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Some general results on fractional Banach sets

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Abstract: The gamma function which is expressed by an improper integral is used to establish the fractional difference operators and fractional Banach sets. In this study, we achieve some comprehensive and complementary results related to characterizations of the matrix classes of fractional Banach sets. We also obtain some identities or inequalities for the Hausdorff measure of noncompactness of the corresponding matrix operators, and finally find the necessary and sufficient conditions for those matrix operators to be compact.

Key words: Hausdorff measure of noncompactness, gamma function, fractional operator, compact operator

1. Introduction

Measure of noncompactness is a very useful concept in functional analysis and applied mathematics, for instance, in the fixed point theory studies, ordinary, partial, and fractional differential and integral equations, and characterizations of compact operators between Banach spaces. Hausdorff measure of noncompactness is established to study modulus of noncompact convexity which is important in the geometry of Banach and Hilbert spaces.

Difference sets of sequences $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ were introduced by Kızmaz [13], as the domain of forward difference matrix Δ in the spaces ℓ_∞, c and c_0 of bounded, convergent, and null sequences, respectively. The difference sequence spaces bv_p were investigated as the domain of backward difference matrix $\Delta^{(1)}$ in the space ℓ_p of absolutely p-summable sequences for the case $1 \leq p \leq \infty$ by Başar and Altay [6] and for the case $0 < p < 1$ by Altay and Başar [1], respectively. The idea of constructing new difference sequence spaces has been developed by many researchers based on some newly defined infinite matrices in [2, 14–16, 27, 28]. We refer to the textbook [5] for a comprehensive study about summability theory, and to the papers [1, 6, 7, 11, 20, 21] for Hausdorff measure of noncompactness for matrix operators, difference sequence spaces, matrix transformations, and related topics. Some remarkable and important results about visualization and animations for the topologies of certain sequence spaces were illustrated in the papers [18, 19, 31]. The authors achieved those results applying their software package. Those results have important and interesting applications in crystallography.

Fractional difference sequence spaces have recently been introduced and studied in the literature [3, 4]. Fractional operators, their properties, and certain fractional sequence spaces have appeared in those papers. Necessary and sufficient conditions for the classes of compact matrix operators in $(\ell_p(\Delta^{(\tilde{\alpha})}), \ell_\infty)$ for $1 \leq p \leq \infty$ were obtained in [26]. In addition to these characterizations, Hausdorff measure of noncompactness was applied

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2010 *AMS Mathematics Subject Classification:* 46B45, 47B37 to establish the necessary and sufficient conditions for a matrix operator to be a compact operator in the classes (X,Y), where $X = \ell_p(\Delta^{(\widetilde{\alpha})})$ $(1 \leq p < \infty)$ and Y is any of the spaces $c_0, c,$ and ℓ_1. The necessary and sufficient compactness conditions for a matrix operator from fractional sets of sequences $c_0(\Delta^{(\alpha)})$, $c(\Delta^{(\widetilde{\alpha})})$, and $\ell_\infty(\Delta^{(\widetilde{\alpha})})$ to the classical sets of sequences have been very recently determined in [25]. Fractional Banach set of difference sequences $\ell(\Delta^{(\widetilde{\alpha})}, p)$ was geometrically characterized and its modular structure was investigated in [24]. In addition to uniform Opial, (β), (L) and (H) properties, reflexivity and convexity of this set were also investigated. The idea of extreme points was used to determine the necessary and sufficient conditions for the set $\ell(\Delta^{(\widetilde{\alpha})}, p)$ to be rotund. We remark that these properties play an important role in fixed point theory.

The sets of sequences $\ell_1(\Psi) = \{\sigma \in \omega : \Psi\sigma \in \ell_1\}$, $c_0(\Psi) = \{\sigma \in \omega : \Psi\sigma \in c_0\}$, and $c(\Psi) = \{\sigma \in \omega : \Psi\sigma \in c\}$ were considered in the papers [9, 10], where ω denotes the space of all complex valued sequences. Certain results on matrix mappings and compact operators on $\ell_1(\Psi)$, $c_0(\Psi)$, and $c(\Psi)$ were generalized using measures of noncompactness in those papers.

In this work, we give some comprehensive and complementary results by characterizing the matrix classes $(c_0(\Delta^{(\widetilde{\alpha})}), c_0(\Psi))$, $(c(\Delta^{(\widetilde{\alpha})}), c(\Psi))$, and $(\ell_1(\Delta^{(\widetilde{\alpha})}), \ell_p(\Psi))$, where Ψ is an arbitrary triangle and $1 \leq p < \infty$. We establish identities or estimates for the Hausdorff measures of noncompactness of matrix operators $L_A \in \mathcal{B}(c_0(\Delta^{(\widetilde{\alpha})}), c_0(\Psi))$, $L_A \in \mathcal{B}(c(\Delta^{(\widetilde{\alpha})}), c(\Psi))$, and $L_A \in \mathcal{B}(\ell_1(\Delta^{(\widetilde{\alpha})}), \ell_1(\Psi))$, and obtain necessary and sufficient compactness conditions for those operators.

2. Notions and notations

We give the following notions, notations, and definitions that are needed throughout the paper.

The gamma function of a real number n (except zero and the negative integers) is defined by an improper integral:

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt.$$

It is known that for any natural number n, $\Gamma(n+1) = n!$, and $\Gamma(n+1) = n\Gamma(n)$ holds for any real number $n \notin \{0, -1, -2, ...\}$. The fractional difference operator for a fraction $\tilde{\alpha}$ have been defined in [4] as $\Delta^{(\tilde{\alpha})}(\sigma_n) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{\Gamma(\tilde{\alpha}-i+1)} \sigma_{n-i}$. It is assumed that this series is convergent for $\sigma \in \omega$. This infinite sum becomes a finite sum if $\tilde{\alpha}$ is a nonnegative integer. We use the usual convention that any term with a negative subscript is equal to naught, throughout the paper.

The inverse $\Delta^{(-\tilde{\alpha})} = (\Delta_{nk}^{(-\tilde{\alpha})})$ of fractional difference triangle $\Delta^{(\tilde{\alpha})} = (\Delta_{nk}^{(\tilde{\alpha})})$

$$\Delta_{nk}^{(\tilde{\alpha})} = \begin{cases} (-1)^{n-k} \frac{\Gamma(\tilde{\alpha}+1)}{(n-k)!\Gamma(\tilde{\alpha}-n+k+1)} & (0 \leq k \leq n) \\ 0 & (k > n) \end{cases}$$

is given in [3] by

$$\Delta_{nk}^{(-\tilde{\alpha})} = \begin{cases} (-1)^{n-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(n-k)!\Gamma(-\tilde{\alpha}-n+k+1)} & (0 \leq k \leq n) \\ 0 & (k > n). \end{cases}$$

A matrix $\Psi = (\psi_{nk})_{n,k=0}^{\infty}$ is said to be a triangle if $\psi_{nk} = 0$ for all $k > n$ and $\psi_{nn} \neq 0$ $(n = 0, 1 \ldots)$. Let Ψ be a triangle, then we write S for its inverse and $R = S^t$ for the transpose of S.

Let λ and μ be subsets of ω, then $\lambda(\Psi)=\{\sigma\in\omega:\Psi\sigma\in\lambda\}$ denotes the matrix domain of Ψ in λ and (λ,μ) is the class of all infinite matrices that map λ into μ.

Note that, $e=(e_k)$ and $e^{(n)}=(e_k^{(n)})$ $(n=0,1,\dots)$ are the sequences with $e_k=1$ for all k, and $e_n^{(n)}=1$ and $e_k^{(n)}=0$ for $k\neq n$.

A BK space is a Banach space with continuous coordinates. A BK space $\lambda\supset\phi$ is said to have AK if every sequence $\sigma=(\sigma_k)_{k=0}^{\infty}\in\lambda$ has a unique representation $\sigma=\lim_{m\to\infty}\sigma^{[m]}$, where ϕ denotes the set of all finitely non-zero sequences and $\sigma^{[m]}=\sum_{n=0}^{m}\sigma_n e^{(n)}$ is the m^{th} section of the sequence σ.

Let λ and μ be BK spaces and $\mathcal{B}(\lambda,\mu)$ denote the set of all bounded linear operators $L:\lambda\to\mu$. Then we have $(\lambda,\mu)\subset\mathcal{B}(\lambda,\mu)$, that is, every $\Psi\in(\lambda,\mu)$ defines an operator $L_\Psi\in\mathcal{B}(\lambda,\mu)$, where $L_\Psi(\sigma)=\Psi\sigma$ for all $\sigma\in\lambda$ (see [30, Theorem 4.2.8]). Let λ have AK, then we have $\mathcal{B}(\lambda,\mu)\subset(\lambda,\mu)$, that is, every $L\in\mathcal{B}(\lambda,\mu)$ is given by a matrix $\Psi\in(\lambda,\mu)$ such that $\Psi\sigma=L(\sigma)$ for all $\sigma\in\lambda$ (see [10, Lemma 1.1]).

Consider now the following fractional difference sequence spaces:

$$\begin{aligned} c_0(\Delta^{(\tilde{\alpha})}) &= \{\sigma\in\omega:\lim_{n\to\infty}\Delta^{(\tilde{\alpha})}(\sigma_n)=0\};\\ c(\Delta^{(\tilde{\alpha})}) &= \{\sigma\in\omega:\lim_{n\to\infty}\Delta^{(\tilde{\alpha})}(\sigma_n)\text{ exists}\};\\ \ell_1(\Delta^{(\tilde{\alpha})}) &= \{\sigma\in\omega:\sum_{n=0}^{\infty}\left|\Delta^{(\tilde{\alpha})}(\sigma_n)\right|<\infty\}. \end{aligned}$$

It is known that $\lambda(\Psi)$ is a BK space with $\|.\|_\Psi=\|\Psi(.)\|$ when $(\lambda,\|.\|)$ is a BK space [30, Theorem 4.3.12]. Taking into account this fact, the set $\ell_1(\Delta^{(\tilde{\alpha})})$ is a complete, linear, BK space equipped with the norm $\|\sigma\|=\sum_{n=0}^{\infty}\left|\Delta^{(\tilde{\alpha})}(\sigma_n)\right|$ and the fractional difference sets $c_0(\Delta^{(\tilde{\alpha})})$ and $c(\Delta^{(\tilde{\alpha})})$ are complete, linear, BK spaces equipped with the norm $\|\sigma\|=\sup_n\left|\Delta^{(\tilde{\alpha})}(\sigma_n)\right|$.

All the notions and notations that are given in this section are standard and can also be found in [3, 17, 32].

3. Characterizations of matrix classes

In this section, we characterize the matrix classes $(c_0(\Delta^{(\tilde{\alpha})}),c_0(\Psi))$, $(c(\Delta^{(\tilde{\alpha})}),c(\Psi))$, and $(\ell_1(\Delta^{(\tilde{\alpha})}),\ell_p(\Psi))$, where Ψ is an arbitrary triangle and $1\le p<\infty$.

We first give the following results which play an important role in this study.

BK spaces play an important role in the theory of sequence spaces and matrix transformations since matrix transformations between BK spaces are continuous. Note that well known characterizations of the classical sequence spaces can be found in [29].

Lemma 3.1 *[9, Lemma 2.3] Let μ be an arbitrary subset of ω and λ be a BK space with AK. Also let $S=(s_{jk})$ be the inverse of an infinite matrix A and R be the transpose of S. Then $A\in(\lambda(\Psi),\mu)$ if and only if $\hat{A}\in(\lambda,\mu)$ and $W^{(A_n)}\in(\lambda,\ell_\infty)$ for all $n=0,1,\dots$. Here $\hat{A}$ is the matrix with rows $\hat{A}_n=RA_n$ for $n=0,1,\dots$, and the triangles $W^{(A_n)}$ $(n=0,1,\dots)$ are defined by*

$$w_{mk}^{(A_n)}=\begin{cases}\sum\limits_{j=m}^{\infty}a_{nj}s_{jk} & (0\le k\le m)\\ 0 & (k>m).\end{cases}$$

Furthermore, if $A\in(\lambda(\Psi),\mu)$ then we have $Az=\hat{A}(Tz)$ for all $z\in Z=\lambda(\Psi)$.

Remark 3.2 [22, Remark 3.5] *Let μ be an arbitrary subset of ω. Then we have $A \in (c(\Psi), \mu)$ if and only if*

$$\begin{cases} \hat{A} \in (c_0, \mu) & \textit{and} \\ W^{(A_n)} \in (c, c) & \textit{for all } n \end{cases}$$

and $\hat{A}e - (\alpha_n) \in \mu$ where $\alpha_n = \lim_{m\to\infty} \sum_{k=0}^{m} w_{mk}^{(A_n)}$ for all n.

Remark 3.3 [9, Remark 2.5] *Let λ be a BK space with AK and μ be an arbitrary subset of ω, then we have $A \in (\lambda(\Psi), \mu(\tilde{\Psi}))$ if and only if*

$$\begin{cases} \hat{B} \in (\lambda, \mu) & \textit{and} \\ W^{(B_n)} \in (\lambda, \ell_\infty) & \textit{for all } n \end{cases}$$

or equivalently if and only if

$$\begin{cases} \tilde{\Psi}\hat{A} \in (\lambda, \mu) & \textit{and} \\ W^{(A_n)} \in (\lambda, \ell_\infty) & \textit{for all } n, \end{cases}$$

where $B = \tilde{\Psi}A$, $\hat{b}_{nk} = \sum_{j=k}^{\infty} s_{jk} b_{nj}$ for all n and k, and

$$w_{mk}^{(B_n)} = \begin{cases} \sum\limits_{j=m}^{\infty} s_{jk} b_{nj} & (0 \le k \le m) \\ 0 & (k > m) \end{cases} \qquad (m = 0, 1, \dots).$$

Theorem 3.4 *Let Ψ be a triangle. Then we have $A \in (c_0(\Delta^{(\tilde{\alpha})}), c_0(\Psi))$ if and only if*

$$\sup_n \sum_{k=0}^{\infty} \left| \sum_{i=0}^{n} \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}+j-k+1)} \psi_{ni} a_{ij} \right| < \infty, \tag{3.1}$$

$$\sup_m \sum_{k=0}^{m} \left| \sum_{j=m}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}-j+k+1)} \psi_{ni} a_{ij} \right| < \infty \tag{3.2}$$

and

$$\lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}+j-k+1)} \psi_{ni} a_{ij} = 0. \tag{3.3}$$

Proof We first have

$$A \in \left(c_0(\Delta^{(\tilde{\alpha})}), c_0(\Psi)\right) \text{ if and only if } \begin{cases} \hat{A} \in (c_0, c_0(\Psi)) & \text{and} \\ W^{(A_n)} \in (c_0, c_0) & \text{for all } n \end{cases}$$

by Lemma 3.1 since c_0 is a BK space with AK. Since $A \in (\lambda, \mu(\Psi))$ if and only if $\Psi A \in (\lambda, \mu)$, we have $\hat{A} \in (c_0, c_0(\Psi))$ if and only if $B = \Psi\hat{A} \in (c_0, c_0)$. We also know from the well-known characterization of (c_0, c_0) that

$$B = \Psi\hat{A} \in (c_0, c_0) \text{ if and only if } \begin{cases} \sup\limits_n \sum\limits_{k=0}^{\infty} |b_{nk}| < \infty & \text{and} \\ \lim\limits_{n\to\infty} b_{nk} = 0 & \text{for all } k. \end{cases} \tag{3.4}$$

Taking into account the conditions in (3.4) and the definitions of $\Delta^{(\tilde{\alpha})}$ and triangle B,

$$b_{nk} = \sum_{i=0}^{n} \psi_{ni}\hat{a}_{ik} = \sum_{i=0}^{n}\sum_{j=k}^{\infty}(-1)^{j-k}\frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}+j-k+1)}\psi_{ni}a_{ij}$$

we obtain the conditions (3.1) and (3.3). On the other hand,

$$W^{(A_n)} \in (c_0, c_0) \text{ if and only if } \begin{cases} \sup_m \sum_{k=0}^{m} \left| w_{mk}^{(A_n)} \right| < \infty & \text{and} \\ \lim_m w_{mk}^{(A_n)} = 0 & \text{for all } k. \end{cases}$$

Since the series

$$\sum_{j=k}^{\infty}(-1)^{j-k}\frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}+j-k+1)}a_{ij} \tag{3.5}$$

converges for each k, the condition $\lim_m w_{mk}^{(A_n)} = 0$ becomes redundant. Therefore, we have $W^{(A_n)} \in (c_0, c_0)$ if and only if the condition (3.2) is satisfied. It completes the proof. □

Theorem 3.5 *Let Ψ be a triangle. Then we have $A \in (c(\Delta^{(\tilde{\alpha})}), c(\Psi))$ if and only if*

$$\sup_n \sum_{k=0}^{\infty} \left| \sum_{i=0}^{n}\sum_{j=k}^{\infty}(-1)^{j-k}\frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}+j-k+1)}\psi_{ni}a_{ij} \right| < \infty, \tag{3.6}$$

$$\lim_{n\to\infty} \sum_{i=0}^{n}\sum_{j=k}^{\infty}(-1)^{j-k}\frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}+j-k+1)}\psi_{ni}a_{ij} = \hat{\xi}_k \text{ exists for all } k, \tag{3.7}$$

$$\sup_m \sum_{k=0}^{m} \left| \sum_{j=m}^{\infty}(-1)^{j-k}\frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}+j-k+1)}\psi_{ni}a_{ij} \right| < \infty \text{ for each } n, \tag{3.8}$$

$$\lim_{m\to\infty} \sum_{k=0}^{m}\sum_{i=0}^{n}\sum_{j=k}^{\infty}(-1)^{j-k}\frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}+j-k+1)}\psi_{ni}a_{ij} = \xi_n \text{ exists for each } n, \tag{3.9}$$

$$\lim_{n\to\infty} \left(\sum_{k=0}^{\infty}\sum_{i=0}^{n}\sum_{j=k}^{\infty}(-1)^{j-k}\frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}+j-k+1)}\psi_{ni}a_{ij} - \xi_n \right) = \zeta \text{ exists.} \tag{3.10}$$

Proof We first have

$$A \in \left(c(\Delta^{(\tilde{\alpha})}), c(\Psi)\right) \text{ if and only if } \begin{cases} \hat{A} \in (c_0, c(\Psi)), & \\ \hat{A}e - (\alpha_n) \in c(\Psi) & \text{and} \\ W^{(A_n)} \in (c, c) & \text{for all } n, \end{cases}$$

where $\alpha_n = \lim_{m\to\infty}\sum_{k=0}^{m} w_{mk}^{(A_n)}$ for all n, by the help of Remark 3.2. Since $A \in (\lambda, \mu(\Psi))$ if and only if $\Psi A \in (\lambda, \mu)$, we have $\hat{A} \in (c_0, c(\Psi))$ if and only if $B = \Psi\hat{A} \in (c_0, c)$. We also know from the well known characterization of (c_0, c) that

$$B = \Psi\hat{A} \in (c_0, c) \text{ if and only if } \begin{cases} \sup_n \sum_{k=0}^{\infty} |b_{nk}| < \infty & \text{and} \\ \lim_{n\to\infty} b_{nk} = \hat{\xi}_k & \text{for all } k. \end{cases} \tag{3.11}$$

Taking into account the conditions in (3.11) and the definitions of $\Delta^{(\widetilde{\alpha})}$ and B, we obtain the conditions (3.6) and (3.7). On the other hand, the condition $\hat{A}e - (\alpha_i) \in c(\Psi)$ implies that $(\Psi_n(\hat{A}e) - \Psi_n(\alpha_i)) \in c$ and this implies the condition in (3.10) since $\lim_n \alpha_n$ exists for each n and since

$$\begin{aligned} \Psi_n\left((\alpha_i)_{i=0}^{\infty}\right) &= \sum_{i=0}^{n} \psi_{ni}\alpha_i = \sum_{i=0}^{n} \psi_{ni} \lim_{m\to\infty} \sum_{k=0}^{m} w_{mk}^{(A_i)} \\ &= \lim_{m\to\infty} \sum_{k=0}^{m}\sum_{i=0}^{n}\sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}+j-k+1)} \psi_{ni} a_{ij} \\ &= \xi_n. \end{aligned}$$

Finally, we have

$$W^{(A_n)} \in (c, c) \text{ if and only if } \begin{cases} \sup_m \sum_{k=0}^{m} \left| w_{mk}^{(A_n)} \right| < \infty, & \\ \lim_m w_{mk}^{(A_n)} = \xi_n & \text{for all } k, \\ \lim_m w_{mk}^{(A_n)} \text{ exists} & \text{for all } k, \end{cases} \tag{3.12}$$

which are the conditions (3.8) and (3.9). Note again that the last condition in (3.12) becomes redundant since the series in (3.5) converges for each k. This completes the proof. □

Theorem 3.6 *Let* $1 \le p < \infty$ *and* Ψ *be a triangle. Then we have* $A \in (\ell_1(\Delta^{(\widetilde{\alpha})}), \ell_p(\Psi))$ *if and only if*

$$\sup_k \sum_{n=0}^{\infty} \left| \sum_{i=0}^{n}\sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}-j+k+1)} \psi_{ni} a_{ij} \right|^p < \infty \tag{3.13}$$

and for all $n = 0, 1, \ldots$

$$\sup_{m,k} \left| \sum_{j=m}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}-j+k+1)} a_{nj} \right| < \infty. \tag{3.14}$$

Proof We first have

$$A \in \left(\ell_1(\Delta^{(\widetilde{\alpha})}), \ell_p(\Psi)\right) \text{ if and only if } \begin{cases} \hat{A} \in (\ell_1, \ell_p(\Psi)) & \text{and} \\ W^{(A_n)} \in (\ell_1, \ell_\infty) & \text{for all } n \end{cases}$$

by Lemma 3.1 since ℓ_1 is a BK space with AK. Also, since $A \in (\lambda, \mu(\Psi))$ if and only if $\Psi A \in (\lambda, \mu)$, we have $\hat{A} \in (\ell_1, \ell_p(\Psi))$ if and only if $B = \Psi\hat{A} \in (\ell_1, \ell_p)$. We also know from the well known characterization of (ℓ_1, ℓ_p) that $B = \Psi\hat{A} \in (\ell_1, \ell_p)$ if and only if

$$\sup_k \sum_{n=0}^{\infty} |b_{nk}|^p = \sup_k \sum_{n=0}^{\infty} \left| \sum_{i=0}^{n} \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}-j+k+1)} \psi_{ni} a_{ij} \right|^p < \infty.$$

Hence, we obtain the condition (3.13). On the other hand, $W^{(A_n)} \in (\ell_1, \ell_\infty)$ if and only if

$$\sup_{m,k} \sum_{k=0}^{m} \left| w_{mk}^{(A_n)} \right| = \sup_{m,k} \sum_{k=0}^{m} \left| \sum_{j=m}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}-j+k+1)} a_{nj} \right| < \infty,$$

which is the condition (3.14). □

4. Applications of measures of noncompactness

In this section, we give our main results related to compact operators on the sets of fractional difference sequences.

We recall the definition of Hausdorff measure of noncompactness for operators between Banach spaces.

If λ and μ are infinite–dimensional complex Banach spaces then a linear operator $L : \lambda \to \mu$ is said to be compact if the domain of L is all of λ, and, for every bounded sequence (σ_n) in λ, the sequence $(L(\sigma_n))$ has a convergent subsequence. We denote the class of such operators by $\mathcal{C}(\lambda, \mu)$.

Let (λ, d) be a metric space, $B(b, c) = \{a \in \lambda : d(a, b) < c\}$ denote the open ball of radius $c > 0$ and centre in $b \in \lambda$, and $\mathcal{M}_\lambda$ be the collection of bounded sets in λ. The Hausdorff measure of noncompactness of $Q \in \mathcal{M}_\lambda$ is

$$\chi(Q) = \inf\{\epsilon > 0 : Q \subset \bigcup_{k=1}^{n} B(\sigma_k, \delta_k) : \sigma_k \in \lambda,\ \delta_k < \epsilon,\ 1 \le k \le n,\ n \in \mathbb{N}\}.$$

Let λ and μ be Banach spaces and χ_1 and χ_2 be measures of noncompactness on λ and μ. Then the operator $L : \lambda \to \mu$ is called (χ_1, χ_2)–bounded if $L(Q) \in \mathcal{M}_\mu$ for every $Q \in \mathcal{M}_\lambda$ and there exists a positive constant C such that $\chi_2(L(Q)) \le C\chi_1(Q)$ for every $Q \in \mathcal{M}_\lambda$. If an operator L is (χ_1, χ_2)–bounded then the number $\|L\|_{(\chi_1,\chi_2)} = \inf\{C \ge 0 : \chi_2(L(Q)) \le C\chi_1(Q)$ holds for all $Q \in \mathcal{M}_\lambda\}$ is called the (χ_1, χ_2)–measure of noncompactness of L. In particular, if $\chi_1 = \chi_2 = \chi$, then we write $\|L\|_\chi$ instead of $\|L\|_{(\chi,\chi)}$.

Let λ be a normed space. Then $S_\lambda = \{\sigma \in \lambda : \|\sigma\| = 1\}$ and $\bar{B}_\lambda = \{\sigma \in \lambda : \|\sigma\| \le 1\}$ denote the unit sphere and closed unit ball in λ.

Let λ and μ be Banach spaces and $L \in \mathcal{B}(\lambda, \mu)$. Then we have

$$\|L\|_\chi = \chi\left(L(\bar{B}_\lambda)\right) = \chi\left(L(S_\lambda)\right) \text{ [23, Theorem 2.25]}; \tag{4.1}$$

$$L \in \mathcal{C}(\lambda, \mu) \text{ if and only if } \|L\|_\chi = 0 \text{ [23, Corollary 2.26]}. \tag{4.2}$$

Lemma 4.1 *[23, Theorem 2.23] Let* λ *be a Banach space with Schauder basis* $(b_n)_{n=0}^{\infty}$, $Q \in \mathcal{M}_\lambda$, $P_n : \lambda \to \lambda$ *be the projector onto the linear span of* $\{b_0, b_1, \dots b_n\}$. I *be the identity map on* λ *and* $R_n = I - P_n$ $(n = 0, 1, \dots)$. *Then we have*

$$\frac{1}{a} \cdot \limsup_{n\to\infty} \left(\sup_{\sigma\in Q} \|R_n(\sigma)\| \right) \leq \chi(Q) \leq \limsup_{n\to\infty} \left(\sup_{\sigma\in Q} \|R_n(\sigma)\| \right),$$

where $a = \limsup_{n\to\infty} \|R_n\|$.

Lemma 4.2 *[23, Theorem 2.8] Let* Q *be a bounded subset of the normed space* λ, *where* λ *is* ℓ_p *for* $1 \leq p < \infty$ *or* c_0. *If* $P_n : \lambda \to \lambda$ *is the operator defined by* $P_n(\sigma) = \sigma^{[n]}$ *for* $\sigma = (\sigma_k)_{k=0}^{\infty} \in \lambda$, *then we have*

$$\chi(Q) = \lim_{n} \left(\sup_{\sigma\in Q} \|R_n(\sigma)\| \right).$$

Lemma 4.3 *[10, Lemma 3.5] Let* λ *and* μ *be Banach sequence spaces,* Ψ *be a triangle and* $L \in \mathcal{B}(\lambda, \mu(\Psi))$. *Then we have*

$$\|L\|_{(\chi,\chi_\Psi)} = \|L_\Psi \circ L\|_\chi.$$

We establish identities or estimates for the Hausdorff measures of noncompactness of matrix operators $L_A \in \mathcal{B}(c_0(\Delta^{(\tilde{\alpha})}), c_0(\Psi))$, $L_A \in \mathcal{B}(c(\Delta^{(\tilde{\alpha})}), c(\Psi))$ and $L_A \in \mathcal{B}(\ell_1(\Delta^{(\tilde{\alpha})}), \ell_1(\Psi))$, and obtain necessary and sufficient compactness conditions for those operators.

Theorem 4.4 *Let* Ψ *be triangle and the operator* $L_A \in \mathcal{B}(c_0(\Delta^{(\tilde{\alpha})}), c_0(\Psi))$ *be given by a matrix* $A \in (c_0(\Delta^{(\tilde{\alpha})}), c_0(\Psi))$. *Then we have* $\|L_A\|_\chi = \Upsilon$, *where*

$$\Upsilon = \lim_{r\to\infty} \sup_{n>r} \left(\sum_{k=0}^{\infty} \left| \sum_{i=0}^{n} \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}-j+k+1)} \psi_{ni} a_{ij} \right| \right).$$

Proof Applying equality (4.1) and Lemma 4.2, we have

$$\|L_A\|_\chi = \chi_\Psi \left(L_A(S_\lambda) \right) = \chi \left(\Psi(L_A(S_\lambda)) \right).$$

Then we also obtain

$$\|L_A\|_\chi = \chi(Q) = \lim_{r\to\infty} \left(\sup_{\sigma\in Q} \|(I-P_r)(x)\| \right) = \lim_{r\to\infty} \left(\sup_{\sigma\in S_\lambda} \|(I-P_r)\Psi(Ax)\| \right)$$

by Lemma 4.2 because the set $Q = \Psi(L_A(S_\lambda))$ is included by c_0.

Taking into account the equality $\Psi(A\sigma) = (\Psi A)\sigma$ for each $\sigma \in c_0(\Psi)$, we have

$$\begin{aligned} \|L_A\|_\chi = \chi(Q) &= \lim_{r\to\infty} \left(\sup_{\sigma\in S_\lambda} \|(I-P_r)((\Psi A)\sigma)\| \right) \\ &= \lim_{r\to\infty} \left(\sup_{\sigma\in S_\lambda} \|(0,0,0,\dots,(\Psi A)_{r+1}\sigma,(\Psi A)_{r+2}\sigma,\cdots)\| \right). \end{aligned}$$

Bearing those equalities in mind and the fact that $A \in (\lambda(\Psi), \mu)$ implies $\|L_A\| = \left\|L_{\hat{A}}\right\|$ when λ has AK; and $\|L_A\| = \|A\|$, we have $\|L_A\|_\chi = \varUpsilon$ since $\Psi A \in (c_0(\Delta^{(\widetilde{\alpha})}), c_0)$.

$\square$

Theorem 4.5 *Let Ψ be triangle and the operator $L_A \in \mathcal{B}(c(\Delta^{(\widetilde{\alpha})}), c(\Psi))$ be given by a matrix $A \in (c(\Delta^{(\widetilde{\alpha})}), c(\Psi))$. Then we have*

$$\frac{1}{2} \cdot \varOmega \leq \|L_A\|_\chi \leq \varOmega,$$

where $\varOmega = \lim_{r\to\infty} \sup_{n>r} (\varOmega_1 + \varOmega_2)$;

$$\varOmega_1 = \sum_{k=0}^{\infty} \left| \sum_{i=0}^{n} \sum_{j=k}^{\infty} (-1)^{j-k} \tfrac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}-j+k+1)} \psi_{ni} a_{ij} - \hat{\xi}_k \right|,$$

$$\varOmega_2 = \left| \sum_{k=0}^{\infty} \hat{\xi}_k - \xi_n - \zeta \right|$$

and also $\hat{\xi}_k$, ξ_n and ζ are defined in Theorem 3.5.

Proof Taking into account $A \in (c(\Delta^{(\widetilde{\alpha})}), c(\Psi))$, [8, Theorem 3.7 (a)] and Lemma 4.3 we write

$$\frac{1}{2} \cdot \limsup_{n\to\infty} (\Phi_n(B)) \leq \|L\|_\chi \leq \limsup_{n\to\infty} (\Phi_n(B)),$$

where $B = \Psi A$ and Φ is a matrix given for $n \in \mathbb{N}$ by

$$\Phi(B) = \sum_{k=0}^{\infty} \left| \hat{b}_{nk} - \hat{\xi}_k \right| + \left| \sum_{k=0}^{\infty} \hat{\xi}_k - \xi_n - \zeta \right|,$$

where $\xi_n = \lim_{m\to\infty} w_{mk}^{(B_n)}$ for $n = 0, 1, \ldots$,

$\zeta = \lim_{n\to\infty} (\sum_{k=0}^{\infty} \hat{b}_{nk} - \xi_n)$ and

$\hat{\xi} = (\xi_k)_{k=0}^{\infty}$ with $\hat{\xi}_k = \lim_{n\to\infty} \hat{b}_{nk}$ for $k = 0, 1, \ldots$.

Since $A \in (c(\Delta^{(\widetilde{\alpha})}), c(\Psi))$ if and only if $B = \Psi A \in (c(\Delta^{(\widetilde{\alpha})}), c)$. Applying Remark 3.2, we have $B = \Psi A \in (c(\Delta^{(\widetilde{\alpha})}), c)$ if and only if $\hat{B} \in (c_0, c)$, $W^{(B_n)} \in (c, c)$ for all n and $\hat{B}e - \xi_n \in c$. The condition $\hat{B} \in (c_0, c)$ implies that

$$\lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}-j+k+1)} \psi_{ni} a_{ij} = \hat{\xi}_k \text{ exists for all } k.$$

Also the condition $W^{(B_n)} \in (c, c)$ implies that

$$\lim_{m\to\infty} \sum_{k=0}^{m} \sum_{i=0}^{n} \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}+j-k+1)} \psi_{ni} a_{ij} = \xi_n \text{ exists for each } n.$$

Finally, the condition $\hat{B}e - \xi_n \in c$ implies that

$$\lim_{n\to\infty}\left(\sum_{k=0}^{\infty}\sum_{i=0}^{n}\sum_{j=k}^{\infty}(-1)^{j-k}\frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}+j-k+1)}\psi_{ni}a_{ij}-\xi_n\right)=\zeta \text{ exists.}$$

Therefore, we have the existences of ξ_n, ζ, and $\hat{\xi}$ and therefore $\varOmega$. This completes the proof by Lemma 4.3 and [8, Theorem 3.7 (a)]. $\square$

Theorem 4.6 *Let Ψ be triangle and the operator $L_A \in \mathcal{B}(\ell_1(\Delta^{(\tilde{\alpha})}), \ell_1(\Psi))$ be given by a matrix $A \in (\ell_1(\Delta^{(\tilde{\alpha})}), \ell_1(\Psi))$. Then we have $\|L_A\|_{\chi} = \varLambda$, where*

$$\varLambda = \lim_{r\to\infty}\sup_k\left(\sum_{n=r}^{\infty}|\sum_{i=0}^{n}\sum_{j=k}^{\infty}(-1)^{j-k}\frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}-j+k+1)}\psi_{ni}a_{ij}|\right).$$

Proof Taking into account Lemma 4.3 and the definition of fractional triangle, we have

$$\begin{aligned}
\|L\|_{(\chi_{\Delta^{(\tilde{\alpha})}},\chi_{\Psi})} &= \|L_{\Psi}\circ L_A\|_{(\chi_{\Delta^{(\tilde{\alpha})}},\chi)} \\
&= \|L_{\Psi A}\|_{(\chi_{\Delta^{(\tilde{\alpha})}},\chi)} \\
&= \lim_{r\to\infty}\sup_k\left(\sum_{n=r}^{\infty}|\sum_{i=0}^{n}\sum_{j=k}^{\infty}(-1)^{j-k}\frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}-j+k+1)}\psi_{ni}a_{ij}|\right),
\end{aligned}$$

which completes the proof. $\square$

We close this section with a corollary giving the compactness conditions for the classes $(c_0(\Delta^{(\tilde{\alpha})}), c_0(\Psi))$, $(c(\Delta^{(\tilde{\alpha})}), c(\Psi))$, and $(\ell_1(\Delta^{(\tilde{\alpha})}), \ell_1(\Psi))$, where Ψ is an arbitrary triangle, by Theorems 4.4–4.6, and the statement in (4.2).

Corollary 4.7 *Let Ψ be a triangle. Then the following statements hold:*

(a) If $A \in (c_0(\Delta^{(\tilde{\alpha})}), c_0(\Psi))$ then $L_A \in \mathcal{C}(c_0(\Delta^{(\tilde{\alpha})}), c_0(\Psi))$ if and only if $\varUpsilon = 0$, where $\varUpsilon$ is defined as in Theorem 4.4.

(b) If $A \in (c(\Delta^{(\tilde{\alpha})}), c(\Psi))$ then $L_A \in \mathcal{C}(c(\Delta^{(\tilde{\alpha})}), c(\Psi))$ if and only if $\varOmega = 0$, where $\varOmega$ is defined as in Theorem 4.5.

(c) If $A \in (\ell_1(\Delta^{(\tilde{\alpha})}), \ell_1(\Psi))$ then $L_A \in \mathcal{C}(\ell_1(\Delta^{(\tilde{\alpha})}), \ell_1(\Psi))$ if and only if $\varLambda = 0$, where $\varLambda$ is defined as in Theorem 4.6.

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