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The twelvefold way, the nonintersecting circles problem, and partitions of multisets

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Abstract: Let n be a nonnegative integer and $\mathbb{A} = \{a_1, \dots, a_k\}$ be a multiset with k positive integers such that $a_1 \leqslant \cdots \leqslant a_k$. In this paper, we give a recursive formula for partitions and distinct partitions of positive integer n with respect to a multiset $\mathbb{A}$. We also consider the extension of the twelvefold way. By using this notion, we solve the nonintersecting circles problem, which asks to evaluate the number of ways to draw n nonintersecting circles in the plane regardless of their sizes. The latter also enumerates the number of unlabeled rooted trees with $n+1$ vertices.

Key words: Multiset, partitions and distinct partitions, twelvefold way, nonintersecting circles problem, rooted trees, Wilf partitions

1. Introduction

A *partition* of n is a sequence $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$ of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$ (see [2]). We write $\lambda \vdash n$ to denote that λ is a partition of n. The nonzero integers λ_k in λ are called *parts* of λ. The number of parts of λ is the *length* of λ, denoted by $\ell(\lambda)$, and $|\lambda| = \sum_{k\geqslant 1} \lambda_k$ is the *weight* of λ. More generally, any weakly decreasing sequence of positive integers is called a partition. The partition whose parts are $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$ is usually denoted by $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$. Let $P(n)$ denote the set of all partitions of n. The size of the set $P(n)$ is denoted by the *partition function* $p(n)$; that is, $p(n) = |P(n)|$. In particular, $p(0)$ consists of a single element, the unique empty partition of zero, which we denote by 0. For example, $P(4)$ consists of five elements: $4, 3+1, 2+2, 2+1+1, and 1+1+1+1$. Hence, $p(4) = 5$.

We let $\mathbb{S}$ be a set of natural numbers and $p(n|\mathbb{S})$ denotes the number of partitions of n into elements of $\mathbb{S}$ (that is, the parts of the partitions belonging to $\mathbb{S}$) and $p_\ell(n|\mathbb{S})$ is the number of partitions of n into exactly ℓ parts in $\mathbb{S}$. When $\mathbb{S} = \mathbb{N}$, the set of natural numbers, we denote $p_\ell(n|\mathbb{N})$ by $p_\ell(n)$, i.e. the number of partitions of n into exactly ℓ parts (*or dually, partitions with the largest part equal to* ℓ).

Recall also that a *multiset* $\mathbb{A}$ with the *multiplicity mapping* θ is a collection of some not necessarily different objects such that for each $a \in \mathbb{A}$ the number $\theta(a)$ is the multiplicity of the occurrence of a in $\mathbb{A}$. If $\mathbb{A}$ is a multiset, we denote the set of members of $\mathbb{A}$ by $S(\mathbb{A})$ and we call it *the background set of* $\mathbb{A}$. For a number a_0 and a multiset $\mathbb{A}$, the multiset $\{a_0 a : a \in \mathbb{A}\}$ is denoted by $a_0\mathbb{A}$. We denote the multiset $\{1, 1, \dots, 1\}$ with $\theta(1) = k$ by I_k. We define that $I_0 = \emptyset$. Thus, a multiset $\mathbb{A}$ can be written as $\cup_{i=1}^{\ell} b_i I_{\theta(b_i)}$, where the

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background set $S(\mathbb{A})$ of $\mathbb{A}$ is $\{b_1, \cdots, b_\ell\}$. For two multisets $\mathbb{A}$ with the multiplicity mapping $\theta_{\mathbb{A}}$ and $\mathbb{B}$ with the multiplicity mapping $\theta_{\mathbb{B}}$, we define the multiplicity mapping $\theta_{\mathbb{A}\setminus\mathbb{B}}$ of $\mathbb{A}\setminus\mathbb{B}$ by $\theta_{\mathbb{A}\setminus\mathbb{B}}(a) = \theta_{\mathbb{A}}(a) - \theta_{\mathbb{B}}(a)$ if $\theta_{\mathbb{A}}(a) \geqslant \theta_{\mathbb{B}}(a)$ and $\theta_{\mathbb{A}\setminus\mathbb{B}}(a) = 0$ if $\theta_{\mathbb{A}}(a) < \theta_{\mathbb{B}}(a)$. Moreover, the multiplicity mapping $\theta_{\mathbb{A}\cup\mathbb{B}}$ of $\mathbb{A}\cup\mathbb{B}$ is defined by $\theta_{\mathbb{A}\cup\mathbb{B}}(a) = \theta_{\mathbb{A}}(a) + \theta_{\mathbb{B}}(a)$. In the following, $\theta(\mathbb{A}) = \sum_{a\in\mathbb{A}} a$ for a multiset $\mathbb{A}$.

For any fixed complex number $|q| \leq 1$, any complex number a, and any nonnegative integer n, let

$$(a;q)_n := \begin{cases} \prod_{k=0}^{n-1}(1-aq^k), & n>0 \\ 1, & n=0. \end{cases}$$

Accordingly, let

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) = \lim_{n\to\infty}(a;q)_n.$$

A q*-series* is any series that involves expressions of the form $(a;q)_n$ and $(a;q)_\infty$. The generating function for $p(n)$, discovered by Euler, is given by $\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}$.

The organization of this paper is as follows. In the next section, we consider the number of partitions of n into elements of the multiset $\mathbb{A}$ and the number of partitions of n with distinct parts from the multiset $\mathbb{A}$. As a consequence, we present a recursive formula for Wilf's unsolved problem.* In Section 3, we present an extension of the *twelvefold way*, which was invented by Stanley [11]. As consequences, we give a recurrence relation for B_n, the number of ways to draw n nonintersecting circles in a plane regardless of their sizes. In Section 5, we deal with the ordered and unordered factorizations of natural numbers. In Section 6, we present generating functions for our sequences. We end with Section 6, where we establish connections with Möbius and Euler's totient functions.

2. Partitions and distinct partitions of positive integer n with respect to a multiset

Let n be a nonnegative integer and $\mathbb{A} = \{a_1, \ldots, a_k\}$ be a multiset with k (not necessarily distinct) positive integers. We denote by $D(n|\mathbb{A})$ the number of ways to partition n as $a_1x_1 + \cdots + a_kx_k$, where x_i are positive integers and $x_i \leqslant x_{i+1}$ whenever $a_i = a_{i+1}$. The number of ways to partition n in the form $a_1x_1 + \cdots + a_kx_k$, where x_i are nonnegative integers and $x_i \leqslant x_{i+1}$ whenever $a_i = a_{i+1}$, is also denoted by $D_0(n|\mathbb{A})$. The numbers $D(n|\mathbb{A})$ and $D_0(n|\mathbb{A})$ are called *the natural partition number* and *the arithmetic partition number of* n *with respect to* $\mathbb{A}$.

Lemma 2.1 *Let n be a nonnegative integer and $\mathbb{A}$ be a multiset with the multiplicity mapping θ and the background set $S(\mathbb{A}) = \{b_1, \ldots, b_\ell\}$. Then*

$$D(n|\mathbb{A}) = \sum_{\substack{n=b_1y_1+\cdots+b_\ell y_\ell \\ \theta(b_i)\leqslant y_i,\ i=1,\cdots,\ell}} \prod_{j=1}^{\ell} p_{\theta(b_j)}(y_j).$$

Proof Let $A = \{a_1, \ldots, a_k\}$, where a_i are not necessarily distinct members of A. We can write $n = a_1x_1 + \cdots + a_kx_k$ in the form $n = b_1(x_{11} + \cdots + x_{1\theta(b_1)}) + \cdots + b_\ell(x_{\ell 1} + \cdots + x_{\ell\theta(b_\ell)})$. Putting $y_i = x_{i1} + \cdots + x_{i\theta(b_i)}$,

*Wilf HS. Some unsolved problems. www.math.upenn.edu/∼wilf/website/UnsolvedProblems.pdf.

we have $n = b_1y_1 + \cdots + b_\ell y_\ell$, where $\theta(b_i) \leqslant y_i$ for $i = 1, \cdots, \ell$. Now the number of ways to partition y_i in the form $x_{i1} + \cdots + x_{i\theta(b_i)}$ is $p_{\theta(b_i)}(y_i)$, where $1 \leqslant x_{i1} \leqslant \cdots \leqslant x_{i\theta(b_i)}$ are positive integers. □

Let $p_{\leqslant m}(n)$ denote the number of partitions of positive integer n into at most m parts, and notice that $p_{\leqslant m}(n)$ is equal to the number of partitions of positive integer n into parts that are all $\leqslant m$ in view of conjugate partitions. Then $p_{\leqslant m}(n) = p_0(n) + p_1(n) + \cdots + p_m(n)$. We can state the following result.

Lemma 2.2 *Let n be a nonnegative integer and $\mathbb{A}$ be a multiset with the multiplicity mapping θ and the background set $S(\mathbb{A}) = \{b_1, \cdots, b_\ell\}$. Then*

$$D_0(n|\mathbb{A}) = \sum_{\substack{n=b_1y_1+\cdots+b_\ell y_\ell \\ \theta(b_i)\leqslant y_i,\ i=1,\cdots,\ell}} \prod_{j=1}^{\ell} p_{\leqslant\theta(b_j)}(y_j).$$

Notice that if n is a positive integer and $\mathbb{A}$ is a multiset as $\mathbb{A} = \{1, 1, \ldots, 1\}$ with multiplicity function θ such that $\theta(1) = \ell$, then

$$D(n|\mathbb{A}) = D(n|\{1, 1, \ldots, 1\}) = p_\ell(n). \tag{2.1}$$

That is the number of partitions of positive integer n into exactly ℓ parts. Furthermore, for each multiset $\mathbb{A}$, $D_0(n|\mathbb{A}) = D(n + \theta(\mathbb{A})|\mathbb{A})$, where $\theta(\mathbb{A}) = \sum_{a\in\mathbb{A}} a$.

Proposition 2.3 *Let n be a nonnegative integer and $\mathbb{A}$ be a multiset with the multiplicity mapping θ. Then, for each $a \in \mathbb{A}$,*

$$D(n|\mathbb{A}) = \sum_{\substack{0\leqslant\ell\leqslant\theta(a) \\ a\theta(a)\leqslant n}} D(n - a\theta(a)|\mathbb{A} \setminus aI_\ell),$$

where $D(0|\emptyset) = 1$.

Proof Let $\mathbb{A} = \{a_1, \ldots, a_k\}$. We have $\theta(a)$ occurrence of a in the equation $n = a_1x_1 + \cdots + a_kx_k$. Let $x_{i+1}, \ldots, x_{i+\theta(a)}$ have coefficients a in the equation, $x_{i+1} = \cdots = x_{i+\ell} = 1$ and $x_{i+\ell+1} > 1$, where $\ell = 0, 1, \ldots, \theta(a)$. If we subtract $a\theta(a)$ from both sides of $n = a_1x_1 + \cdots + a_kx_k$, then we get $n - a\theta(a) = a_1x_1 + \cdots + a_ix_i + a_{i+\ell+1}x_{i+\ell+1} + \cdots + a_kx_k$. The number of solutions of this equation is $D(n - a\theta(a)|\mathbb{A} \setminus aI_\ell)$, which completes the proof. □

Example 2.4 *We evaluate $D(17 \mid \{1, 2, 2, 3\})$ and $D_0(17, \{1, 2, 2, 3\})$. Using Proposition 2.3 and Corollary 2.5, we can write*

$$\begin{aligned} D(17 \mid \{1, 2, 2, 3\}) &= D(14 \mid \{1, 2, 2, 3\}) + D(14 \mid \{1, 2, 2\}) \\ &= D(11 \mid \{1, 2, 2, 3\}) + D(11 \mid \{1, 2, 2\}) + 9 \\ &= D(8 \mid \{1, 2, 2, 3\}) + D(8 \mid \{1, 2, 2\}) + 6 + 9 \\ &= 1 + 2 + 6 + 9 = 18. \end{aligned}$$

Moreover,

$$\begin{aligned}D_0(17,\{1,2,2,3\}) &= D(17+8,\{1,2,2,3\})\\ &= D(22,\{1,2,2,3\})+D(22,\{1,2,2\})\\ &= D(19,\{1,2,2,3\})+D(19,\{1,2,2\})+25\\ &= D(16,\{1,2,2,3\})+D(16,\{1,2,2\})+20+25\\ &= D(13,\{1,2,2,3\})+D(13,\{1,2,2\})+12+20+25\\ &= D(10,\{1,2,2,3\})+D(10,\{1,2,2\})+9+12+20+25\\ &= D(7,\{1,2,2,3\})+D(7,\{1,2,2\})+4+9+12+20+25\\ &= 0+2+4+9+12+20+25=72.\end{aligned}$$

Corollary 2.5 *Let n be a positive integer. Then*

$$\begin{aligned}D(n\mid\{1,2\}) &= \lfloor\frac{n-1}{2}\rfloor,\\ D(n\mid\{1,2,2\}) &= \lfloor\frac{n-1}{4}\rfloor(\lfloor\frac{n+1}{2}\rfloor-\lfloor\frac{n+3}{4}\rfloor),\\ D(n\mid\{1,1,2\}) &= \lfloor\frac{3}{2}\lfloor\frac{n-1}{3}\rfloor+\frac{1}{2}\rfloor(\lfloor\frac{n-1}{2}\rfloor-\frac{1}{2}\lfloor\frac{3}{2}\lfloor\frac{n+2}{3}\rfloor\rfloor+\frac{1+(-1)^n}{2}).\end{aligned}$$

Proof Let $n=2k+r$, where $r=1,2$. By Proposition 2.3, we obtain

$$\begin{aligned}D(n|\{1,2\}) &= D(n-2|\{1,2\})+D(n-2|\{1\})=D(n-2|\{1,2\})+1\\ &= D(n-4|\{1,2\})+D(n-2|\{1\})+1=D(n-4|\{1,2\})+2\\ &= D(n-6|\{1,2\})+3=\cdots=D(n-2k|\{1,2\})+k=0+k=\lfloor\frac{n-1}{2}\rfloor.\end{aligned}$$

For the second assertion, let $n=4k+r$, where $r=1,2,3,4$. Then

$$\begin{aligned}D(n|\{1,2,2\}) &= D(n-4|\{1,2,2\})+D(n-4|\{1,2\})+D(n-4|\{1\})\\ &= D(n-4|\{1,2,2\})+\lfloor\frac{n-5}{2}\rfloor+1\\ &= D(n-8|\{1,2,2\})+\lfloor\frac{n-7}{2}\rfloor+\lfloor\frac{n-3}{2}\rfloor\\ &= \cdots=D(n-4k|\{1,2,2\})+\sum_{i=1}^{k}\lfloor\frac{n-(4i-1)}{2}\rfloor\\ &= D(r|\{1,2,2\})+\sum_{i=1}^{k}\lfloor\frac{n-(4i-1)}{2}\rfloor=0+\sum_{i=1}^{k}\lfloor\frac{n-(4i-1)}{2}\rfloor\\ &= k\lfloor\frac{n+1}{2}\rfloor-k(k+1)=\lfloor\frac{n-1}{4}\rfloor(\lfloor\frac{n+1}{2}\rfloor-\lfloor\frac{n+3}{4}\rfloor),\end{aligned}$$

as required. Now let $n = 3k + r$ where $r = 1, 2, 3$. Then

$$\begin{aligned}
D(n \mid \{1,1,2\}) &= D(n-2 \mid \{1,1,2\}) + D(n-2 \mid \{1,2\}) + D(n-2 \mid \{2\}) \\
&= D(n-2 \mid \{1,1,2\}) + \lfloor \frac{n-3}{2} \rfloor + \frac{1+(-1)^n}{2} \\
&= D(n-4 \mid \{1,1,2\}) + \lfloor \frac{n-5}{2} \rfloor + \lfloor \frac{n-3}{2} \rfloor + 2\frac{1+(-1)^n}{2} \\
&= \dots \\
&= D(n - 4(\lfloor \frac{3k+1}{2} \rfloor), \{1,1,2\}) \\
&\quad + \sum_{i=1}^{\lfloor \frac{3k+1}{2} \rfloor} \lfloor \frac{(n-2i)-1}{2} \rfloor + \lfloor \frac{3k+1}{2} \rfloor \frac{1+(-1)^n}{2} \\
&= 0 + \sum_{i=1}^{\lfloor \frac{3k+1}{2} \rfloor} \lfloor \frac{(n-2i)-1}{2} \rfloor + k\frac{1+(-1)^n}{2} \\
&= \lfloor \frac{3k+1}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor - \frac{\lfloor \frac{3k+1}{2} \rfloor (\lfloor \frac{3k+1}{2} \rfloor + 1)}{2} + \lfloor \frac{3k+1}{2} \rfloor \frac{1+(-1)^n}{2}.
\end{aligned}$$

It is enough to note that $k = \lfloor \frac{n-1}{3} \rfloor$. □

Let $Q_m(n)$ be the number of partitions of a positive integer n into exactly m distinct parts. It is not difficult to verify by using Ferrers diagrams that $Q_m(n) = p_{\leqslant m}\left(n - \binom{m+1}{2}\right)$, which means that the number of partitions of positive integer n into exactly m distinct parts equals the number of partitions of $n - \binom{m+1}{2}$ into at most m parts (*or dually, partitions into parts* $\leqslant m$) [4]. Then the generating function of $Q_m(n)$ reads as

$$\sum_{n=0}^{\infty} Q_m(n) q^n = \frac{q^{\binom{m+1}{2}}}{(q;q)_m}.$$

We let $Q(n)$ be the number of all partitions of n into distinct parts.

Let n be a nonnegative integer and $\mathbb{A} = \{a_1, \dots, a_k\}$ be a multiset of k not necessarily distinct positive integers, where $a_1 \leqslant \cdots \leqslant a_k$. We denote by $\Delta(n|\mathbb{A})$ the number of partitions of n as the form $a_1x_1 + \cdots + a_kx_k$, where x_i are distinct positive integers and $x_i < x_{i+1}$ whenever $a_i = a_{i+1}$. The number of partitions of n of the form $a_1x_1 + \cdots + a_kx_k$, where x_i are distinct nonnegative integers and $x_i < x_{i+1}$ whenever $a_i = a_{i+1}$, is also denoted by $\Delta_0(n|\mathbb{A})$. The numbers $\Delta(n|\mathbb{A})$ and $\Delta_0(n|\mathbb{A})$ are called the *natural distinct partition number* and the *arithmetic distinct partition number of* n *with respect to* $\mathbb{A}$.

Lemma 2.6 *Let* n *be a nonnegative integer and* $\mathbb{A}$ *be a multiset with the multiplicity mapping* θ *and the background set* $S(\mathbb{A}) = \{b_1, \dots, b_\ell\}$. *Then*

$$\Delta(n|\mathbb{A}) = \sum_{\substack{n = b_1y_1 + \cdots + b_\ell y_\ell \\ \theta(b_i) \leqslant y_i,\ i=1,\cdots,\ell}} \prod_{j=1}^{\ell} Q_{\theta(b_j)}(y_j).$$

Proof Proof as similar to Lemma 2.1. Let $\mathbb{A} = \{a_1, \dots, a_k\}$, where a_i are k not necessarily distinct members of A. We can write $n = a_1x_1 + \ldots + a_kx_k$ as the form $n = b_1(x_{11} + \ldots + x_{1\theta(b_1)}) + \ldots + b_\ell(x_{\ell 1} + \ldots + x_{\ell\theta(b_\ell)})$.

Putting $y_i = x_{i1} + \ldots + x_{i\theta(b_i)}$ we have $n = b_1y_1 + \ldots + b_\ell y_\ell$, where $\theta(b_i) \leqslant y_i$ for $i = 1, \ldots, \ell$. Now the number of ways to partition y_i into $x_{i1} + \ldots + x_{i\theta(b_i)}$ with $1 \leqslant x_{i1} \leqslant \ldots \leqslant x_{i\theta(b_i)}$ is $Q_{\theta(b_i)}(y_i)$. □

Let $Q_{\leqslant m}(n)$ denote the number of partitions of positive integer n into at most m distinct parts. Then $Q_{\leqslant m}(n) = Q_1(n) + Q_2(n) + \cdots + Q_m(n)$, which leads to the following corollary.

Corollary 2.7 *Let n be a nonnegative integer and $\mathbb{A}$ be a multiset with the multiplicity mapping θ and the background set $S(\mathbb{A}) = \{b_1, \ldots, b_\ell\}$. Then*

$$\Delta_0(n|\mathbb{A}) = \sum_{\substack{n=b_1y_1+\cdots+b_\ell y_\ell \\ \theta(b_i)\leqslant y_i,\ i=1,\cdots,\ell}} \prod_{j=1}^{\ell} Q_{\leqslant\theta(b_j)}(y_j).$$

Corollary 2.8 *Let n be a nonnegative integer and $\mathbb{A}$ be a multiset with the multiplicity mapping θ and the background set $S(\mathbb{A}) = \{b_1, \ldots, b_\ell\}$. Then*

$$\Delta\left(n|\mathbb{A}\right) = D\left(n + \theta(\mathbb{A}) - \sum_{i=1}^{\ell} b_i \binom{\theta(b_i)+1}{2} |\mathbb{A}\right).$$

Proof Let n be a nonnegative integer and $\mathbb{A} = \{a_1, \ldots, a_k\}$ be a multiset with k not necessarily distinct positive integers, where $a_1 \leqslant \cdots \leqslant a_k$. $\Delta(n|\mathbb{A})$ is the number of partitions of n as the form $n = a_1x_1 + \cdots + a_kx_k$, where x_i are distinct positive integers and $x_i < x_{i+1}$ whenever $a_i = a_{i+1}$. We can write

$$n = b_1(x_{11} + \cdots + x_{1\theta(b_1)}) + \cdots + b_\ell(x_{\ell 1} + \cdots + x_{\ell\theta(b_\ell)}).$$

Putting $y_i = x_{i1} + \cdots + x_{i\theta(b_i)}$, the number of partitions of y_i into exactly $\theta(b_i)$ distinct parts equals $Q_{\theta(b_i)}(y_i)$ for $i = 1, \ldots, \ell$. By Corollary (2.8), we get

$$Q_{\theta(b_i)}(y_i) = p_{\leqslant\theta(b_i)}\left(y_i + \binom{\theta(b_i)+1}{2}\right).$$

Then we can write

$$\begin{aligned} n &= b_1y_1 + b_2y_2 + \cdots + b_\ell y_\ell \\ &= b_1\left(y_1 - \binom{\theta(b_1)+1}{2}\right) + \cdots + b_\ell\left(y_\ell - \binom{\theta(b_\ell)+1}{2}\right) \\ &= b_1y_1 + b_2y_2 + \cdots + b_\ell y_\ell - \sum_{i=1}^{\ell} b_i\binom{\theta(b_i)+1}{2}. \end{aligned}$$

Then we can conclude

$$\Delta(n|\mathbb{A}) = D_0\left((n - \sum_{i=1}^{\ell} b_i\binom{\theta(b_i)+1}{2}|\mathbb{A}\right) = D\left(n + \theta(\mathbb{A}) - \sum_{i=1}^{\ell} b_i\binom{\theta(b_i)+1}{2}|\mathbb{A}\right),$$

as claimed. □

It is easy to see that if n is a positive integer and $\mathbb{A}$ is the multiset $\{1,1,\ldots,1\}$, with multiplicity function $\theta(1)=\ell$, then $\Delta(n|\mathbb{A})=\Delta(n|\{1,1,\ldots,1\})=Q_\ell(n)$. Furthermore, for each multiset $\mathbb{A}$, $\Delta_0(n|\mathbb{A})=\Delta(n+\theta(\mathbb{A})|\mathbb{A})$, where, $\theta(\mathbb{A})=\sum_{a\in\mathbb{A}}a$.

Herbert Wilf posed some unsolved problems.[†] Wilf's Sixth Unsolved Problem regards "the set of partitions of positive integer n for which the (nonzero) multiplicities of its parts are all different". We refer to these as *Wilf partitions* and $T(n)$ for the set of Wilf partitions. For example, there exist 4 Wilf partitions of $n=4$:

$$4=(1)4;\quad 2+2=(2)2;\quad 2+1+1=(1)2+(2)1;\quad 1+1+1+1=(4)1.$$

Then $|T(4)|=4$. Let $\mathbb{A}=\{a_1,a_2,\ldots,a_k\}$ be a set of nonnegative integers. We denote $T(n|\mathbb{A})$ for the number of Wilf partitions of positive integers n as the form $a_1x_1+a_2x_2+\cdots+a_kx_k$, where x_i are positive distinct integers. Furthermore, if we put $\mathbb{A}=\mathbb{N}$, the set of natural numbers, then $T(n|\mathbb{A})=|T(n)|$.

Proposition 2.9 *Let n be a nonnegative integer and $\mathbb{A}=\{a_1,\ldots,a_k\}$ be a multiset with the background set $S(\mathbb{A})=\{b_1,\ldots,b_\ell\}$. Then*

$$\Delta(n|\mathbb{A})=\Delta(n-\theta(\mathbb{A})|\mathbb{A})+\sum_{i=1}^{\ell}\Delta(n-\theta(\mathbb{A})|\mathbb{A}\setminus\{b_i\}).$$

Moreover, $\Delta(n|\mathbb{A})=0$ when $n<\sum_{i=1}^{k}(k+1-i)a_i$.

Proof At most one of the x_is can be 1. If there is no x_i with $x_i=1$ then we can write $n-\theta(A)=a_1(x_1-1)+\cdots+a_k(x_k-1)$ and there are $\Delta(n-\theta(A),A)$ solutions for this equation under the required conditions. Moreover, if $x_j=1$ for some j, then other x_is are greater that 1 and thus we can write

$$n-\theta(A)=a_1(x_1-1)+\cdots+a_{j-1}(x_{j-1}-1)+a_{j+1}(x_{j+1}-1)+\cdots+a_k(x_k-1).$$

There are $\Delta(n-\theta(A)|\mathbb{A}\setminus\{b_i\})$ solutions for the latter equation, where $b_i=a_j$. The other parts are obvious. □

Corollary 2.10 *Let n be a nonnegative integer and $\mathbb{A}=\{a_1,\ldots,a_k\}$ be a set of nonnegative integers. Then $T(n|\mathbb{A})$ is given by $T(n|\mathbb{A})=\sum_{i=0}^{k}T(n-\theta(\mathbb{A})|\mathbb{A}\setminus\{a_i\})$ with $b_0=\emptyset$.*

Example 2.11 *We evaluate $\Delta(18|\{1,2,2,3\})$. By Proposition 2.9, we have*

$$\begin{aligned}\Delta(18|\{1,2,2,3\}) &= \Delta(10|\{1,2,2,3\})+\Delta(10|\{2,2,3\})+\Delta(10|\{1,2,3\})+\Delta(10|\{1,2,2\})\\ &= 0+0+\Delta(4|\{1,2,3\})+\Delta(4|\{2,3\})+\Delta(4|\{1,3\})+\Delta(4|\{1,2\})\\ &\quad +\Delta(5|\{1,2,2\})+\Delta(5|\{2,2\})+\Delta(5|\{1,2\})\\ &= 0+0+0+0+0+1+0+0+2=3.\end{aligned}$$

The 3 solutions are

$$\begin{aligned}18 &= 1\times\mathbf{3}+2\times\mathbf{2}+2\times\mathbf{4}+3\times\mathbf{1}=1\times\mathbf{5}+2\times\mathbf{2}+2\times\mathbf{3}+3\times\mathbf{1}\\ &= 1\times\mathbf{4}+2\times\mathbf{1}+2\times\mathbf{3}+3\times\mathbf{2}.\end{aligned}$$

[†]Wilf HS. Some unsolved problems. www.math.upenn.edu/∼wilf/website/UnsolvedProblems.pdf.

Corollary 2.12 *Let n be a positive integer. Then*

$$\Delta(n \mid \{1,1\}) = \lfloor \frac{n-1}{2} \rfloor \text{ and } \Delta(n \mid \{1,2\}) = \lfloor \frac{n-1}{3} \rfloor + \lfloor \frac{n-1}{6} \rfloor.$$

Proof Let $n = 2k + r$, where $r = 1, 2$. Using Proposition 2.9 we can write

$$\begin{aligned}\Delta(n \mid \{1,1\}) &= \Delta(n-2 \mid \{1,1\}) + \Delta(n-2 \mid \{1\}) \\ &= \Delta(n-2 \mid \{1\}) + 1 \\ &= \Delta(n-4 \mid \{1,1\}) + \Delta(n-4 \mid \{1\}) + 1 \\ &= \dots \\ &= \Delta(n-2k \mid \{1,1\}) + k = 0 + k = \lfloor \frac{n-1}{2} \rfloor.\end{aligned}$$

Now let $n = 3k + r$, where $r = 1, 2, 3$. Thus,

$$\begin{aligned}\Delta(n \mid \{1,2\}) &= \Delta(n-3 \mid \{1,2\}) + \Delta(n-3 \mid \{1\}) + \Delta(n-3 \mid \{2\}) \\ &= \Delta(n-3 \mid \{1,2\}) + \Delta(n-3 \mid \{2\}) + 1 \\ &= \Delta(n-6 \mid \{1,2\}) + \Delta(n-6 \mid \{1\}) + \Delta(n-6 \mid \{2\}) + \Delta(n-3 \mid \{2\}) + 1 \\ &= \Delta(n-6 \mid \{1,2\}) + \Delta(n-6 \mid \{2\}) + \Delta(n-3 \mid \{2\}) + 2 \\ &= \dots \\ &= \Delta(n-3k \mid \{1,2\}) + \sum_{i=1}^{\lfloor \frac{n-1}{3} \rfloor} \Delta(n-3i \mid \{2\}) + k \\ &= 0 + \sum_{i=1}^{\lfloor \frac{n-1}{3} \rfloor} \Delta(n-3i \mid \{2\}) + \lfloor \frac{n-1}{3} \rfloor.\end{aligned}$$

If $n = 3k$ then $k - i$ is even and so

$$\sum_{i=1}^{\lfloor \frac{n-1}{3} \rfloor} \Delta(n-3i \mid \{2\}) + \lfloor \frac{n-1}{3} \rfloor = \lfloor \frac{n-1}{6} \rfloor + \lfloor \frac{n-1}{3} \rfloor.$$

Similarly, we have the result for the cases $n = 3k + 1$ and $n = 3k + 2$. □

3. The twelvefold way

The twelvefold way gives the number of mappings f from the set N of n objects to set K of k objects (putting balls from the set N into boxes in the set K). Richard Stanley invented the twelvefold way [11]. Consider n (un)labeled balls and k (un)labeled cells. There are four cases, $\mathbf{U} \to \mathbf{L}, \mathbf{L} \to \mathbf{U}, \mathbf{L} \to \mathbf{L}, \mathbf{U} \to \mathbf{U}$, for arrangements of **L**abeled or **U**nlabeled balls $\xrightarrow{in}$ **L**abeled or **U**nLabeled boxes. Here Labeled means distinguishable and Unlabeled means indistinguishable. If we want to partition these balls into these cells we are faced with the following twelve problems (see Table). In the Table, $(k)_n := k(k-1)\cdots(k-n+1)$ is *Pochhammer's*

Table. The twelvefold way.

Elements of N	Elements of K	f unrestricted	f one-to-one	f onto
L	**L**	k^n	$(k-n+1)_n$	$k!\left\{ {n \atop k} \right\}$
U	**L**	$\binom{n+k-1}{n}$	$\binom{k}{n}$	$\binom{n-1}{n-k}$
L	**U**	$\sum_{i=1}^{k} \left\{ {n \atop i} \right\}$	$\delta_{k\leqslant n}$	$\left\{ {n \atop k} \right\}$
U	**U**	$\sum_{i=1}^{i} p_i(n)$	$\delta_{k\leqslant n}$	$p_k(n)$

symbol or *falling factorial*, for $k, n \in \mathbb{N}$, $\left\{ {n \atop k} \right\}$ denotes the *Stirling number of the second kind* or the number of partitions of the set $\{1, 2, \ldots, n\}$ into exactly k nonempty subsets, which is equal to $\sum_{i=1}^{k}(-1)^i\binom{k}{i}(k-i)^n$, and the number $\left\{ {n \atop k} \right\}$ satisfies the recursive relation $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}$ and $\delta_{k\leqslant n} := \begin{cases} 1 & \text{when } n \leqslant k, \\ 0 & \text{when } n > k. \end{cases}$ Now we consider a new problem as an extension and unification of the above problems. Consider $b_1+b_2+\cdots+b_n$ balls with b_1 balls Labeled 1, b_2 balls Labeled 2, and so on, $c_1+c_2+\cdots+c_k$ cells with c_1 cells Labeled 1, c_2 cells Labeled 2, and so on. We denote the situation of these balls and cells by the two multisets $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ of balls and $\mathcal{C} = \{c_1, \ldots, c_k\}$ of cells. Let the number of mappings $\mathcal{F}$ from the multiset $\mathcal{B}$ of balls to the multiset $\mathcal{C}$ of cells be called *the mixed twelvefold way (or dually, the number of ways to partition the multiset $\mathcal{B}$ of balls into the multiset $\mathcal{C}$ of cells).* We denote the number of *unrestricted* mappings of $\mathcal{F}$ by $\Gamma_0(\mathcal{B}|\mathcal{C})$. Also, we denote the number of *onto* mappings of $\mathcal{F}$, that is, *the number of ways to partition the multiset $\mathcal{B}$ of balls into the multiset $\mathcal{C}$ of cells, such that the cells are nonempty* by $\Gamma(\mathcal{B}|\mathcal{C})$.

Theorem 3.1 *Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ and $\mathcal{C} = \{c_1, \ldots, c_k\}$ be two multisets whose members are positive integers. The number of unrestricted mappings $\mathcal{F}$ from the multiset $\mathcal{B}$ to $\mathcal{C}$ is given by*

$$\Gamma_0(\mathcal{B}|\mathcal{C}) = \sum_{\substack{b_1=n_1+\cdots+n_k \\ 0\leqslant n_i\leqslant b_1}} \sum_{\substack{(C_1,\cdots,C_k) \\ \theta(C_i)\leqslant c_i}} \Big(\prod_{j=1}^{k}\Delta(n_j|C_j)\Big)\Gamma_0\big(\mathcal{B}\setminus\{b_1\}|(\bigcup_{i=1}^{k}C_i)\cup(\bigcup_{\substack{i=1 \\ \theta(C_i)<c_i}}^{k}\{c_i-\theta(C_i)\}\big),$$

where $\Gamma_0(\emptyset, A) = 1$ for each multiset A of nonnegative integers.

Proof First we distribute the b_1 balls Labeled 1 into cells. Let n_i be the number of balls in cells Labeled i for $i = 1, \ldots, k$. Thus, we can write $b_1 = n_1 + \cdots + n_k$. When we put n_i balls in cell Labeled i, the c_i cells Labeled i are partitioned into different types. Suppose that we have ℓ_{ij} cells Labeled i with x_{ij} balls Labeled 1. Here, $c_i = \ell_{i1}x_{i1} + \cdots + \ell_{it}x_{it} + r_i$, where r_i is the number of cells Labeled i that are still empty. Let $C_i = \{\ell_{i1}, \ldots, \ell_{it}\}$. Thus, $\theta(C_i) \leqslant c_i$ and there are $\Delta(n_i|C_i)$ situations in which the types of the c_i cells Labeled i change into ℓ_{i1} cells of the first type, say Labeled ℓ_{i1}, $\ldots$, ℓ_{it} cells of the tth type, say Labeled it, and r_i empty cells the $t+1$st type, say Labeled $i(t+1)$. We can therefore say that after distributing the b_1 balls Labeled 1 into cells we have the multiset $\mathcal{B}\setminus\{b_1\}$ of balls and the multiset

$$(\bigcup_{i=1}^{k}C_i)\cup(\bigcup_{\substack{i=1 \\ \theta(C_i)<c_i}}^{k}\{c_i-\theta(C_i)\}),$$

of cells. The number of ways putting of these balls into these cells is

$$\Gamma_0\big(\mathcal{B}\setminus\{b_1\}|(\bigcup_{i=1}^{k} C_i)\cup(\bigcup_{\substack{i=1\\ \theta(C_i)<c_i}}^{k}\{c_i-\theta(C_i)\}\big),$$

which completes the proof. □

Theorem 3.2 *Let $\mathcal{B}=\{b_1,\ldots,b_n\}$ and $\mathcal{C}=\{c_1,\ldots,c_k\}$ be two mulitsets whose members are positive integers. The number of onto mappings of $\mathcal{F}$ from the multiset $\mathcal{B}$ to $\mathcal{C}$ is given by*

$$\Gamma(\mathcal{B}|\mathcal{C})=\sum_{\ell=0}^{k}\sum_{1\leqslant i_1<\cdots<i_\ell\leqslant k}(-1)^{\ell}\Gamma_0\big(\mathcal{B}|\bigcup_{i=1}^{\ell}\big((\mathcal{C}\setminus\{c_{i_j}\})\cup(\{c_{i_j}-1\})\big)\big).$$

Proof Let $\mathcal{E}_i$ be the set of situations in which some of the cells Labeled i are empty. Then the number of the elements of $\mathcal{E}_{i_1}\cap\cdots\cap\mathcal{E}_{i_\ell}$ is $\Gamma_0\big(\mathcal{B}|\bigcup_{i=1}^{\ell}\big((\mathcal{C}\setminus\{c_{i_j}\})\cup(\{c_{i_j}-1\})\big)\big)$. Now the inclusion exclusion principle implies the result. □

Let n and k be positive integers. Consider $\mathcal{B}=\{1,2,\ldots,n\}$, the set of n Unlabeled balls, and $\mathcal{C}=\{1,2,\ldots,k\}$, the set of k Unlabeled cells. Also, let $\mathcal{I}_k=\{1,1,\ldots,1\}$ be a multiset with multiplicity mapping m, such that $\theta(1)=k$. Then we conclude the following result about the number of unrestricted or onto mappings of $\mathcal{F}$, from the set $\mathcal{B}$ or $\mathcal{I}_k$ to the set $\mathcal{C}$ or $\mathcal{I}_k$. Then:

i) $\Gamma(\mathcal{B}|\mathcal{C})=p_k(n)$ and $\Gamma_0(\mathcal{B}|\mathcal{C})=p_k(n+k)$.

ii) $\Gamma(\mathcal{B}|\mathcal{I}_k)=\binom{n-1}{k-1}$ and $\Gamma_0(\mathcal{B}|\mathcal{I}_k)=\binom{n+k-1}{k-1}$.

iii) $\Gamma(\mathcal{I}_n|\mathcal{C})=\left\{{n \atop k}\right\}$ and $\Gamma_0(\mathcal{I}_n|\mathcal{C})=\sum_{i=1}^{k}\left\{{n \atop i}\right\}$.

iv) $\Gamma(\mathcal{I}_n|\mathcal{I}_k)=k!\left\{{n \atop k}\right\}$ and $\Gamma_0(\mathcal{I}_n|\mathcal{I}_k)=k^n$.

Corollary 3.3 *Let n and k be positive integers. Then $p_k(n)=\sum_{\theta(\mathcal{C})=k}\Delta(n|\mathcal{C})$, where the summation is taken over all multisets $\mathcal{C}$ whose members are positive integers.*

Proof Using Theorems 3.1 and 3.2, we can write

$$\begin{aligned}
p_k(n) &= \Gamma(\{1,2,\ldots,n\}|\{1,2,\ldots,k\})\\
&= \Gamma_0(\{1,2,\ldots,n\}|\{1,2,\ldots,k\})-\Gamma_0(\{1,2,\ldots,n\}|\{1,2,\ldots,k-1\})\\
&= \sum_{\theta(\mathcal{C})\leqslant k}\Delta(n|\mathcal{C})-\sum_{\theta(\mathcal{C})\leqslant k-1}\Delta(n|\mathcal{C})=\sum_{\theta(\mathcal{C})=k}\Delta(n|\mathcal{C}),
\end{aligned}$$

as claimed. □

4. The nonintersecting circles problem

To solve the nonintersecting circles problem, let us assume the following notations. Let n be a positive integer. We denote the set of all multisets $\mathbb{A} = \{a_1, \ldots, a_k\}$ such that there are distinct positive integers $x_1, \ldots, x_k$ with $n = a_1x_1 + \cdots + a_kx_k$, where $x_i < x_{i+1}$ whenever $a_i = a_{i+1}$, by $\mathcal{A}_{n,k}$. Recall that for an $\mathbb{A} \in \mathcal{A}_{n,k}$ there are $\Delta(n|\mathbb{A})$ solutions $(x_1, \ldots, x_k)$ satisfying the above condition. We denote the set of these $(x_1, \ldots, x_k)$ by $\mathcal{X}_{\mathbb{A}}$.

Note that the number of $(n_1, \ldots, n_r)$ with $1 \leqslant n_1 \leqslant \cdots \leqslant n_r \leqslant s$ is given by

$$\sum_{i=1}^{k} \binom{r-1}{i-1}\binom{s}{i} = \sum_{i=1}^{k} \binom{r-1}{r-i}\binom{s}{i} = \binom{r+s-1}{r}. \tag{4.1}$$

The nonintersecting circles problem asks to evaluate the number of ways to draw n nonintersecting circles in a plane regardless of their sizes. For example, if we use the symbol ( ) for a circle then there are four such ways for 3 circles ( )( )( ), (( )( )), (( ))( ), ((( ))) and nine ways for 4 circles,

( )( )( )( ), (( )( )( )), (( )( ))( ), ((( )( ))), (( ))( )( ), ((( ))( )), ((( )))( ), (((( )))), (( ))(( )).

If we denote this number by B_n then we can see that $B_0 = B_1 = 1, B_2 = 2, B_3 = 4, B_4 = 9$, $B_5 = 20$, and so on.

Theorem 4.1 *Let B_n be the number of ways to draw n nonintersecting circles in a plane regardless of their sizes. Then*

$$B_n = \sum_{k=1}^{\lfloor\sqrt{2n}\rfloor} \sum_{\mathbb{A}=\{a_1,\cdots,a_k\}\in\mathcal{A}_{n,k}} \sum_{(x_1,\cdots,x_k)\in\mathcal{X}_A} \prod_{i=1}^{k} \binom{B_{x_i-1}+a_i-1}{a_i}.$$

Proof Given n, let us draw our circles in ℓ parts with y_i circles in the ith part. We can assume that $y_1 \leqslant \cdots \leqslant y_\ell$. Thus, $n = y_1 + \cdots + y_\ell$. We can rewrite it in the form $n = a_1x_1 + \cdots + a_kx_k$ such that $x_i < x_{i+1}$ whenever $a_i = a_{i+1}$. This shows that we have a_i parts with x_i circles of the form $(x_i - 1)$ where ( ) denotes a circle containing $x_i - 1$ circles. We can form the a_i parts of the form $(x_i - 1)$ in $\binom{B_{x_i-1}+a_i-1}{a_i}$ ways. The latter is true since we may put $r = a_i$ and $s = B_{x_i-1}$ in 4.1. Note that a single form $(x_i - 1)$ can be drawn in B_{x_i-1} ways.

Now notice the fact that the maximum of k occurs when $a_1 = \cdots = a_k = 1$. Since we have $1 \leqslant x_1 < \cdots < x_k$ in this case, we can therefore deduce that $\frac{k(k+1)}{2} \leqslant n$. Thus, the maximum value of k is $\lfloor\sqrt{2n}\rfloor$. □

Example 4.2 *For $n = 6$ we have*

$$\begin{aligned} \mathcal{A}_{6,1} &= \{\{\mathbf{1}\}, \{\mathbf{2}\}, \{\mathbf{3}\}, \{\mathbf{6}\}\}, \\ \mathcal{A}_{6,2} &= \{\{\mathbf{1},\mathbf{1}\}, \{\mathbf{1},\mathbf{2}\}, \{\mathbf{1},\mathbf{3}\}, \{\mathbf{1},\mathbf{4}\}, \{\mathbf{2},\mathbf{2}\}\}, \\ \mathcal{A}_{6,3} &= \{\{\mathbf{1},\mathbf{1},\mathbf{1}\}\}. \end{aligned}$$

We can therefore write

$$\begin{aligned} 6 &= \mathbf{1}\times 6 = \mathbf{2}\times 3 = \mathbf{3}\times 2 = \mathbf{6}\times 1 = \mathbf{1}\times 1 + \mathbf{1}\times 5 = \mathbf{1}\times 2 + \mathbf{1}\times 4 \\ &= \mathbf{1}\times 4 + \mathbf{2}\times 1 = \mathbf{1}\times 3 + \mathbf{3}\times 2 = \mathbf{1}\times 2 + \mathbf{4}\times 1 = \mathbf{2}\times 1 + \mathbf{2}\times 2 \\ &= \mathbf{1}\times 1 + \mathbf{1}\times 2 + \mathbf{1}\times 3. \end{aligned}$$

Thus,

$$\begin{aligned} B_6 &= \binom{B_5}{1} + \binom{B_2+1}{2} + \binom{B_1+2}{3} + \binom{B_0+5}{6} \\ &+ \binom{B_0}{1}\binom{B_4}{1} + \binom{B_1}{1}\binom{B_3}{1} + \binom{B_3}{1}\binom{B_0+1}{2} + \binom{B_2}{1}\binom{B_1+2}{3} \\ &+ \binom{B_1}{1}\binom{B_0+3}{4} + \binom{B_0+1}{2}\binom{B_1+1}{2} + \binom{B_0}{1}\binom{B_1}{1}\binom{B_2}{1} \\ &= 20+3+1+1+9+4+4+2+1+1+2 = 48. \end{aligned}$$

The number of ways to draw 6 *nonintersecting circles in a plane regardless of their sizes is thus equal to* 48*.*

A rooted tree may be defined as a free tree in which some vertex has been distinguished as the *root*. We can see some values of a rooted tree for positive integer n in [10].

Corollary 4.3 *Let* n *be a positive integer. Then* B_n *is the number of unlabeled rooted tree with* $n+1$ *vertices.*

Proof There is a one-to-one correspondence between n nonintersecting circles and an unlabeled rooted tree with $n+1$ vertices. It is enough to draw a circle for each nonroot vertex and put a circle inside another one if the second one is the parent of the first one. □

5. Ordered and unordered factorizations of natural numbers

An *ordered factorization* of a positive integer n is a representation of n as an ordered product of integers, each factor greater than 1. For positive integer $\ell, k \geqslant 1$ we denote the number of the ordered factorizations of positive integer n into exactly k factors, such that each factor $\geqslant \ell$ by $\mathcal{H}(n;k,\ell)$. We use $\mathcal{H}(n)$ to represent the number of all ordered factorizations of the positive integer n *(in analogy with compositions for sum)*. For example, $\mathcal{H}(12) = 8$, since we have the factorizations $12, 2\times 6, 6\times 2, 3\times 4, 4\times 3, 2\times 2\times 3, 2\times 3\times 2$, and $3\times 2\times 2$. By the definition, $\mathcal{H}(1) = 0$, but in some situations it is useful to set $\mathcal{H}(1) = 1$ or $\mathcal{H}(1) = \frac{1}{2}$ [5].
Every integer $n > 1$ has a canonical factorization into distinct prime numbers $p_1, p_2, \ldots, p_r$, namely

$$n = p^{\alpha_1} p_2^{\alpha} \ldots p^{\alpha_r}; \qquad 1 < p_1 < p_2 < \ldots < p_r. \tag{5.1}$$

Many problems involving factorisatio numerorum depend only on the set of exponents in 5.1, $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$. MacMahon [7] developed the theory of compositions of *multipartite numbers* from this perspective and indeed considered these problems throughout his career [8], but Andrews suggested the more modern terminology *vector compositions* [2]. A general formula for $\mathcal{H}(n,k,2)$ of ordered factorizations of positive integer n such that each factor is larger than 2 was given by MacMahon in [7]. Now we give another proof for $\mathcal{H}(n,k,2)$ and $\mathcal{H}(n,k,1)$ with the above results.

Theorem 5.1 *Let* $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ *be a positive integer. Then the number of ordered factorizations of* n *into* k *factors such that each factor* $\geqslant 1$ *and* $\alpha_1 + \ldots + \alpha_n \geqslant k \geqslant 1$ *is given by*

$$\mathcal{H}(n,k,1) = \sum_{i=1}^{\alpha_1+\ldots+\alpha_r} \Gamma_0(\{\alpha_1,\ldots,\alpha_r\}, I_i) = \sum_{i=1}^{\alpha_1+\ldots+\alpha_r} \prod_{j=1}^{n} \binom{\alpha_j + i - 1}{i-1}. \tag{5.2}$$

Also, the number of unordered factorizations of n *into* k *factors such that each factor* $\geqslant 2$ *is given by*

$$\begin{aligned}\mathcal{H}(n,k,2) &= \sum_{i=1}^{\alpha_1+\ldots+\alpha_r} \Gamma(\{\alpha_1,\ldots,\alpha_r\},I_i)\\ &= \sum_{i=1}^{\alpha_1+\ldots+\alpha_r}\sum_{\ell=0}^{i}(-1)^\ell\binom{i}{\ell}\prod_{j=1}^{n}\binom{\alpha_j+i-\ell-1}{i-\ell-1}. \end{aligned} \tag{5.3}$$

Proof It is sufficient to use Theorems 3.2 and 3.1. Suppose that for $1\leqslant j\leqslant n$ we have α_j balls labeled p_j and we want to put these balls into k different cells. There is a one-to-one correspondence between these situations and unordered factorizations of positive integer n as the form $n=n_1\times n_2\times\ldots\times n_k$ such that each factor $\geqslant 1$. In fact, we can consider n_j as the product of the balls in cell j. There are $\binom{\alpha_j+k-1}{k-1}$ ways to put balls labeled p_j. Thus, the first part is obvious.

For the second part, let E_r be the set of all situations in which cell r is empty, where $1\leqslant r\leqslant k$. Then we have

$$|E_{r_1}\cap\ldots\cap E_{r_i}|=\prod_{j=1}^{n}\binom{\alpha_j+k-i-1}{k-i-1},\quad 1\leqslant i\leqslant k-1.$$

Thus, the principle of inclusion and exclusion implies the result. □

Let $\mathcal{F}(n;k,\ell)$ denote the number of unordered factorizations of a positive integer n into exactly k factors, such that every factor $\geqslant \ell$. This means that the number of ways can be written as positive integer n as the product $n=n_1\times n_2\times\ldots\times n_k$, where $n_1\geqslant n_2\geqslant\ldots\geqslant n_k\geqslant\ell$. We call $\mathcal{F}(n)$ *the unordered Factorization function of* n (in analogy with partitions function $p(n)$ for sum). For example, $\mathcal{F}(12)$ corresponds to $2\times 6, 2\times 2\times 3, 3\times 4$, and 12. The sequence $\mathcal{F}(n)$ is listed in [10].

Now, by using Theorems 3.2 and 3.1, we conclude the following proposition.

Proposition 5.2 *Let* $n=p_1^{\alpha_1}\ldots p_r^{\alpha_r}$ *be a positive integer. Then the number of unordered factorizations of* n *into* k *factors such that each factor* $\geqslant 1$ *and* $\alpha_1+\ldots+\alpha_n\geqslant k\geqslant 1$ *is given by*

$$\mathcal{F}(n,k,1)=\sum_{i=1}^{\alpha_1+\ldots+\alpha_n}\Gamma_0(\{\alpha_1,\ldots,\alpha_n\},\{i\}).$$

Also, the number of unordered factorizations of n *into* k *factors such that each factor is greater* 1 *is given by*

$$\mathcal{F}(n,k,2)=\sum_{i=1}^{\alpha_1+\ldots+\alpha_n}\Gamma(\{\alpha_1,\ldots,\alpha_n\},\{i\}).$$

6. Generating function of $D(n|A)$

In this section, by using the generating function, we obtain the values of $D(n|A)$ for a special multiset.

Theorem 6.1 *Let* n *be a nonnegative integer and* $\mathbb{A}=\{a_1,\ldots,a_k\}$ *be a multiset with the multiplicity mapping* θ *and the background set* $S(\mathbb{A})=\{b_1,\ldots,b_\ell\}$*, where* $\theta(b_i)=m_i$*. The generation function of* $D(n|A)$ *is given* *by*

$$\sum_{n=0}^{\infty} D(n|A)x^n = \prod_{i=1}^{\ell}\prod_{j=1}^{m_i} \frac{x^{b_i}}{1-x^{b_i(m_i-j+1)}}. \tag{6.1}$$

Proof For each $1 \le i \le \ell$, we want a monotonically nondecreasing sequence $n_{i,1} \le n_{i,2} \le \cdots \le n_{i,m_i}$. For $2 \le j \le m_i$, we make the change of variables as follows: $d_{i,1} = n_{i,1}$ and $d_{i,j} = n_{i,j} - n_{i,j-1}$ for $j = 2, 3, \ldots, m_i$. Then the monotonically nondecreasing condition on the $(n_{i,j})_j$ becomes $d_{i,1} \ge 1$ and $d_{i,j} \ge 0$ for $1 \le j \le m_i$. Observe that

$$\begin{aligned}\sum_{j=1}^{m_i} n_{i,j} &= (d_{i,1}) + (d_{i,1} + d_{i,2}) + \cdots + (d_{i,1} + d_{i,2} + \cdots + d_{i,m_i}) \\ &= \sum_{j=1}^{m_i} (m_i - j + 1) d_{i,j}.\end{aligned}$$

Then $D(n|A)$ is the number of ways of choosing all these $d_{i,j}$ such that

$$\begin{aligned} n = \sum_{i=1}^{\ell} b_i \sum_{j=1}^{m_i} n_{i,j} &= \sum_{i\in I} b_i \sum_{j=1}^{m_i} (m_i - j + 1) d_{i,j} \\ &= \sum_{i=1}^{\ell}\left(b_i m_i d_{i,1} + \sum_{j=2}^{m_i} b_i (m_i - j + 1) d_{i,j}\right),\end{aligned}$$

where $d_{i,1} \ge 1$ $(1 \le i \le \ell)$ and $d_{i,j} \ge 0$ $(1 \le i \le \ell,\ 2 \le j \le m_i)$. Thus, the generating function for $D(n|A)$ is

$$\prod_{i=1}^{\ell}\left(\frac{x^{b_i m_i}}{1-x^{b_i m_i}} \prod_{j=2}^{m_i} \frac{1}{1-x^{b_i(m_i-j+1)}}\right),$$

as required. □

By (2.1), we have the following corollary.

Corollary 6.2 *Let n be a positive integer and $A = \{1, 1, \cdots, 1\}$ be a multiset for which $\theta(1) = \ell$. Then*

$$\sum_{n=0}^{\infty} D(n|A)x^n = \frac{x^{\ell}}{(x;x)_{\ell}}.$$

Proof We can rewrite the generating function of $D(n|A)$ more simply:

$$\begin{aligned}\prod_{i\in I}\left(\frac{x^{b_i m_i}}{1-x^{b_i m_i}} \prod_{j=2}^{m_i} \frac{1}{1-x^{b_i(m_i-j+1)}}\right) &= \prod_{i\in I} \frac{x^{b_i m_i}}{1-x^{b_i m_i}} \frac{1}{\prod_{j=2}^{m_i} 1-x^{b_i(m_i-j+1)}} \\ &= \prod_{i\in I} \frac{x^{b_i m_i}}{\prod_{j=1}^{m_i} 1-x^{b_i(m_i-j+1)}} \\ &= \prod_{i\in I}\prod_{j=1}^{m_i} \frac{x^{b_i}}{1-x^{b_i(m_i-j+1)}}.\end{aligned}$$

Consider multiset $A = \{1, 1, \cdots, 1\}$ such that $\theta(1) = \ell$. Put $m_i = \ell$ and $b_i = 1$, and then

$$\sum_{n=0}^{\infty} D(n|A)x^n = \frac{x}{1-x^{\ell}} \cdot \frac{x}{1-x^{\ell-1}} \cdot \cdots \cdot \frac{x}{1-x},$$

as claimed. □

Now we obtain another generating function for $D(n|A)$ by using hypergeometric series.

Let n be a nonnegative integer and $A = \{a_1, a_2, \ldots, a_k\}$ be a multiset. Let $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$ be a positive solution of the system $n = a_1 n_1 + \ldots + a_k n_k$, such that $n_i = n_{i-1} + s_i$ where s_i is nonnegative integers for $1 \leq i \leq k$. For $|q| < 1$, we can write

$$\begin{aligned}
\sum_{n=0}^{\infty} D(n, A)q^n &= \sum_{1 \leqslant n_1 \leqslant \ldots \leqslant n_k} q^{a_1 n_1 + \ldots + a_k n_k} \\
&= \sum_{1 \leqslant n_1 \leqslant \ldots \leqslant n_k} (q^{a_1})^{n_1} (q^{a_2})^{n_2} \ldots (q^{a_k})^{n_k} \\
&= \sum_{1 \leqslant n_1 \leqslant \ldots \leqslant n_{k-1}} (q^{a_1})^{n_1} (q^{a_2})^{n_2} \ldots (q^{a_{k-1}})^{n_{k-1}} (q^{a_k})^{n_{k-1}+s_k} \\
&= \sum_{1 \leqslant n_1 \leqslant \ldots \leqslant n_{k-1}} (q^{a_1})^{n_1} (q^{a_2})^{n_2} \ldots (q^{a_{k-1}+a_k})^{n_{k-1}} \frac{q^{a_k}}{1-q^{a_k}} \\
&= \sum_{1 \leqslant x_1 \leqslant \ldots \leqslant x_{k-2}} (q^{a_1})^{n_1} (q^{a_2})^{n_2} \ldots (q^{a_{k-2}+a_{k-1}+a_k})^{n_{k-2}} \frac{q^{a_k+a_{k-1}}}{(1-q^{a_k+a_{k-1}})(1-q^{a_k})} \\
&= \ldots \\
&= \frac{q^{\ell}}{(1-q^{a_1+a_2+\ldots+a_k})(1-q^{a_2+\ldots+a_k})\ldots(1-q^{a_{k-1}+a_k})(1-q^{a_k})}.
\end{aligned}$$

Corollary 6.3 *Let n be a nonnegative integer and $A = \{1, 2, 2, \cdots, 2\}$ be a multiset with $\theta(A) = 2\ell + 1$. The generation function of $D(n|A)$ is given by $\sum_{n=0}^{\infty} D(n|A)x^n = \frac{x}{1-x}E_{\ell}(n)$, where $E_{\ell}(n)$ is the number of partitions of positive integer n with even parts to at most ℓ parts.*

Corollary 6.4 *Let n be a positive integer and $A = \{1, 1, \cdots, 1, 2, 2, \cdots, 2\}$ be a multiset with ℓ-times one and d times two. Then $\sum_{n=0}^{\infty} D(n|A)x^n = p_{\ell}(n)E_d(n)$.*

Example 6.5 *The generating functions for multisets $\{1, 1, 2\}$, $\{1, 3, 3\}$, and $\{1, 2, 3\}$ are*

$$\begin{aligned}
\sum_{n=0}^{\infty} D(n|\{1, 1, 2\}))x^n &= x^2 + x^3 + 2x^4 + 2x^5 + 3x^6 + 3x^7 + 4x^8 + 4x^9 + \cdots, \\
\sum_{n=0}^{\infty} D(n|\{1, 3, 3\}))x^n &= x^7 + x^8 + x^9 + 2x^{10} + 2x^{11} + 2x^{12} + 3x^{13} + 3x^{14} + 3x^{15} \\
&+ 4x^{16} + 4x^{17} + 4x^{18} + 5x^{19} + 5x^{20} + 5x^{21} + 6x^{22} + \cdots, \\
\sum_{n=0}^{\infty} D(n|\{1, 2, 3\}))x^n &= x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + 8x^{13} + \cdots.
\end{aligned}$$

Let n be a positive integer and $A=\{a_1,a_2,\cdots,a_k\}$ be a multiset. We denote the number of partitions of n as $n=a_1n_1+a_2n_2+\ldots+a_kn_k$, for which n_i are odd by $Do(n|A)$.

Theorem 6.6 *Let n be a positive integer and $A=\{a_1,a_2,\cdots,a_k\}$ be multiset. Then*

$$Do(2n|A)=\sum_{\substack{0\leq\theta(A)\leq n\\ \theta(A) is even}} D(2n-\theta(A)|A)D(\theta(A)|A),$$

where $\theta(A)=\sum_{i=1}^{k}a_i$.

Proof Let n be positive integer. We have $2n=a_1n_1+a_2n_2+\ldots+a_kn_k$, where $n_i=2r_i+1$ are odd. We can write the following:

$$\begin{aligned}2n&=a_1(2r_1+1)+a_2(2r_2+1)+\ldots+a_k(2r_k+1)\\&=2r_1a_1+2r_2a_2+\ldots+2r_ka_k+a_1+a_2+\ldots+a_k.\end{aligned}$$

Since $2n$ is even, put $a_1+a_2+\ldots+a_k=\theta(A)$, where $\theta(A)$ is even. Then the number of natural partitions of $2n$ to odd parts is equal to the number of natural partitions of $\theta(A)$ and the number of natural partitions of $n-\theta(A)$. □

7. Relatively prime $D(n|A)$

Definition 7.1 *Let n be a positive integer and $A=\{a_1,a_2,\ldots,a_k\}$ be a multiset. We say that $D(n,A)$ is relatively prime if its parts form a relatively prime set; that is, if we partition n as $n=a_1n_1+a_2n_2+\ldots+a_kn_k$ then $(n_1,n_2,\cdots,n_k)=1$. We denote the number of such partitions of n with $D^r(n,A)$.*

Example 7.2 *We evaluate the relatively prime natural number of $n=11$ with respect to multiset $\{1,1,2\}$ and we have*

$$\begin{aligned}&11=1\times\mathbf{1}+1\times\mathbf{2}+2\times\mathbf{4}, && 11=1\times\mathbf{1}+1\times\mathbf{4}+2\times\mathbf{3}\\&11=1\times\mathbf{1}+1\times\mathbf{6}+2\times\mathbf{2}, && 11=1\times\mathbf{1}+1\times\mathbf{8}+2\times\mathbf{1}\\&11=1\times\mathbf{2}+1\times\mathbf{3}+2\times\mathbf{3}, && 11=1\times\mathbf{2}+1\times\mathbf{5}+2\times\mathbf{2}\\&11=1\times\mathbf{2}+1\times\mathbf{7}+2\times\mathbf{1}, && 11=1\times\mathbf{3}+1\times\mathbf{4}+2\times\mathbf{2}\\&11=1\times\mathbf{3}+1\times\mathbf{6}+2\times\mathbf{1}, && 11=1\times\mathbf{1}+1\times\mathbf{6}+2\times\mathbf{2}\\&11=1\times\mathbf{4}+1\times\mathbf{5}+2\times\mathbf{1}.\end{aligned}$$

Then $D^r(11,\{1,1,2\})=10$.

Lemma 7.3 *Let n be a positive integer and $A=\{a_1,a_2\}$. If $a_1=a_2=a$ then $D_0(n,\{a_1,a_2\})=\lfloor\frac{n}{2a}\rfloor+1$, and if $a_1\neq a_2$ then $D_0(n,\{a_1,a_2\})=\lfloor\frac{n+a_1+a_2-1}{a_1a_2}\rfloor$.*

Theorem 7.4 *Let n be a nonnegative integer. For multiset $A=\{a_1,a_2,\ldots,a_k\}$, we have*

$$D^r(n,A)=\sum_{d|n}\mu(d)D\left(\frac{n}{d},A\right), \tag{7.1}$$

where $\mu(d)$ is the Möbius function.

Proof For nonnegative integers n, k, we have $D(n, A) = \sum_{d|n} D^r\left(\frac{n}{d}, A\right)$, and by the *Möbius inversion formula* we have that $D^r(n, A) = \sum_{d|n} \mu(d) D\left(\frac{n}{d}, A\right)$, as required. □

Corollary 7.5 *Let n be a nonnegative integer and $A = \{a_1, a_2\}$. If $a_1 = a_2 = a$, then*

$$D_0^r(n, \{a_1, a_2\}) = \frac{1}{2a} \lfloor \varphi(n) \rfloor,$$

where $\varphi(n)$ is the Euler totient function.

Proof Let n, k be nonnegative integers and $p_1^{\alpha_1} \dots p_k^{\alpha_k}$ be the prime decomposition of n. By Lemma 7.3 and Theorem 7.4, we have $D^r(n, A) = \sum_{d|n} \mu(d) \left(\lfloor \frac{n}{2ad} \rfloor + 1 \right)$. If $2ad|n$ then $\lfloor \frac{n}{2ad} \rfloor$ is an integer and recall that $\sum_{d|n} \varphi(n) = n$ and $\sum_{d|n} \mu(d) = \lfloor \frac{1}{n} \rfloor$. By the Möbius inversion we have

$$\sum_{d|n} \mu(d) \lfloor \frac{n}{2ad} \rfloor = \frac{1}{2a} \varphi(n).$$

Now, if $2ad \nmid n$, we have

$$\begin{aligned}\sum_{d|n} \mu(d) \lfloor \frac{n}{2ad} \rfloor &= \sum_{d|n} \mu(d) \left(\frac{n}{2ad} - \frac{1}{2a}\right) = \sum_{d|n} \mu(d) \left(\frac{n}{2ad}\right) - \sum_{d|n} \mu(d) \left(\frac{1}{2a}\right) \\ &= \frac{1}{2a} \varphi(n) - \frac{1}{2a} \sum_{d|n} \mu(d) = \frac{1}{2a} \varphi(n),\end{aligned}$$

as claimed. □

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