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On λ-pseudo q-bi-starlike functions

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Abstract: Making use of the λ-pseudo-q-differential operator, we aim to investigate a new, interesting class of bi-starlike functions in the conic domain. Furthermore, we obtain certain sharp bounds of the Fekete–Szegö functional for functions belonging to this class.

Key words: Fekete–Szegö inequality, bi-starlike functions, q-differential operator

1. Introduction

Let $\mathcal{A}$ denote the family of functions analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and given by the following Taylor–Maclaurin series:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots . \tag{1.1}$$

We denote by $\mathcal{S}$ the class of starlike functions $f \in \mathcal{A}$, which are univalent in $\mathbb{U}$ (e.g., see [1, 4, 5, 9, 11]).

Let $\mathcal{S}^*(\beta)$ be the usual subclass of starlike functions $\mathcal{S}$ of order β, $0 \le \beta < 1$, so that $f \in \mathcal{S}^*(\beta)$ if and only if, for $z \in \mathbb{U}$,

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \beta.$$

For $\alpha > 0$, let $\mathcal{B}(\alpha)$ denote the class of Bazilevič functions defined in the open unit disk $\mathbb{U}$, normalized by the condition $f(0) = f'(0) - 1 = 0$, and such that, for $z \in \mathbb{U}$,

$$Re\left(f'(z)\left(\frac{zf(z)}{z}\right)^{\alpha-1}\right) > 0.$$

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The class $\mathcal{B}(\alpha)$ reduces to the starlike function and bounded turning function whenever $\alpha = 0$ and $\alpha = 1$, respectively. This class is extended to $\mathcal{B}(\alpha, \beta)$, which satisfies the geometric condition

$$Re\left(\frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}}\right) > \beta,$$

where α is a nonnegative real number and $0 \leq \beta < 1$. This class of functions was intensively studied by Singh [18] and considered subsequently by London and Thomas [14]. Recently, Babalola [3] introduced a new subclass $\mathcal{L}_\lambda(\beta)$ of λ-pseudo-starlike functions of order β satisfying the geometric condition

$$Re\left(\frac{z(f'(z))^\lambda}{f(z)}\right) > \beta, \qquad (z \in \mathbb{U}, 0 \leq \beta < 1, \lambda \geq 1).$$

We note that, if $\lambda = 1$, we have the class of starlike functions of order β, which in this context is 1-pseudo-starlike functions of order β. If $\beta = 0$, we simply write $\mathcal{L}_\lambda$ instead of $\mathcal{L}_\lambda(0)$. For $\lambda = 2$, we note that functions in $\mathcal{L}_2(\beta)$ are defined by

$$Re\left(f'(z)\frac{zf'(z)}{f(z)}\right) > \beta, \qquad (z \in \mathbb{U}),$$

which is a product combination of geometric expression for bounded turning and starlike functions, an interesting analytic presentation on univalent functions in the open unit disk $\mathbb{U}$. Joshi et al. [8] defined the subclasses $S_\Sigma^\lambda(k, \alpha)$ and $S_\Sigma^\lambda(k, \beta)$ of bi-univalent functions associated with λ-bi-pseudo-starlike functions in the unit disk $\mathbb{U}$. Recently, Altinkaya and Özkan [2] introduced the subclasses $\mathcal{L}_\lambda(\beta)$ and $\mathcal{L}_\lambda(\beta, \phi)$ of Sălăgean type λ-pseudo-starlike functions. For these function classes, they found upper bounds for the initial coefficients as well as Fekete–Szegö inequalities.

Definition 1.1 *Let $\mathcal{P}$ be analytic and normalized Carathèodory functions with positive real part in $\mathbb{U}$. Let $\mathcal{P}(p_k)(0 \leq k < \infty)$ denote the family of functions p, such that $p \in \mathcal{P}$ and $p \prec \mathcal{P}$ in $\mathbb{U}$, where p_k maps the unit disk conformally onto the domain Ω_k such that $1 \in \Omega_k$ and $\partial\Omega_k$ is defined by*

$$\partial\Omega_k = \{u + iv : u^2 = k^2(u-1)^2 + k^2v^2\}.$$

Moreover, Ω_k is elliptic for $k > 1$, hyperbolic when $0 < k < 1$, and parabolic for $k = 1$ and it covers the right half plane when $k = 0$. The extremal functions of class $\mathcal{P}(p_k)(0 \leq k < \infty)$ were presented and investigated by Kanas et al. in [12] and [13]. Obviously,
for $k = 0$, we have

$$p_0(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + 2z^3 + 2z^4 + ...,$$

for $k = 1$, we have

$$p_1(z) = 1 + \frac{2}{\pi^2}log^2\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

and for $0 < k < 1$ and $A = A(k) = (2/\pi)arccos\ k$, we have

$$p_k(z) = 1 + \frac{2}{1-k^2}sinh^2\left(A(k)\,arc\,tanh\sqrt{z}\right).$$

By virtue of

$$p(z) = \frac{zf'(z)}{f(z)} \prec p_k(z)$$

or

$$p(z) = 1 + \frac{zf''(z)}{f'(z)} \prec p_k(z),$$

and the properties of domains, we have

$$Re(p(z)) > Re(p_k(z)) > \frac{k}{k+1}.$$

The q-differential operator plays a vital role in the theory of geometric function theory. The various subclasses of the normalized analytic function class $\mathcal{A}$ have been studied from different view points. Both q-calculus and fractional calculus provide important tools that have been used in order to investigate various subclasses of $\mathcal{A}$. Historically speaking, the firm footing of the usage of q-calculus in the context of geometric function theory was provided and q-hypergeometric functions were first used in geometric function theory in a book chapter by Srivastava (see, for details, [19, p. 347 et seq.]). Ismail et al. [6] introduced the class of generalized complex functions via q-calculus on some subclasses of analytic functions. Recently, Purohit and Raina [16] investigated applications of the fractional q-calculus operator to define new classes of functions that are analytic in unit disk $\mathbb{U}$ (see, for details, [7], [10], and [20]–[23]).

For $0 < q < 1$, the q-derivative of a function $f \in \mathcal{A}$ given by (1.1) is defined as follows:

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z} \qquad (z \neq 0), \tag{1.2}$$

and $D_q f(0) = f'(0), D_q^2 f(z) = D_q(D_q f(z))$. From (1.1), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k \, z^{k-1}, \tag{1.3}$$

where

$$[k]_q = \frac{1-q^k}{1-q}. \tag{1.4}$$

As $q \to 1^-$, $[k]_q \to k$. For a function $g(z) = z^k$, we observe that

$$D_q(g(z)) = D_q(z^k) = \frac{1-q^k}{1-q} \, z^{k-1} = k \, z^{k-1},$$

$$\lim_{q \to 1^-} (D_q(g(z))) = k \, z^{k-1} = g'(z),$$

where g' is the ordinary derivative.

We define the Sălăgean q-differential operator (also refer to [10]) using the q-differential operator as follows:

$$\mathcal{D}_q^0 f(z) = f(z),$$

$$\mathcal{D}_q^1 f(z) = z\mathcal{D}_q f(z),$$

$$\mathcal{D}_q^n f(z) = z\mathcal{D}_q(\mathcal{D}_q^{n-1} f(z)),$$

$$\mathcal{D}_q^n f(z) = z + \sum_{k=2}^{\infty} [k]_q^n a_k z^k \qquad (n \in \mathbb{N}_0, z \in \mathbb{U}). \tag{1.5}$$

We note that $\lim_{q \longrightarrow 1^-}$

$$\mathcal{D}^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \qquad (n \in \mathbb{N}_0, z \in \mathbb{U}). \tag{1.6}$$

Definition 1.2 *Let $0 \leq k < 1, \lambda \geq 1, n \in \mathbb{N}_0, 0 < q < 1$. For $p_k(z)$ as defined in Definition 1.1, the function f given by (1.1) belongs to $\mathcal{S}_{\lambda,k}^q(p_k)$ if*

$$\left(\frac{z[((\mathcal{D}_q^n f)z)']^\lambda}{(\mathcal{D}_q^n f)z}\right) \prec p_k(z) \qquad (z \in \mathbb{U}). \tag{1.7}$$

Let $\phi(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...(c_1 > 0)$ be an analytic function with positive real part on $\mathbb{U}$.

Definition 1.3 *For $\lambda \geq 1, 0 < q < 1$, we say a function f given by (1.1) belongs to the class $\mathcal{S}_{\lambda,\varphi}^q(\phi)$ if it satisfies the quasi-subordination condition*

$$\left(\frac{z[((\mathcal{D}_q^n f)z)']^\lambda}{(\mathcal{D}_q^n f)z}\right) \prec_q \phi(z) - 1 \qquad (z \in \mathbb{U}). \tag{1.8}$$

In order to derive our main results, we use the following lemma.

Lemma 1.4 *[15] Let $w(z) = w_1 z + w_2 z^2 + w_3 z^3 + ... \in \mho$ such that $|w(z)| < 1$ in $\mathbb{U}$. If t is a complex number, then*

$$\left|w_2 + t w_1^2\right| \leq max\{1, |t|\}.$$

The inequality is sharp for the function $w(z) = z$ or $w(z) = z^2$.

In this paper, motivated by the earlier work of Babalola [3] and Altinkaya and Özkan [2], we introduce a new approach for studying a subclass of λ-pseudo bi-starlike functions using the q-differential operator and estimate the Fekete–Szegö body of the coefficient using subordination [17].

2. Main results

We investigate $\left|a_3 - \sigma a_2^2\right|$ for the function $f \in \mathcal{A}$ for the class $\mathcal{S}_{\lambda,k}^{q}(p_k)$ associated with conical domains.

Theorem 2.1 *Let* $0 \leq k < 1, \lambda \geq 1, 0 < q < 1$ *and* $p_k(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...$ *defined in Definition 1.1. If the function* f *given by (1.1) belongs to* $\mathcal{S}_{\lambda,k}^{q}(p_k)$*, then for any complex* σ *we have*

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{p_1}{(3\lambda-1)[3]_q^n} max\left\{1, \left|\frac{p_2}{p_1} + \frac{p_1(4\lambda - 1 - 2\lambda^2)[2]_q^{2n} - \sigma p_1(3\lambda-1)[3]_q^n}{\{(2\lambda-1)[2]_q^n\}^2}\right|\right\}. \tag{2.1}$$

Proof By (1.7), we have

$$\frac{z[((\mathcal{D}_q^n f)z)']^{\lambda}}{(\mathcal{D}_q^n f)z} = p_k(z) \qquad\qquad (z \in \mathbb{U}). \tag{2.2}$$

We note that

$$z[((\mathcal{D}_q^n f)z)']^{\lambda} = z + 2\lambda[2]_q^n a_2 z^2 + \left(3\lambda[3]_q^n a_3 + 2\lambda(\lambda-1)a_2^n[2]_q^{2n}\right) z^3 + ... \tag{2.3}$$

and

$$p_k(w(z))(\mathcal{D}_q^n f)(z) = z + \left(p_1 w_1 + [2]_q^n a_2\right) z^2 + \left(p_1 w_2 + p_2 w_1^2 + [2]_q^n a_2 p_1 w_1 + [3]_q^n a_3\right) z^3 + \dots . \tag{2.4}$$

Comparing coefficients of (2.2), (2.3), and (2.4), we obtain

$$a_2 = \frac{p_1 w_1}{(2\lambda-1)[2]_q^n} \tag{2.5}$$

and

$$a_3 = \frac{p_1 w_2}{(3\lambda-1)[3]_q^n} + \frac{p_2 w_1^2}{(3\lambda-1)[3]_q^n} + \frac{\left(4\lambda - 1 - 2\lambda^2\right) p_1 w_1^2}{(3\lambda-1)(2\lambda-1)^2[3]_q^n}. \tag{2.6}$$

Hence, by (2.5) and (2.6), we get the following:

$$a_3 - \sigma a_2^2 = \frac{p_1}{(3\lambda-1)[3]_q^n}\left(w_2 + \vartheta w_1^2\right),$$

where

$$\vartheta = \left|\frac{p_2}{p_1} + \frac{p_1(4\lambda - 1 - 2\lambda^2)[2]_q^{2n} - \sigma p_1(3\lambda-1)[3]_q^n}{\{(2\lambda-1)[2]_q^n\}^2}\right|. \tag{2.7}$$

Using Lemma 1.4 and equation (2.7), we yield (2.1). This completes the proof. □

Corollary 2.2 *Let* $f \in \mathcal{S}_{\lambda,k}^{q}(p_k)$*, then*

$$|a_2| = \frac{p_1}{(2\lambda-1)[2]_q^n} \tag{2.8}$$

and

$$|a_3| \leq \frac{p_1}{(3\lambda-1)[3]_q^n} max\left\{1, \left|\frac{p_2}{p_1} + \frac{p_1(4\lambda - 1 - 2\lambda^2)}{(2\lambda-1)^2[2]_q^{2n}}\right|\right\}, \tag{2.9}$$

where $0 \leq k < 1, \lambda \geq 1, 0 < q < 1$.

For the class of functions $f \in \mathcal{S}^q_{\lambda,\varphi}(\phi)$, we can prove the following:

Theorem 2.3 *Let $\lambda \geq 1, 0 < q < 1$. If the function f given by (1.1) belongs to $\mathcal{S}^q_{\lambda,\varphi}(\phi)$, then for any complex σ we have*

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{1}{3\lambda[3]_q^n}\left(c_1 + max\left\{c_1, \left|\frac{2\lambda(2-\lambda)[2]_q^{2n} - 3\sigma\lambda[3]_q^n}{4\lambda^2[2]_q^{2n}}\right| c_1^2 + |c_2|\right\}\right). \tag{2.10}$$

Proof If $f \in \mathcal{S}^q_{\lambda,\varphi}(\phi)$, then

$$\frac{z[((\mathcal{D}_q^n f)z)']^\lambda}{(\mathcal{D}_q^n f)z} = \varphi(z)\left(\phi(z) - 1\right) \qquad (z \in \mathbb{U}). \tag{2.11}$$

We have

$$z[((\mathcal{D}_q^n f)z)']^\lambda = z + 2\lambda[2]_q^n a_2 z^2 + \left(3\lambda[3]_q^n a_3 + 2\lambda(\lambda-1)[2]_q^{2n} a_2^2\right) z^3 + \cdots$$

and

$$\varphi(z)\left(\phi(z) - 1\right)(\mathcal{D}_q^n f)(z) = c_1 A_0 w_1 z^2 + \left(c_1 A_1 w_1 + A_0\left(c_1 w_2 + c_2 w_1^2 + [2]_q^n c_1 A_0 w_1 a_2\right)\right) z^3 + \cdots. \tag{2.12}$$

From (2.11) and (2.12), it is easily seen that

$$a_2 = \frac{c_1 A_0 w_1}{2\lambda[2]_q^n}, \tag{2.13}$$

$$a_3 = \frac{c_1 A_1 w_1}{3\lambda[3]_q^n} + \frac{c_1 A_0 w_2}{3\lambda[3]_q^n} + \frac{A_0}{3\lambda[3]_q^n}\left(c_2 - \frac{(2-\lambda)c_1^2 A_0}{2\lambda}\right) w_1^2, \tag{2.14}$$

and

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{1}{3\lambda[3]_q^n}\left[|c_1 A_1 w_1| + |c_1 A_0 \Psi|\right], \tag{2.15}$$

where

$$\Psi = \left\{w_2 - \left(\frac{(2-\lambda)c_1 A_0}{2\lambda} + \frac{3\lambda c_1 A_0 w_1^2 \sigma [3]_q^n}{4\lambda^2[2]_q^{2n}} - \frac{c_2}{c_1}\right) w_1^2\right\}. \tag{2.16}$$

Since φ is analytic in $\mathbb{U}$, using the inequalities $|A_n| \leq 1$ and $|w_1| \leq 1$, we get

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{c_1}{3\lambda[3]_q^n}\left[|1 + |\Phi|\right], \tag{2.17}$$

where

$$\Phi = \left|w_2 - \left(-\frac{c_2}{c_1} - \left[\frac{(2-\lambda)c_1}{2\lambda} + \frac{3\sigma\lambda[3]_q^n c_1}{4\lambda^2[2]_q^{2n}}\right] c_1\right) w_1^2\right|. \tag{2.18}$$

Applying Lemma 1.4 and equation (2.18) yields result (2.10). □

Corollary 2.4 *Let* $f \in \mathcal{S}^{q}_{\lambda,\varphi}(\phi)$, *then*

$$|a_2| \leq \frac{c_1 A_0}{2\lambda[2]_q^n} \tag{2.19}$$

and

$$|a_3| \leq \frac{1}{3\lambda[3]_q^n}\left(c_1 + max\left\{c_1, \left|\frac{(2-\lambda)c_1^2}{2\lambda}\right| + |c_2|\right\}\right), \tag{2.20}$$

where $\lambda \geq 1, 0 < q < 1$.

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