Singly generated invariant subspaces in the Hardy space on the unit ball

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Abstract: In this paper, we give a complete characterization of singly generated invariant subspaces in the Hardy space on the unit ball. Then we construct a singly generated invariant subspace that cannot be generated by a single inner function, contrary to the one-variable case where every invariant subspace is generated by a single inner function. Some important properties of invariant subspaces are also determined for singly generated invariant subspaces.

Key words: Invariant subspace, Hardy space, ball, inner function, factorization

1. Introduction

In his well-known paper [3], Beurling showed that every invariant subspace M of the Hardy space $H^2(\mathbb{D})$ on the unit disc $\mathbb{D}$ is of the form $M = fH^2(\mathbb{D})$ for some inner function f, i.e. is generated by a single inner function. This type of invariant subspace is called Beurling-type. However, in the case of several variables, the structure of invariant subspaces cannot be characterized in such a simple form. Although it is clear that the subspaces generated by an inner function are also invariant, determining all invariant subspaces in the several-variable case is difficult. For example, in the case of the polydisc $\mathbb{D}^n$ $(n > 1)$ it is clear that the subspaces generated by an inner function are also invariant in $H^2(\mathbb{D}^n)$. However, Rudin [9] showed that there are invariant subspaces of $H^2(\mathbb{D}^2)$ that are not even finitely generated. In his book [9, p. 78], he posed the following question: "One may ask for a classification or an explicit description (in some sense) of all invariant subspaces of $H^2(\mathbb{D}^n)$". This question has been extensively studied by various authors in different contexts, but it is still open. Recently, the authors gave a partially answer to the question by determining the structure of singly generated invariant subspaces [7].

In this paper, inspired by [7], we deal with the case of the unit ball B^n. In this case, the existence of nonconstant inner functions was a rather elusive problem for quite some time. There was a time when nonconstant inner functions were thought not to exist. In 1982, Alexandrov [2] and Løw [8] proved independently the existence of nonconstant inner functions in the unit ball. After this investigation it is natural to ask for *a classification or an explicit description of all invariant subspaces of $H^2(B^n)$ for arbitrary n*, as that of Rudin in the polydisc case.

A subspace M of the Hardy space $H^2(B^n)$ on the unit ball B^n is called *invariant* if (a) M is a closed linear subspace of $H^2(B^n)$ and (b) $f \in M$ implies $z_i f \in M$ for $i = 1, \ldots, n$; i.e. multiplication by the variables

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$z_1, z_2, \ldots, z_n$ maps M into M. The smallest invariant subspace of $H^2(B^n)$ that contains a given f is denoted by M_f and M_f is called the subspace generated by f if $M_f = fH^2(B^n)$.

It is clear that a subspace of $H^2(B^n)$ generated by an inner function is invariant and there exist a lot of invariant subspaces not generated by a single inner function, e.g., invariant subspaces of finite codimension [4, 5]. In this study we partially answer the question above by giving a complete characterization of singly generated invariant subspaces in the unit ball. Then we construct a singly generated invariant subspace that cannot be generated by any single inner function. In view of these results, it is seen that the structure of invariant subspaces in the unit ball is much more complicated than the one-variable case where every invariant subspace is generated by a single inner function.

Before beginning, let us recall some required facts.

Throughout this paper, $n > 1$ is a positive integer, and $\mathbb{C}^n$ is the vector space of all ordered n-tuples $z = (z_1, \ldots, z_n)$ of complex numbers with inner product $\langle z, w \rangle = \sum z_i \bar{w}_i$, norm $|z| = \langle z, z \rangle^{1/2}$, and corresponding unit ball

$$B^n = \{z \in \mathbb{C}^n : |z| < 1\},$$

whose boundary is the unit sphere

$$S^n = \{z \in \mathbb{C}^n : |z| = 1\}.$$

On S^n, there is a unique rotation-invariant positive measure σ so that $\sigma(S^n) = 1$. If f is in $L^1(S^n)$, the usual Lebesgue space with respect to σ, then its Poisson integral $P[f]$ is the function

$$P[f] = \int_{S^n} P(z, w) f(w) d\sigma(w), \ \ z \in B^n,$$

where $P(z, w)$ is the Poisson kernel.

The Hardy space on the unit ball $H^2(B^n)$ is defined as the space of all holomorphic functions f in B^n for which

$$||f||_2 = \sup_{0 \leq r < 1} \left\{ \int_{S^n} |f(rw)|^2 d\sigma(w) \right\}^{1/2} < \infty.$$

The radial limits $f^*(w)$ of $f \in H^2(B^n)$ exist a.e. on S^n and satisfy the integrability condition $\log |f^*| \in L^1(S^n)$ if $f \not\equiv 0$ [11, p. 85].

Every $f \in H^2(B^n)$ satisfies the inequality

$$\log |f| \leq P[\log |f^*|].$$

This is proved exactly as in polydiscs [9, Theorem 3.3.5].

$H^\infty(B^n)$ is the space of all bounded holomorphic functions f in B^n; $||f||_\infty = \sup_{z \in B^n} |f(z)| < \infty$. An *inner* function in B^n is a function $f \in H^\infty(B^n)$ with $|f^*| = 1$ a.e. on S^n. A function $f \in H^2(B^n)$ is said to be *outer* if

$$\log |f(0)| = \int_{S^n} \log |f^*| d\sigma.$$

If f is outer, then $\log |f| = P[\log |f^*|]$. This implies that f has no zero in B^n, and $\log |f|$ and $P[\log |f^*|]$ are real parts of holomorphic functions in B^n. The proofs are the same as that in the polydisc case [9, p. 73].

The following theorem is also required in this study.

Theorem 1.1 *[12, Theorem 4.2, p. 113] If ψ is a positive, bounded, and lower semicontinuous function on S^n, then there is a zero-free $f \in H^\infty(B^n)$ such that $|f^*| = \psi$ a.e. on S^n.*

For further information on Hardy space on the ball, see [11, 13].

2. Main results

Definition 2.1 *A function $f \in H^\infty(B^n)$ with $1/f^* \in L^\infty(S^n)$ is called* a generalized inner function.

Every inner function in the unit ball is a generalized inner function. Similar to the problem of existence of nonconstant inner functions in the unit ball given by Rudin [11, p. 403], we can naturally ask the following question:

Question: Does there exist a nonconstant generalized inner function in the unit ball that is not inner?

We have a positive answer by [12, Example 6.7]: Let V be a dense open circular set in S^n with $\sigma(V) < 1$. There is $f \in H^\infty(B^n)$ such that $|f^*| = 1$ a.e. on V, $|f^*| = \frac{1}{2}$ a.e. on $S^n \backslash V$.

It is clear that the subspace $\varphi H^2(B^n)$ of $H^2(B^n)$ with some inner function φ is invariant. After the above answer, it is natural to ask whether there exists any function f that is not necessarily inner such that $f H^2(B^n)$ is invariant. The following theorem describes the class of all such functions.

Theorem 2.2 *Let $f \in H^\infty(B^n)$. The subspace $f H^2(B^n)$ of $H^2(B^n)$ is invariant if and only if f is a generalized inner function.*

Proof Let M_f denote the bounded linear operator on $H^2(B^n)$ given by $M_f(g) = fg$ for any $g \in H^2(B^n)$. Suppose that the subspace $f H^2(B^n)$ is invariant. Since $\mathrm{Ker} M_f = \{0\}$ and the image of M_f, $f H^2(B^n)$ is closed, and M_f is bounded below; that is, there exists a number $\delta > 0$ such that $||M_f g||_2 \geq \delta ||g||_2$ for any $g \in H^2(B^n)$. Then $|f^*| \geq \delta$ a.e. on S^n. In fact, assume that $\sigma\{\xi \in S^n : |f^*(\xi)| < \delta\} > 0$. Then $\sigma\{\xi \in S^n : |f^*(\xi)| < \delta_0\} > 0$ for some $\delta_0 \in (0, \delta)$. Let us fix such a δ_0 and put $E = \{\xi \in S^n : |f^*(\xi)| < \delta_0\}$. We can construct a sequence of continuous functions $\{\varphi_j\}_{j \geq 1}$ defined on S^n such that $0 < \varphi_j \leq 1$ and $\lim_{j \to \infty} \varphi_j = \chi_E$ a.e. on S^n, where χ_E denotes the characteristic function of E. By Theorem (1.1) for each j there exists a function $g_j \in H^\infty(B^n) \subset H^2(B^n)$ such that $|g_j^*| = \varphi_j$ a.e. on S^n. We get

$$\delta^2 \int_{S^n} |g_j^*|^2 d\sigma \leq \int_{S^n} |f^*|^2 |g_j^*|^2 d\sigma$$

for all j. Applying the Lebesgue dominated convergence theorem, we have

$$\delta^2 \sigma(E) \leq \int_E |f^*|^2 d\sigma \leq \delta_0^2 \sigma(E) < \delta^2 \sigma(E),$$

and we get a contradiction. Hence, $|f^*| \geq \delta$ a.e. on S^n, i.e. $1/f^* \in L^\infty(S^n)$. Conversely, suppose that f is a generalized inner function. It is clear that for the invariance of $f H^2(B^n)$, it is enough to show the closedness of the subspace $f H^2(B^n)$. For this, we show that M_f is bounded below, and that means there exists a number

$c > 0$ such that $||M_f g||_2 \geq c||g||_2$ for any $g \in H^2(B^n)$. In fact, for any $g \in H^2(B^n)$ we obtain

$$||g||_2 = ||f^{-1}fg||_2 \leq ||f^{-1}||_\infty \cdot ||fg||_2 = ||f^{-1}||_\infty \cdot ||M_f g||_2,$$

and the proof is complete. □

This theorem shows that every singly generated invariant subspace M of $H^2(B^n)$ is of the form $M = fH^2(B^n)$ for some generalized inner function f. This is a generalization of Beurling's theorem. It is clear that the class of inner functions is contained in the class of generalized inner functions. Then the following is a natural question to ask:

Question: Is every singly generated invariant subspace of $H^2(B^n)$ generated by an inner function?

The following construction leads to a negative answer:

Theorem 2.3 *For $n > 1$, there exists a generalized inner function f such that $fH^2(B^n) \neq IH^2(B^n)$ for any inner function I.*

Proof We can take a function $h \in C(S^n)$ such that $h > 0$ everywhere and the Poisson integral of $\log h$ is not pluriharmonic. By Theorem (1.1), there exists a function $f \in H^\infty(B^n)$ such that $|f^*| = h$ almost everywhere on S^n. Suppose that $fH^2(B^n) = IH^2(B^n)$ for some inner function I. Then $fI^{-1}H^2(B^n) = H^2(B^n)$, and hence fI^{-1} is outer (the proof is the same as that in the polydisc case [9, p. 74, Theorem 4.4.6]). It follows that the equality

$$\operatorname{Re}\log(fI^{-1}) = \log|fI^{-1}| = P[\log|(fI^{-1})^*|] = P[\log|f^*|] = P[\log h]$$

is satisfied, showing that $P[\log h]$ is pluriharmonic, and we get a contradiction, and the proof is complete. □

Proposition 2.4 *Let f_1 and f_2 be two generalized inner functions that have no zero in B^n. The invariant subspaces $f_1 H^2(B^n)$ and $f_2 H^2(B^n)$ satisfy the following conditions:*

(a) *$f_1 H^2(B^n) \subset f_2 H^2(B^n)$ if and only if f_1/f_2 is a generalized inner function.*

(b) *$f_1 H^2(B^n) = f_2 H^2(B^n)$ if and only if $f_1/f_2, f_2/f_1 \in H^\infty(B^n)$.*

Proof

(a) Assume that $f_1 H^2(B^n) \subset f_2 H^2(B^n)$. Then $f_1 \in f_2 H^2(B^n)$ and thus $f_1 = f_2 g$ for some $g \in H^2(B^n)$. Moreover, $f_1 \in H^\infty(B^n)$ and $1/f_2^* \in L^\infty(S^n)$ give $g = f_1/f_2 \in H^\infty(B^n)$. On the other hand, if $1/f_1^* \in L^\infty(S^n)$, then $1/g^* = (f_2/f_1)^* \in L^\infty(S^n)$. Conversely, if $f_1 = f_2 h$, where $h \in H^\infty(B^n)$ with $1/h^* \in L^\infty(S^n)$, then $f_1 f = f_2(hf)$ for any $f \in H^2(B^n)$ and therefore $f_1 H^2(B^n) \subset f_2 H^2(B^n)$.

(b) It is easily seen from (a).

□

Beurling's theorem also states (in the one-variable case) that a function $f \in H^\infty(\mathbb{D})$ is outer if and only if $fH^2(\mathbb{D})$ is dense in $H^2(\mathbb{D})$. For the unit ball case one part of this theorem holds if $n > 1$: f is outer if $fH^2(B^n)$ is dense in $H^2(B^n)$. The proof is the same as that in the polydisc case [9, Theorem 4.4.6, p. 74]. The following result gives a class of functions for which Beurling's theorem holds for $n > 1$.

Corollary 2.5 *If f is a generalized inner function, then f is outer if and only if $fH^2(B^n) = H^2(B^n)$.*

Proof   If $fH^2(B^n) = H^2(B^n)$, then that the function f is outer is obvious by [9, Theorem 4.4.6, p. 74]. Conversely, suppose that f is outer. Then

$$\log|f| = P[\log|f^*|], \tag{2.1}$$

which implies that f has no zero in B^n and hence $1/f$ is analytic in B^n. On the other hand, multiplying equality (2.1) by (-1), we obtain that the same equality is true for $1/f$; that is, $\log|1/f| = P[\log|(1/f)^*|]$. Since $1/f^* \in L^\infty(S^n)$, $\log|(1/f)^*| \in L^\infty(S^n)$, and then its Poisson integral $P[\log|(1/f)^*|]$ is bounded on B^n. Hence, $1/f$ is bounded on B^n, and $1/f \in H^\infty(B^n)$. Applying Proposition (2.4) (b) to $f_1 = f$ and $f_2 = 1$, we have $fH^2(B^n) = H^2(B^n)$. □

Two invariant subspaces M_1 and M_2 are said to be unitarily equivalent if there is a unitary operator $U : M_1 \to M_2$ such that $U(\theta f) = \theta(Uf)$ for $\theta \in H^\infty(B^n)$ and $f \in M_1$. In order to classify invariant subspaces, unitary equivalence is a natural equivalence relation. In the one-variable case, all invariant subspaces are unitarily equivalent to $H^2(\mathbb{D})$ by Beurling's theorem. However, it is known that there exist many equivalence classes of invariant subspaces in the polydisc case. Agrawal et al. [1] studied the question of unitary equivalence of invariant subspaces in this case. In the following theorem, we show unitary equivalence of the singly generated invariant subspaces in the unit ball case.

Theorem 2.6   (a) *Let f_1 and f_2 be two generalized inner functions. The invariant subspaces $f_1H^2(B^n)$ and $f_2H^2(B^n)$ are unitarily equivalent if $|f_1^*| = |f_2^*|$ a.e. on S^n.*

(b) *Any invariant subspace M of $H^2(B^n)$ unitarily equivalent to a singly generated invariant subspace is also singly generated.*

Proof

(a) Let $|f_1^*| = |f_2^*|$ a.e. on S^n. Taking the multiplication operator $M_\psi : f_1H^2(B^n) \to f_2H^2(B^n)$, where $\psi := f_2/f_1$-unimodular function, as a desired unitary operator, we easily obtain the unitary equivalence of these subspaces.

(b) If the invariant subspace M of $H^2(B^n)$ is unitarily equivalent to $fH^2(B^n)$ for some generalized inner function f, then, by Lemma 1 in [1] (this is also true for the unit ball case), there exists a unimodular function ψ such that $M = \psi fH^2(B^n)$ and so the subspace M is an invariant subspace generated by $g := \psi f \in H^\infty(B^n)$ with $1/g^* \in L^\infty(S^n)$.

□

It is well known that every nonzero function f in $H^2(\mathbb{D})$ has a factorization $f = gh$, where g is inner and h is outer. This is no longer true for $H^2(B^n)$ for $n > 1$ (see [6]). One can ask whether a generalized statement may hold true, namely that every nonzero function f in $H^2(B^n)$ can be written as a product $f = gh$, where h is still an outer function, but where g is now a *generalized* inner function. However, according to a result of Rudin [10, Rem. (b), p. 59], for any $n \geq 2$, there exists an f in $H^2(B^n)$, $f \not\equiv 0$, such that for no nonzero g in $H^\infty(B^n)$ the quotient g/f is holomorphic in B^n. Assume that such an f factors as $f = gh$ with g a generalized inner function and h an outer function. Since outer functions have no zeros in B^n, the quotient

g/f is holomorphic, which contradicts Rudin's result. Thus, the generalized factorization does not hold for this $f \in H^2(B^n)$. We do not know whether a generalized factorization holds true for arbitrary bounded $f \not\equiv 0$ on B^n.

Problem Can every nonzero function f in $H^\infty(B^n)$ be written as a product $f = gh$, where g is a generalized inner function and h is an outer function?

Acknowledgments

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