}{2\epsilon}} + |h|)\|f\|_{L^p(\omega)}.$$

Taking $\epsilon = \frac{2}{\log \frac{1}{|h|}}$ and letting $h \to 0$, we have the uniform equicontinuity (condition (C2)) of G. Thus, we complete the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Since $b \in CMO \subset BMO$, by Theorem B, we have that the commutator $[b, g_{\lambda,\alpha}^*]$ is continuous in $L^p(\omega)$. Therefore, for any bounded subset $F \subset L^p(\omega)$, it suffices to prove that $\{[b, g_{\lambda,\alpha}^*]f : f \in F, b \in CMO\}$ is a strongly precompact subset. Similar to Theorem 1.1, according to a density argument, it will be enough to prove that $P = \{[b, g_{\lambda,\alpha}^*]f : f \in F, b \in C_c^\infty(\mathbb{R}^n)\}$ is strongly precompact. By Lemma 2.3, we need to show that conditions (C1)–(C3) hold uniformly for subset P. First, by Theorem B, it is easy to check condition (C1):

$$\sup_{f\in F} \|[b, g_{\lambda,\alpha}^*]f\|_{L^p(\omega)} \preceq \|b\|_{BMO} \sup_{f\in F} \|f\|_{L^p(\omega)} < \infty.$$

Next, we verify condition (C3):

$$\left(\int_{|x|>R} \left|[b, g_{\lambda,\alpha}^*]f(x)\right|^p \omega(x)dx\right)^{\frac{1}{p}} \to 0 (R \to \infty).$$

We decompose the function as follows:

$$[b,g_{\lambda,\alpha}^{*}](f)(x)=\left(\int_0^\infty\int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\sup_{\phi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}[b(x)-b(z)]\phi_t(y-z)f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac12}$$
$$+\left(\int_0^\infty\int_{|x-y|\geq t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\sup_{\phi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}[b(x)-b(z)]\phi_t(y-z)f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac12}$$
$$\leq[b,G_\alpha](f)(x)+C\sum_{i=1}^\infty 2^{-\frac{in\lambda}{2}}[b,S_{\alpha,i}](f)(x), \tag{4.1}$$

where

$$[b,S_{\alpha,i}](f)(x)=\left(\int_0^\infty\int_{2^{i-1}t\leq|x-y|<2^it}\sup_{\phi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}[b(x)-b(z)]\phi_t(y-z)f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac12}.$$

First, we claim that

$$\left(\int_{|x|>R}|[b,S_{\alpha,i}]f(x)|^p\,\omega(x)dx\right)^{\frac1p}\preceq 2^{in}R^{n(q-p)}\|f\|_{L^p(\omega)}(q<p). \tag{4.2}$$

Noticing that $\lambda>2$ and $q<p$, if claim (4.2) holds, combining (3.1), we will get

$$\left(\int_{|x|>R}\left|[b,g_{\lambda,\alpha}^{*}]f(x)\right|^p\omega(x)dx\right)^{\frac1p}\preceq\left(\int_{|x|>R}|[b,G_\alpha]f(x)|^p\,\omega(x)dx\right)^{\frac1p}+C\sum_{i=1}^\infty 2^{-\frac{in\lambda}{2}}\left(\int_{|x|>R}|[b,S_{\alpha,i}]f(x)|^p\,\omega(x)dx\right)^{\frac1p}$$
$$\preceq R^{n(q-p)}\|f\|_{L^p(\omega)}\to 0(R\to\infty).$$

Now we prove claim (4.2). Suppose supp $b\subset\{z:|z|<\eta\}$. Let us choose $R>2\eta$. For $|x|>R>2\eta$, we have $b(x)=0$ and $|x|>2|z|$. Thus, $|x|\leq|x-z|+|z|\leq|x-z|+\frac12|x|$, and we have $|x|\leq 2|x-z|$. Therefore,

$$\begin{aligned}
[b,S_{\alpha,i}]f(x) &\leq \left(\int_0^\infty\int_{|x-y|<2^it}\sup_{\phi\in\mathcal{C}_\alpha}\left|\int_{\mathbb{R}^n}(b(x)-b(z))\phi_t(y-z)f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{\frac12}\\
&\preceq \left(\int_0^\infty\int_{|x-y|<2^it}\left|\int_{|y-z|\leq t,|z|<\eta}|b(z)f(z)|dz\right|^2\frac{dydt}{t^{3n+1}}\right)^{\frac12}\\
&\leq \int_{|z|<\eta}\left(\int_0^\infty\int_{|x-y|\leq 2^it,|y-z|\leq t}|b(z)f(z)|^2\frac{dydt}{t^{3n+1}}\right)^{\frac12}dz\\
&\preceq \int_{|z|<\eta}\left(\int_{\frac{|x-z|}{2^{i+1}}}^\infty\frac{dt}{t^{2n+1}}\right)^{\frac12}|b(z)f(z)|\,dz\\
&\preceq 2^{in}\|b\|_\infty\int_{|z|<\eta}\frac{|f(z)|}{|x-z|^n}dz\preceq\frac{2^{in}}{|x|^n}\int_{|z|<\eta}|f(z)|dz.
\end{aligned}$$

Then, as in the proof of (3.1), we have (4.2). So, we have proved condition (C3). At last, we are in the position to prove

$$\sup_{f\in F}\|[b,g^*_{\lambda,\alpha}]f(\cdot+h)-[b,g^*_{\lambda,\alpha}]f(\cdot)\|_{L^p(\omega)}\to 0(h\to 0). \tag{4.3}$$

Similar to (3.2), first we obtain that

$$\begin{aligned}
&|[b,g^*_{\lambda,\alpha}]f(x+h)-[b,g^*_{\lambda,\alpha}]f(x)|\le\left(\int_0^\infty\!\!\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\sup_{\phi\in\mathcal{C}_\alpha}|I(x,y,t)|^2\frac{dydt}{t^{3n+1}}\right)^{\frac12}\\
&\le\left(\int_0^\infty\!\!\int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\sup_{\phi\in\mathcal{C}_\alpha}|I(x,y,t)|^2\frac{dydt}{t^{3n+1}}\right)^{\frac12}\\
&\quad+\left(\int_0^\infty\!\!\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\sup_{\phi\in\mathcal{C}_\alpha}|I(x,y,t)|^2\frac{dydt}{t^{3n+1}}\right)^{\frac12}\\
&:=V+VI,
\end{aligned}\tag{4.4}$$

where

$$I(x,y,t)=\int\phi(\frac{y+h-z}{t})[b(x+h)-b(z)]f(z)dz-\int\phi(\frac{y-z}{t})[b(x)-b(z)]f(z)dz.$$

For V, as the estimate of (3.2) in Theorem 1.1, we obtain

$$\|V\|_{L^p(\omega)}\le\left\|\left(\int_0^\infty\int_{|x-y|<t}\sup_{\phi\in\mathcal{C}_\alpha}|I(x,y,t)|^2\frac{dydt}{t^{3n+1}}\right)^{\frac12}\right\|_{L^p(\omega)}\to 0(h\to 0). \tag{4.5}$$

Now we estimate VI. For every x, as the proof of Theorem 1.1, we divide $I(x,y,t)$ into four parts; see (3.3). Therefore,

$$\begin{aligned}
&\left(\int_0^\infty\!\!\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\sup_{\phi\in\mathcal{C}_\alpha}|I(x,y,t)|^2\frac{dydt}{t^{3n+1}}\right)^{\frac12}\le\left(\int_0^\infty\!\!\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\sup_{\phi\in\mathcal{C}_\alpha}|I|^2\frac{dydt}{t^{3n+1}}\right)^{\frac12}\\
&\quad+\left(\int_0^\infty\!\!\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\sup_{\phi\in\mathcal{C}_\alpha}|II|^2\frac{dydt}{t^{3n+1}}\right)^{\frac12}\\
&\quad+\left(\int_0^\infty\!\!\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\sup_{\phi\in\mathcal{C}_\alpha}|III|^2\frac{dydt}{t^{3n+1}}\right)^{\frac12}\\
&\quad+\left(\int_0^\infty\!\!\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\sup_{\phi\in\mathcal{C}_\alpha}|IV|^2\frac{dydt}{t^{3n+1}}\right)^{\frac12}\\
&:=I_3+I_4+I_5+I_6.
\end{aligned}\tag{4.6}$$

First, we estimate I_3. Noticing that $\lambda > 2$, by the Minkowski inequality, we obtain that

$$\begin{aligned}
I_3 \leq & \sum_{i=1}^{\infty} \left(\int_0^{\infty} \int_{2^{i-1}t \leq |x-y| < 2^i t} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} \right. \\
& \left. \cdot \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\substack{|x-z|<2^{1/\epsilon}|h| \\ |y-z| \leq t}} \phi(\frac{y-z}{t})[b(z)-b(x)]f(z)dz \right|^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} \\
\preceq & \|\nabla b\|_\infty \sum_{i=1}^{\infty} 2^{-\frac{in\lambda}{2}} \int_{|x-z|<2^{1/\epsilon}|h|} \left(\int_{\frac{|x-z|}{2^{i+1}}}^{\infty} \int_{|y-z| \leq t} \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} |x-z||f(z)|dz \\
\preceq & \sum_{i=1}^{\infty} 2^{-\frac{in}{2}(\lambda-2)} \int_{|x-z|<2^{1/\epsilon}|h|} \frac{|f(z)|}{|x-z|^{n-1}} dz \\
\preceq & \sum_{j=1}^{\infty} \int_{2^{-j+1/\epsilon}|h| \leq |x-z| < 2^{-j+1+1/\epsilon}|h|} \frac{|f(z)|}{|x-z|^{n-1}} dz \\
\preceq & \sum_{j=1}^{\infty} \frac{1}{(2^{-j+1/\epsilon}|h|)^{n-1}} \int_{|x-z| \leq 2^{-j+1+1/\epsilon}|h|} |f(z)|dz \\
\preceq & 2^{n+1/\epsilon}|h| \sum_{j=1}^{\infty} 2^{-j} Mf(x) \\
\preceq & 2^{1/\epsilon}|h| Mf(x).
\end{aligned}$$

Using the boundedness of M on $L^p(\omega)$, we obtain that

$$\left(\int_{\mathbb{R}^n} |I_3|^p \omega(x) dx \right)^{\frac{1}{p}} \preceq 2^{1/\epsilon}|h| \|f\|_{L^p(\omega)}. \tag{4.7}$$

Then we estimate I_4. Since $\lambda > 2$, we get that

$$\begin{aligned}
I_4 \leq & \sum_{i=1}^{\infty} \left(\int_0^{\infty} \int_{2^{i-1}t \leq |x-y| < 2^i t} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} \right. \\
& \left. \cdot \sup_{\phi \in \mathcal{C}_\alpha} \left| \int_{\substack{|x-z|<2^{1/\epsilon}|h| \\ |y+h-z| \leq t}} \phi(\frac{y+h-z}{t})[b(z)-b(x+h)]f(z)dz \right|^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} \\
\preceq & \|\nabla b\|_\infty \sum_{i=1}^{\infty} 2^{-\frac{in\lambda}{2}} \int_{|x-z|<2^{1/\epsilon}|h|} \left(\int_{\frac{|x+h-z|}{2^{i+1}}}^{\infty} \int_{|y+h-z| \leq t} \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} |x+h-z||f(z)|dz \\
\preceq & \sum_{i=1}^{\infty} 2^{-\frac{in}{2}(\lambda-2)} \int_{|x+h-z|<2^{1/\epsilon+1}|h|} \frac{|f(z)|}{|x+h-z|^{n-1}} dz \\
\preceq & \sum_{j=1}^{\infty} \int_{2^{-j+1+1/\epsilon}|h| \leq |x+h-z| < 2^{-j+2+1/\epsilon}|h|} \frac{|f(z)|}{|x+h-z|^{n-1}} dz
\end{aligned}$$

$$\preceq \sum_{j=1}^{\infty} \frac{1}{(2^{-j+1+1/\epsilon}|h|)^{n-1}} \int_{|x+h-z|\leq 2^{-j+2+1/\epsilon}|h|} |f(z)|dz$$

$$\preceq 2^{n+1/\epsilon}|h| \sum_{j=1}^{\infty} 2^{-j} Mf(x)$$

$$\preceq 2^{1/\epsilon}|h|Mf(x).$$

Using the boundedness of M on $L^p(\omega)$ again, we have that

$$\left(\int_{\mathbb{R}^n} |I_4|^p \omega(x)dx\right)^{\frac{1}{p}} \preceq 2^{1/\epsilon}|h|\|f\|_{L^p(\omega)}. \tag{4.8}$$

Now we estimate I_5. As the estimate of III in Theorem 1.1, we will divide the integral into three parts again. Denote

$$A_1 = \{z \in \mathbb{R}^n : |y-z| \leq t, |y-z+h| \leq t\},$$

$$A_2 = \{z \in \mathbb{R}^n : |y-z| \leq t, |y-z+h| > t\},$$

$$A_3 = \{z \in \mathbb{R}^n : |y-z| > t, |y-z+h| \leq t\}.$$

Then, if $z \in \mathbb{R}^n \setminus A_1 \cup A_2 \cup A_3$, we have $I_5 = 0$. We first estimate the integral over A_1. Noticing $\lambda > 3 + \frac{2\alpha}{n} > 2 + \frac{2\alpha}{n}$, by the Minkowski inequality, we have that

$$\Bigg(\int_0^{\infty}\int_{|x-y|\geq t} \left(\frac{t}{t+|x-y|}\right)^{n\lambda}$$

$$\cdot \sup_{\phi\in\mathcal{C}_\alpha} \left|\int_{A_1\cap\{z||x-z|\geq 2^{1/\epsilon}|h|\}} \left[\phi(\frac{y+h-z}{t}) - \phi(\frac{y-z}{t})\right][b(x+h)-b(z)]f(z)dz\right|^2 \frac{dydt}{t^{3n+1}}\Bigg)^{\frac{1}{2}}$$

$$\preceq \|b\|_\infty \sum_{i=1}^{\infty} 2^{-\frac{in\lambda}{2}} \left(\int_0^{\infty}\int_{2^{i-1}t\leq|x-y|<2^it} \left|\int_{A_1\cap\{z||x-z|\geq 2^{1/\epsilon}|h|\}} \left|\frac{h}{t}\right|^{\alpha} |f(z)|dz\right|^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}$$

$$\preceq |h|^{\alpha} \sum_{i=1}^{\infty} 2^{-\frac{in\lambda}{2}} \int_{|x-z|\geq 2^{1/\epsilon}|h|} \left(\int_{t>\frac{|x-z|}{2^{i+1}}} \int_{|y-z|\leq t} \frac{1}{t^{3n+2\alpha+1}} dydt\right)^{\frac{1}{2}} |f(z)|dz$$

$$\preceq \sum_{i=1}^{\infty} 2^{-\frac{in\lambda}{2}} 2^{i(n+\alpha)} |h|^{\alpha} \int_{|x-z|\geq 2^{1/\epsilon}|h|} \frac{|f(z)|}{|x-z|^{n+\alpha}} dz$$

$$\preceq \sum_{i=1}^{\infty} 2^{-\frac{in}{2}(\lambda-2-\frac{2\alpha}{n})} \sum_{j=0}^{\infty} |h|^{\alpha} \int_{2^{j+1/\epsilon}|h|\leq|x-z|\leq 2^{j+1+1/\epsilon}|h|} \frac{|f(z)|}{|x-z|^{n+\alpha}} dz$$

$$\preceq 2^{-\frac{\alpha}{\epsilon}} \sum_{j=0}^{\infty} 2^{-j\alpha} Mf(x) \preceq 2^{-\frac{\alpha}{\epsilon}} Mf(x).$$

Using the boundedness of M on $L^p(\omega)$, we have that

$$\left(\int_{\mathbb{R}^n} |I_5|^p \omega(x)dx\right)^{\frac{1}{p}} \preceq 2^{-\frac{\alpha}{\epsilon}} \|f\|_{L^p(\omega)}, over A_1. \tag{4.9}$$

The integrals over A_2 and A_3 are symmetric, so we only give the details over A_2. We divide the proof in three cases.

Case 1. If $|y-z| \le \min\{\frac{1}{3}|x-z|, 2|h|\}$, then for any t satisfying $2^{i-1}t \le |x-y| < 2^i t$, on one hand, we can deduce that $2^i t > |x-y| = |x-z+z-y| \ge |x-z| - |y-z| \ge \frac{2}{3}|x-z| \Rightarrow t > \frac{1}{3\cdot 2^{i-1}}|x-z|$. On the other hand, $|x-z| = |x-y+y-z| \ge |x-y| - |y-z| \ge |x-y| - \frac{1}{3}|x-z| \ge 2^{i-1}t - \frac{1}{3}|x-z| \Rightarrow t \le \frac{4}{3\cdot 2^{i-1}}|x-z|$. Therefore, we get

$$\frac{1}{3\cdot 2^{i-1}}|x-z| \le t \le \frac{4}{3\cdot 2^{i-1}}|x-z|.$$

Then, by the Minkowski inequality and $\lambda > 2 + \frac{1}{n}$, it follows that

$$\begin{aligned}
&\left(\int_0^\infty \int_{|x-y|\ge t} \left(\frac{t}{t+|x-y|}\right)^{n\lambda}\right.\\
&\quad \left. \cdot \sup_{\phi\in\mathcal{C}_\alpha} \left| \int_{A_2\cap|x-z|\ge 2^{1/\epsilon}|h|} \phi(\frac{y-z}{t})[b(x+h)-b(z)]f(z)dz \right|^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}\\
&\le 2\|b\|_\infty \sum_{i=1}^\infty 2^{-\frac{in\lambda}{2}} \left(\int_0^\infty \int_{2^{i-1}t\le|x-y|<2^i t} \left| \int_{\substack{|x-z|\ge 2^{1/\epsilon}|h|\\ |y-z|\le t<|y-z+h|}} |f(z)|dz \right|^2 \frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}\\
&\preceq \sum_{i=1}^\infty 2^{-\frac{in\lambda}{2}}\\
&\quad \cdot \int_{|x-z|\ge 2^{1/\epsilon}|h|} \left(\int_{\frac{1}{3\cdot 2^{i-1}}|x-z|\le t\le \frac{4}{3\cdot 2^{i-1}}|x-z|} \int_{|y-z|\le 2|h|} \frac{1}{|y-z|^{n-1}} \frac{1}{t^{2n+2}} dydt\right)^{\frac{1}{2}} |f(z)|dz\\
&\preceq |h|^{\frac{1}{2}} \sum_{i=1}^\infty 2^{-\frac{in\lambda}{2}} \int_{|x-z|\ge 2^{1/\epsilon}|h|} \left(\int_{\frac{1}{3\cdot 2^{i-1}}|x-z|\le t\le \frac{4}{3\cdot 2^{i-1}}|x-z|} \frac{1}{t^{2n+2}} dt\right)^{\frac{1}{2}} |f(z)|dz\\
&\preceq \sum_{i=1}^\infty 2^{-\frac{in\lambda}{2}} 2^{i(n+\frac{1}{2})} |h|^{\frac{1}{2}} \int_{|x-z|\ge 2^{1/\epsilon}|h|} \frac{1}{|x-z|^{n+\frac{1}{2}}} |f(z)|dz\\
&\preceq \sum_{i=1}^\infty 2^{-\frac{in}{2}(\lambda-2-\frac{1}{n})} |h|^{\frac{1}{2}} \sum_{j=0}^\infty \int_{2^{j+1/\epsilon}|h|\le|x-z|\le 2^{j+1+1/\epsilon}|h|} \frac{1}{|x-z|^{n+\frac{1}{2}}} |f(z)|dz\\
&\preceq 2^{-\frac{1}{2\epsilon}} \sum_{j=0}^\infty 2^{-\frac{j}{2}} Mf(x)\\
&\preceq 2^{-\frac{1}{2\epsilon}} Mf(x).
\end{aligned} \tag{4.10}$$

Case 2. If $2|h| \le |y-z| \le \frac{1}{3}|x-z|$, then

$$t < |y-z+h| \le |y-z| + |h| \le \frac{3}{2}|y-z|$$

and

$$|x-y| \ge |x-z| - |y-z| \ge \frac{2}{3}|x-z|.$$

Noticing that $\lambda > 3 + \frac{2\alpha}{n} > 2 + \frac{1}{n}$, by the Minkowski inequality, we obtain that

$$\begin{aligned}
&\left(\int_0^\infty\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\cdot\sup_{\phi\in\mathcal{C}_\alpha}\left|\int_{A_2\cap|x-z|\ge 2^{1/\epsilon}|h|}\phi(\frac{y-z}{t})[b(x+h)-b(z)]f(z)dz\right|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}\\
&\le 2\|b\|_\infty\left(\int_0^\infty\int_{|x-y|\ge t}\left(\frac{t}{|x-y|}\right)^{n\lambda}\left|\int_{\substack{|x-z|\ge 2^{1/\epsilon}|h|\\ |y-z|\le t<|y-z+h|}}|f(z)|dz\right|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}\\
&\preceq \int_{|x-z|\ge 2^{1/\epsilon}|h|}\left(\int_{2|h|\le|y-z|\le\frac{1}{3}|x-z|}|y-z|^{n\lambda}\int_{|y-z|\le t<|y-z+h|}\frac{1}{t^{3n+1}}dtdy\right)^{\frac{1}{2}}\frac{|f(z)|}{|x-z|^{\frac{n\lambda}{2}}}dz\\
&\preceq \int_{|x-z|\ge 2^{1/\epsilon}|h|}\frac{|f(z)|}{|x-z|^{\frac{n\lambda}{2}}}\left(\int_{|y-z|\le\frac{1}{3}|x-z|}\frac{|h|}{|y-z|^{(3n-n\lambda+1)}}dy\right)^{\frac{1}{2}}dz\\
&\preceq |h|^{\frac{1}{2}}\int_{|x-z|\ge 2^{1/\epsilon}|h|}\frac{1}{|x-z|^{n+\frac{1}{2}}}|f(z)|dz\\
&\preceq |h|^{\frac{1}{2}}\sum_{j=0}^\infty\int_{2^{j+1/\epsilon}|h|\le|x-z|\le 2^{j+1+1/\epsilon}|h|}\frac{1}{|x-z|^{n+\frac{1}{2}}}|f(z)|dz\\
&\preceq 2^{-\frac{1}{2\epsilon}}\sum_{j=0}^\infty 2^{-\frac{j}{2}}Mf(x) \preceq 2^{-\frac{1}{2\epsilon}}Mf(x). \qquad (4.11)
\end{aligned}$$

Case 3. If $|y-z| > \frac{1}{3}|x-z|$, then $|y-z| > \frac{1}{3}|x-z| \ge \frac{1}{3}2^{1/\epsilon}|h| \ge \frac{4}{3}|h|$. It follows that $|h| < \frac{3}{4}|y-z|$. Noticing that $\lambda > 0$, by the Minkowski inequality, we have that

$$\begin{aligned}
&\left(\int_0^\infty\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\sup_{\phi\in\mathcal{C}_\alpha}\left|\int_{A_2}\phi(\frac{y-z}{t})][b(x+h)-b(z)f(z)dz\right|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}\\
&\le 2\|b\|_\infty\sum_{i=1}^\infty 2^{-\frac{in\lambda}{2}}\left(\int_0^\infty\int_{2^{i-1}t\le|x-y|<2^it}\left|\int_{\substack{|x-z|\ge 2^{1/\epsilon}|h|\\ |y-z|\le t<|y-z+h|}}|f(z)|dz\right|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}\\
&\preceq \sum_{i=1}^\infty 2^{-\frac{in\lambda}{2}}\int_{|x-z|\ge 2^{1/\epsilon}|h|}\left(\int_{|y-z|>\frac{1}{3}|x-z|}\int_{|y-z|\le t<|y-z+h|}\frac{dtdy}{t^{3n+1}}\right)^{\frac{1}{2}}|f(z)|dz
\end{aligned}$$

$$\preceq \sum_{i=1}^{\infty} 2^{-\frac{in\lambda}{2}} \int_{|x-z|\geq 2^{1/\epsilon}|h|} \left(\int_{|y-z|>\frac{1}{3}|x-z|} \left| \frac{1}{|y-z|^{3n}} - \frac{1}{|y+h-z|^{3n}} \right| \right)^{\frac{1}{2}} |f(z)|dz$$

$$\leq |h|^{1/2} \int_{|x-z|\geq 2^{1/\epsilon}|h|} \frac{1}{|x-z|^{n+1/2}} |f(z)|dz$$

$$\preceq |h|^{1/2} \sum_{j=0}^{\infty} \int_{2^{j+1/\epsilon}|h|\leq|x-z|\leq 2^{j+1+1/\epsilon}|h|} \frac{1}{|x-z|^{n+1/2}} |f(z)|dz$$

$$\preceq 2^{-\frac{1}{2\epsilon}} \sum_{j=0}^{\infty} 2^{-\frac{j}{2}} Mf(x) \preceq 2^{-\frac{1}{2\epsilon}} Mf(x). \tag{4.12}$$

Combining (4.10), (4.11), and (4.12), using the boundedness of M on $L^p(\omega)$ again, we have that

$$\left(\int_{\mathbb{R}^n} |I_5|^p \omega(x) dx \right)^{\frac{1}{p}} \preceq 2^{-\frac{1}{2\epsilon}} \|f\|_{L^p(\omega)}, over A_2. \tag{4.13}$$

For the last part,

$$I_6 = \left(\int_0^\infty \int_{|x-y|\geq t} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} \right.$$

$$\left. \cdot \sup_{\phi\in\mathcal{C}_\alpha} \left| \int_{|x-z|\geq 2^{1/\epsilon}|h|} \phi(\frac{y-z}{t}) [b(x)-b(x+h)] f(z) dz \right|^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}}$$

$$= |b(x)-b(x+h)|$$

$$\cdot \left(\int_0^\infty \int_{|x-y|\geq t} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} \sup_{\phi\in\mathcal{C}_\alpha} \left| \int_{|x-z|\geq 2^{1/\epsilon}|h|} \phi(\frac{y-z}{t}) f(z) dz \right|^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}}$$

$$\preceq \|\nabla b\|_\infty |h| \left(\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|} \right)^{n\lambda} \sup_{\phi\in\mathcal{C}_\alpha} \left| \int_{|x-z|\geq 2^{1/\epsilon}|h|} \phi(\frac{y-z}{t}) f(z) dz \right|^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}}$$

$$:= \|\nabla b\|_\infty |h| g^*_{\lambda,\alpha,\epsilon,h} f(x).$$

For any $1 < q < p$, we claim that

$$g^*_{\lambda,\alpha,\epsilon,h} f(x) \preceq M(g^*_{\lambda,\alpha} f)(x) + (M(|f|^q)(x))^{\frac{1}{q}} + Mf(x). \tag{4.14}$$

Let Q denote the cube centered at x and with diameter $r = 2^{1/\epsilon}|h|/4$. Furthermore, let $f_1(x) = f\chi_{4Q}(x)$ and $f_2(x) = f(x) - f_1(x)$. Then we get

$$\begin{aligned} g^*_{\lambda,\alpha,\epsilon,h}f(x) &= \frac{1}{|Q|}\int_Q |g^*_{\lambda,\alpha,\epsilon,h}f(x)|d\xi \\ &\leq \frac{1}{|Q|}\int_Q |g^*_{\lambda,\alpha}f(\xi)|d\xi + \frac{1}{|Q|}\int_Q |g^*_{\lambda,\alpha}f_1(\xi)|d\xi \\ &\quad + \frac{1}{|Q|}\int_Q |g^*_{\lambda,\alpha}f_2(\xi) - g^*_{\lambda,\alpha,\epsilon,h}f(x)|d\xi \\ &\leq M(g^*_{\lambda,\alpha}f)(x) + I_7 + I_8. \end{aligned}$$

Similar to (4.1), according to the pointwise relationship between $G_{\alpha,\beta}$ with different β, we have that

$$\begin{aligned} g^*_{\lambda,\alpha}f(x) &\leq G_\alpha f(x) + C\sum_{i=1}^{\infty} 2^{-\frac{in\lambda}{2}} G_{\alpha,2^i}(f)(x) \\ &\leq G_\alpha f(x) + C\sum_{i=1}^{\infty} 2^{-\frac{in\lambda}{2}+i(\frac{3n}{2}+\alpha)} G_\alpha(f)(x) \\ &\preceq G_\alpha f(x). \end{aligned}$$

The last inequality holds for $\lambda > 3 + \frac{2\alpha}{n}$. Then, by Theorem A, we have that $g^*_{\lambda,\alpha}$ is bounded from $L^p(\omega)$ to itself. For any $1 < q < p$, combining Hölder's inequality, we get that

$$\begin{aligned} I_7 &\leq \frac{1}{|Q|^{1/q}}\left(\int_Q |g^*_{\lambda,\alpha}f_1(\xi)|^q d\xi\right)^{\frac{1}{q}} \leq \frac{1}{|Q|^{1/q}}\left(\int_Q |f_1(\xi)|^q d\xi\right)^{\frac{1}{q}} \\ &\preceq (M(|f|^q)(x))^{\frac{1}{q}}. \end{aligned}$$

Now let us estimate I_8. Similar to Lemma 2.2, we can compute that

$$\begin{aligned} |g^*_{\lambda,\alpha}f_2(\xi) - g^*_{\lambda,\alpha,\epsilon,h}f(x)| = &\Bigg| \left(\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{n\lambda} \sup_{\phi\in\mathcal{C}_\alpha}\left|\int_{|\xi-x+y-z|\leq t}\phi(\frac{\xi-x+y-z}{t})f_2(z)dz\right|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}} \\ &- \left(\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{n\lambda} \sup_{\phi\in\mathcal{C}_\alpha}\left|\int_{\substack{|x-z|\geq 2^{1/\epsilon}|h|\\ |y-z|\leq t}}\phi(\frac{y-z}{t})f(z)dz\right|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}\Bigg| \\ \leq &\Bigg(\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{n\lambda} \sup_{\phi\in\mathcal{C}_\alpha}\Bigg|\int_{|\xi-x+y-z|\leq t}\phi(\frac{\xi-x+y-z}{t})f_2(z)dz \\ &- \int_{\substack{|x-z|\geq 2^{1/\epsilon}|h|\\ |y-z|\leq t}}\phi(\frac{y-z}{t})f(z)dz\Bigg|^2\frac{dydt}{t^{3n+1}}\Bigg)^{\frac{1}{2}} \\ \leq &\Bigg(\int_0^\infty\int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda} \sup_{\phi\in\mathcal{C}_\alpha}\Bigg|\int_{|\xi-x+y-z|\leq t}\phi(\frac{\xi-x+y-z}{t})f_2(z)dz \\ &- \int_{\substack{|x-z|\geq 2^{1/\epsilon}|h|\\ |y-z|\leq t}}\phi(\frac{y-z}{t})f(z)dz\Bigg|^2\frac{dydt}{t^{3n+1}}\Bigg)^{\frac{1}{2}} \end{aligned}$$

$$+\left(\int_0^\infty\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\sup_{\phi\in\mathcal{C}_\alpha}\left|\int_{|\xi-x+y-z|\le t}\phi(\frac{\xi-x+y-z}{t})f_2(z)dz\right.\right.$$
$$\left.\left.-\int_{\substack{|x-z|\ge 2^{1/\epsilon}|h|\\ |y-z|\le t}}\phi(\frac{y-z}{t})f(z)dz\right|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}$$
$$:=I_{81}+I_{82}.$$

By I_2 in Theorem 1.1, we have that

$$I_{81}=I_2\preceq Mf(x).$$

Now we estimate I_{82} :

$$I_{82}\le\left(\int_0^\infty\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\sup_{\phi\in\mathcal{C}_\alpha}\left|\int_{\substack{|x-z|\ge 2^{1/\epsilon}|h|\\ |y-z|\le t\\ |\xi-x+y-z|>t}}\phi(\frac{y-z}{t})f(z)dz\right|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}$$
$$+\left(\int_0^\infty\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\sup_{\phi\in\mathcal{C}_\alpha}\left|\int_{\substack{|x-z|\ge 2^{1/\epsilon}|h|\\ |y-z|>t\\ |\xi-x+y-z|\le t}}\phi(\frac{\xi-x+y-z}{t})f(z)dz\right|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}$$
$$+\left(\int_0^\infty\int_{|x-y|\ge t}\left(\frac{t}{t+|x-y|}\right)^{n\lambda}\right.$$
$$\left.\cdot\sup_{\phi\in\mathcal{C}_\alpha}\left|\int_{\substack{|x-z|\ge 2^{1/\epsilon}|h|\\ |y-z|\le t\\ |\xi-x+y-z|\le t}}\left[\phi(\frac{\xi-x+y-z}{t})-\phi(\frac{y-z}{t})\right]f(z)dz\right|^2\frac{dydt}{t^{3n+1}}\right)^{\frac{1}{2}}$$
$$:=J_1+J_2+J_3.$$

Noticing $\xi\in Q$, similar to the proof of I_5 over A_2, we get

$$\begin{aligned}J_1 &\le \int_{\xi\in Q,|x-z|\ge 2^{1/\epsilon}|h|}\frac{|\xi-x|^{\frac{1}{2}}}{|x-z|^{n+\frac{1}{2}}}|f(z)|dz\\ &\preceq (2^{1/\epsilon}|h|)^{\frac{1}{2}}\int_{|x-z|\ge 2^{1/\epsilon}|h|}\frac{1}{|x-z|^{n+\frac{1}{2}}}|f(z)|dz\\ &\preceq Mf(x).\end{aligned}$$

Similarly, we obtain

$$J_2\preceq Mf(x).$$

For J_3, similar to I_5 over A_1, since $\lambda > 2 + \frac{2\alpha}{n}$, we have

$$\begin{aligned} J_3 &\leq \sum_{i=1}^{\infty} 2^{-\frac{in\lambda}{2}} \left(\int_0^{\infty} \int_{2^{i-1}t \leq |x-y| < 2^i t} \left| \int_{\substack{|x-z| \geq 2^{1/\epsilon}|h| \\ |y-z| \leq t \\ |\xi - x + y - z| \leq t}} \left| \frac{\xi - x}{t} \right|^{\alpha} |f(z)| dz \right|^2 \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} \\ &\leq (2^{1/\epsilon}|h|)^{\alpha} \sum_{i=1}^{\infty} 2^{-\frac{in\lambda}{2}} \int_{|x-z| \geq 2^{1/\epsilon}|h|} \left(\int_{t > \frac{|x-z|}{2^{i+1}}} \int_{|y-z| \leq t} \frac{dydt}{t^{3n+2\alpha+1}} \right)^{\frac{1}{2}} |f(z)| dz \\ &\leq (2^{1/\epsilon}|h|)^{\alpha} \sum_{i=1}^{\infty} 2^{-\frac{in}{2}(\lambda - 2 - \frac{2\alpha}{n})} \int_{|x-z| \geq 2^{1/\epsilon}|h|} \frac{|f(z)|}{|x-z|^{n+\alpha}} dz \\ &\preceq Mf(x). \end{aligned}$$

By the estimates of J_1, J_2, and J_3, we obtain

$$I_{82} \preceq Mf(x).$$

Combining I_7, I_{81}, and I_{82}, we have claim (4.14). Then, applying the claim (4.14) for $1 < q < p$, with the $L^p(\omega)(p > 1)$ boundedness of M, M^q, and $g^*_{\lambda,\alpha}$, we have

$$\left(\int_{\mathbb{R}^n} |I_6|^p \omega(x) dx \right)^{\frac{1}{p}} \preceq |h| \|f\|_{L^p(\omega)}. \tag{4.15}$$

By (4.6), (4.7), (4.8), (4.9), (4.13), and (4.15), we obtain

$$\|VI\|_{L^p(\omega)} \preceq (2^{1/\epsilon}|h| + 2^{-\frac{\alpha}{\epsilon}} + 2^{-\frac{1}{2\epsilon}} + |h|) \|f\|_{L^p(\omega)}.$$

Taking $\epsilon = \frac{2}{\log \frac{1}{|h|}}$ and letting $h \to 0$, then we have

$$\|VI\|_{L^p(\omega)} \to 0.$$

Then, combining (4.4) and (4.5), we have the uniform equicontinuity (Condition (C2)) of $g^*_{\lambda,\alpha}$. Thus, we complete the proof of Theorem 1.2.

Acknowledgments

This work was completed with the support of the National Natural Science Foundation of China (Grant No. 11501516, 11561057 and 11671362), Natural Science Foundation of Zhejiang Province (Grant No. LQ15A010003), and Natural Science Foundation of Jiangxi Province (Grant No. 20151BAB211002).

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