TÜBİTAK

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 813 – 832
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doi:10.3906/mat-1806-23

Complete flat cone metrics on punctured surfaces

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Received: 04.06.2018 • **Accepted/Published Online:** 11.02.2019 • **Final Version:** 27.03.2019

Abstract: We prove that each complete flat cone metric on a surface with regular or irregular punctures can be triangulated with finitely many types of triangles. We derive the Gauss–Bonnet formula for this kind of cone metrics. In addition, we prove that each free homotopy class of paths has a geodesic representative.

Key words: Flat metric, the Gauss–Bonnet formula, surfaces with punctures, the Hopf–Rinow theorem

1. Introduction

Flat cone metrics appear in several areas of mathematics. For example, they are studied in the Teichmüller theory through quadratic differentials and in dynamics of billiard tables [5, 10]. These objects are also interesting for their own sake. Classification of certain families of these metrics may yield interesting results in several areas of mathematics such as hypergeometric functions and (real and complex) hyperbolic geometry [2, 4, 6, 7, 11, 12]. In addition, regarding combinatorial triangulations or quadrangulations as cone metrics as in [12], one can parametrize certain families of dessins d'enfants. See [1, 17, 19].

Flat cone metrics on compact surfaces have been studied well. We know that there is a length-minimizing path between any two points of such a surface. Also, each free homotopy class of loops on a compact surface with a flat metric contains a length-minimizing geodesic. Indeed, these properties follow from the general theory of length spaces [3, 8]. Furthermore, the Gauss–Bonnet formula holds for these surfaces, and they can be triangulated with finitely many triangles. See [13–15].

The Teichmüller theory is related with the theory of cone metrics in a natural way. Let S be a closed, orientable surface, $x_1, \ldots, x_n \in S$. Pick $a_1, \ldots a_n \in \mathbb{R}$ so that $\sum_{i=1}^{n} a_i = 2\pi\chi(S)$, where $\chi(S)$ is Euler characteristics of S. Consider the curvature divisor

$$D = \sum_{i=1}^{n} a_i x_i.$$

It is known that each conformal class on S includes a flat metric with n singular points of divisor D. Furthermore, this metric is unique up to homothety. See the papers by Troyanov [13, 14] in the case where $a_i < 2\pi$ for each $1 \leq i \leq n$, otherwise see the paper by Hulin and Troyanov [16].

Flat surfaces *regular* punctures have been also studied well. By a regular puncture on a flat surface, we mean a puncture which has a neighborhood isometric to that of point at infinity of a cone. The Gauss–

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2000 *AMS Mathematics Subject Classification:* 51F99, 57M50

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Bonnet formula holds for the surfaces with regular punctures. Also, there is length-minimizing geodesic in any homotopy class of loops in such a surface. In addition, these surfaces may be triangulated with finitely many types of triangles.

Our objective is to verify that any complete flat metric on a given surface, with regular or irregular punctures, has the above-mentioned properties. Let $\bar{S}$ be a surface with a complete flat cone metric. We summarize the results of the present paper as follows.

1. In Section 3, we show that $\bar{S}$ can be triangulated with finitely many types of triangles.
2. In Section 4, we show that a variant of the Gauss–Bonnet formula holds for $\bar{S}$.
3. In Section 5, we show that each loop on $\bar{S}$ has a geodesic representative in its free homotopy class.

We want to study complete flat metrics at the highest level of generality. The surfaces that we consider are of finite type and may have punctures and boundary. We do not omit the surfaces having punctured boundary components from our discussion. Therefore, we start with introducing a convenient notation.

1.1. Notation

Definition 1. *Let S be a compact, connected topological surface perhaps with boundary B. Let $\mathfrak{l}, \mathfrak{p}, \mathfrak{l}', \mathfrak{p}'$ be finite disjoint subsets of S so that*

- *$\mathfrak{l}$ and $\mathfrak{p}$ are subsets of the interior of S,*
- *$\mathfrak{p}', \mathfrak{l}'$ are subsets of B.*

An element in $\mathfrak{l}$ will be called labeled interior point. An element in $\mathfrak{p}$ will be called punctured interior point. Other points in interior of S called ordinary interior points. An element in B will be called boundary point. An element in $\mathfrak{l}'$ will be called a labeled boundary point. An element in $\mathfrak{p}'$ will be called punctured boundary point. Other points in boundary will be called ordinary boundary points. A doubly labeled surface, shortly DL surface, is the tuple

$$(S, B, \mathfrak{l}, \mathfrak{p}, \mathfrak{l}', \mathfrak{p}')$$

Also, we will use the following notation:

1. $S_B = S - B$.
2. $S_{\mathfrak{l}} = S - \mathfrak{l}$
3. $S_{B,\mathfrak{l}} = S - (B \cup \mathfrak{l})$
4. ...

We will denote a doubly labeled surface $(S, B, \mathfrak{l}, \mathfrak{p}, \mathfrak{l}', \mathfrak{p}')$ as S^L. Underlying compact surface of S^L will simply be denoted as S.

DL surfaces can be considered to be punctured surfaces, with puncture set $\mathfrak{p} \cup \mathfrak{p}'$. Indeed, $S_{\mathfrak{p},\mathfrak{p}'}$ is the punctured surface that we consider. Observe that punctured and labeled points may also lie in boundary.

2. Flat DL surfaces

Flat compact surfaces can be triangulated with finitely many triangles. For noncompact surfaces, we need to modify the definition of triangulation. The reason for this is that punctured surfaces may require infinitely many triangles and arbitrary triangulations possibly induce noncomplete cone metrics.

Definition 2. *An Euclidean triangulation of a DL surface S^L is a set of pairs $\mathfrak{T} = \{(T_\alpha, f_\alpha)_{\alpha\in A}\}$ where each T_α is a compact subset of $S_{\mathfrak{p},\mathfrak{p}'}$ and $f_\alpha : T_\alpha \to \mathbb{R}^2$ is a homeomorphism onto a nondegenarete triangle $f_\alpha(T_\alpha)$ in the Euclidean plane. T_α is called a triangle. Let e be a subset of T_α. e is called an edge if $f_\alpha(e)$ is an edge for the Euclidean triangle $f_\alpha(T_\alpha)$. Similary, $v \in T_\alpha$ is called a vertex if $f_\alpha(v)$ is a vertex of the triangle $f_\alpha(T_\alpha)$. The Euclidean triangulation also satisfies the following properties:*

1. $S_{\mathfrak{p},\mathfrak{p}'} = \cup_{\alpha\in A} T_\alpha$
2. *If $\alpha \neq \beta$, then $T_\alpha \cap T_\beta$ is either empty or an edge, or a vertex.*
3. *If $T_\alpha \cap T_\beta$ is not empty, then there is a $g_{\alpha\beta} \in \mathfrak{E}(2)$ (the group of isometries of Euclidean plane) so that $f_\alpha = g_{\alpha\beta} f_\beta$ on the intersection.*
4. *(Local finiteness) Each compact subset of $S_{\mathfrak{p},\mathfrak{p}'}$ intersect with finitely many triangles, edges, and vertices.*
5. *Set of triangles $f_\alpha(T_\alpha)$ consists of finitely many isometry classes of Euclidean triangles.*

Observe that our definition is a generalization of the one given in [15]. We just added two more conditions: (4) and (5). Note that Euclidean triangulations on compact surfaces always have these properties. We will show that these triangulations induce complete flat cone metrics on DL surfaces and DL surfaces with complete cone metrics can be triangulated. See Proposition 2 and Theorem 3.

The Notions of angle and curvature for DL surfaces having Euclidean triangulations

Definition 3. *Let S^L be a DL surface together with a Euclidean triangulation. Let x be a vertex in S_B. x is called a point having angle θ if*

$$\theta = \theta(x) = \sum_{j=1}^{k} \phi_j, \tag{1}$$

where $\phi_1, \dots, \phi_k$ are angles of the triangles incident to x, at the vertex x. The curvature at x is

$$\kappa = \kappa(x) = 2\pi - \theta(x). \tag{2}$$

Similarly, let y be a vertex in $B - \mathfrak{p}'$. y is called a point having angle θ if

$$\theta = \theta(y) = \sum_{j=1}^{r} \phi'_j, \tag{3}$$

where $\phi'_1, \dots, \phi'_r$ are angles of the triangles incident to y, at the vertex y. The curvature at y is

$$\kappa = \kappa(y) = \pi - \theta(y). \tag{4}$$

Curvature at the points which are not vertices is defined to be 0. If a point, either on the boundary or not, has curvature 0, then it is called nonsingular, otherwise it is called singular.

Definition 4. *A flat doubly labeled (FDL) surface* $(S^L, \mathfrak{T})$ *is a DL surface* S^L *together with an Euclidean triangulation* $\mathfrak{T}$ *such that its set of singular points is* $\mathfrak{l} \cup \mathfrak{l}'$.

In Section 4, we will extend the notions of the curvature and the angle to the punctured interior and punctured boundary points.

2.1. Induced length structure

An FDL surface $(S^L, \mathfrak{T})$ has natural area measure which coincides with the 2 dimensional Lebesque measure at each triangle T_α. Also, as in [15], we can define the length $l(c)$ of a curve $c:\ [a,b] \to S_{\mathfrak{p},\mathfrak{p}'}$ $(a, b \in \mathbb{R},\ a < b)$ as follows:

- If c is contained in a triangle T_α of $\mathfrak{T}$, then $l(c)$ is its Euclidean length.
- If c is concatenation of two curves c_1 and c_2, then $l(c) = l(c_1) + l(c_2)$.

When there is no risk of confusion, we will refer curves on $S_{p,p'}$ as curves on S^L. Also, we will use the notation $[a,b] \to S^L$ instead of $[a,b] \to S_{p,p'}$.

Lemma 1. *Let* S^L *be an FDL surface. Any two points in* $S_{\mathfrak{p},\mathfrak{p}'}$ *can be joined by a curve of finite length.*

Proof Take two points $x, y \in S_{\mathfrak{p},\mathfrak{p}'}$ and a curve $c:\ [0,1] \to S^L$ joining them. Since image of the curve is compact it is contained in a finite number of triangles. One can easily construct a finite length curve joining x and y which lies in the union of these triangles.

Consider the following function $d:\ S_{\mathfrak{p},\mathfrak{p}'} \times S_{\mathfrak{p},\mathfrak{p}'} \to \mathbb{R}$

$$d(x,y) = \inf\{l(\alpha):\ \alpha \text{ is a curve joining x to y }\}. \tag{5}$$

Proposition 1. d *is a metric on* $S_{\mathfrak{p},\mathfrak{p}'}$*:*

1. $d(x,x) = 0$ *for all* $x \in S_{\mathfrak{p},\mathfrak{p}'}$.
2. $\infty > d(x,y) > 0$, *when* $x \neq y$.
3. $d(x,y) = d(y,x)$ *for all* $x, y \in S_{\mathfrak{p},\mathfrak{p}'}$.
4. $d(x,y) + d(y,z) \geq d(x,z)$ *for all* $x, y, z \in S_{\mathfrak{p},\mathfrak{p}'}$.

Proof (1), (3), and (4) are obvious. (2) follows from local finiteness property of the Euclidean triangulations and Lemma 1.

If there is no risk of confusion, we will refer to this metric as a metric on S^L instead of a metric on $S_{\mathfrak{p},\mathfrak{p}'}$.

Now we state an elementary lemma from Euclidean geometry. See Figure 1.

Lemma 2. *Let* H *be hinge of two triangles. Let* E *be a line segment on* H *joining two edges which are not adjacent. The length of* E *is greater than or equal to one of the altitudes of the triangles.*

Proof We assume that $\alpha \geq \beta$. It follows that the length of E is greater than or equal to the length of L, which is greater than or equal to the length of h.

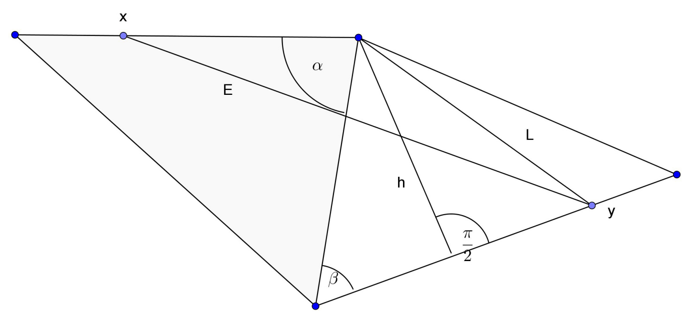

Figure 1. Length of E is greater than or equal to length of h.

Lemma 3. *Let (S^L, d) be an FDL surface with induced metric d. Let T_α be a triangle on it, and δ be the minimum of altitudes of the triangles on it. Let $x \in T_\alpha$, and y be a point which is not in T_α or triangles intersecting T_α. Then it follows that*

$$d(x, y) > \delta.$$

Proof Let

$$U = \{x \in S_{\mathfrak{p},\mathfrak{p}'} : \text{ } x \text{ is in one of the triangles intersecting with } T_\alpha\}.$$

Consider the following subset of U:

$$V = \{x \in U : x \text{ is in an edge which does not intersect with } T_\alpha\}.$$

See Figure 2. Note that

1. if $S^L \neq U$, then $S^L - V$ is disconnected,

2. if T is a triangle of U so that its edge e is in V, then distance between a point in T_α and a point in e is greater than or equal to δ. See Lemma 2.

Take a curve joining y to x. It follows that the curve and V intersect. Hence, $d(x, y)$ is strictly greater than the distance between the sets V and T_α. Since the distance between V and T_α is greater than or equal to δ, it follows that $d(x, y) > \delta$.

Proposition 2. (S^L, d) *is complete metric space.*

Proof Let $x_1, \ldots, x_n, \ldots$ be a Cauchy sequence in $S_{\mathfrak{p},\mathfrak{p}'}$. There exists $m \in \mathbb{Z}^+$ such that for all $n \geq m$ $d(x_n, x_m) < \delta$, where δ is minimum of the lengths of the altitudes of all triangles $T \in \mathfrak{T}$. Let T_α be one of the triangles which contains x_m. By locally finiteness, the set of all triangles incident to T_α, either from a vertex or from an edge, is finite. Consider the following compact set:

$$U = \{x \in S_{\mathfrak{p},\mathfrak{p}'} : \text{ } x \text{ is in one of the triangles incident to } T_\alpha\}.$$

If $U = S^L$, then S^L is compact and so it is complete. If $U \neq S^L$, Lemma 3 implies that U contains a ball of radius δ around x_m. Hence, $x_m, x_{m+1}, \ldots$ are contained in U. Since U is compact, this sequence converges to some element in U. This means that the sequence $(x_n)_{n=1}^{\infty}$ is convergent.

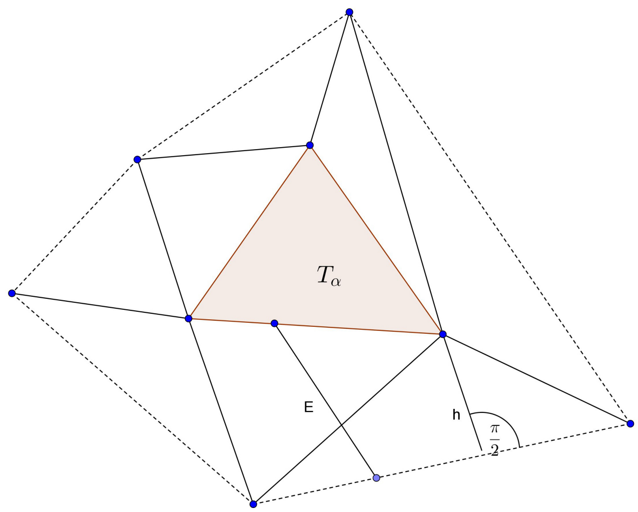

Figure 2. The triangular neighborhood U of T_α. Dashed line segments correspond to V.

Remark 1. *(S^L, d) is a length space, see [8].*

Proposition 3. *The following properties hold for an FDL surface $S_{\mathfrak{p},\mathfrak{p}'}$.*

1. *Given any two points on $S_{\mathfrak{p},\mathfrak{p}'}$, there exists a path joining them which has minimum length.*
2. *If $\mathfrak{p} \cup \mathfrak{p}'$ is not empty then $\mathfrak{T}$ contains infinitely many triangles.*
3. *If $\mathfrak{p} \cup \mathfrak{p}'$ is not empty then d is unbounded.*

Proof We prove each of the above items separately.

1. This follows from the Hopf–Rinow theorem for complete length spaces.
2. If there are finitely many triangles then $S_{\mathfrak{p},\mathfrak{p}'}$ is compact. This is impossible.
3. A complete metric space together with a bounded metric is compact.

2.2. Cones

A cone having angle θ, or equivalently curvature $\kappa = 2\pi - \theta$, is the set

$$\{(r, \psi) : r \in \mathbb{R}^{\geq 0}, \psi \in \mathbb{R}/\theta\mathbb{Z}\} \tag{6}$$

with the metric

$$\mu = dr^2 + r^2 d\psi^2. \tag{7}$$

See [13] for more information about cones. A cone can be considered to be an FDL sphere with one punctured and one labeled point. The point $(0, 0)$ is called vertex or the origin of the cone in the Euclidean plane. We will denote it by v_0. Since a cone can be considered to be a piecewise flat surface, it makes sense to talk about the punctured point or the *point at infinity*. We will denote this point as v_∞.

Definition 5. *Consider a cone with angle $\theta > 0$.*

1. $\kappa(v_\infty) = 2\pi + \theta$ *is called curvature at* v_∞.
2. $\theta(v_\infty) = -\theta$ *is called the angle at* v_∞.

Remark 2. *Observe that* $\kappa(v_0) + \kappa(v_\infty) = 4\pi$*: the Gauss–Bonnet formula for the sphere holds.*

A cone with angle θ will be denoted by C_θ.

Definition 6. *A section of a cone of angle θ is the once punctured disk obtained by cutting a cone of angle θ along a geodesic directing from its origin, and will be denoted as V_θ.*

A section of cone can be regarded as an FDL disk with one punctured point and one labeled point at its boundary. As usual, angle and curvature at the labeled point are θ and $\pi - \theta$, respectively. For the punctured point, angle and curvature at the punctured point are $-\theta$ and $\pi + \theta$, respectively. Hence, the Gauss–Bonnet formula for the closed disk holds. See Figure 3.

Definition 7. *A section of a cylinder, $I(r)$, is the twice punctured disk together with a metric which is isometric to an infinite strip in the Euclidean plane. Width of the strip, r, is called width of the cut.*

A section of cylinder can be regarded as an FDL disk with two punctured points at its boundary. By definition, the angle and the curvature at each of the punctured points are 0 and π, respectively. Hence, the Gauss–Bonnet formula for the closed disks holds. See the Figure 3.

Definition 8. *A cylinder of width r, C_{0r}, is a metric space obtained by identifying edges of a cut of a cylinder having width r through opposite points.*

Observe that a cylinder can be considered to be an FDL sphere with two punctured points. By convention, angles at these punctures are 0. We can also call a cylinder a cone of angle 0. Also, again by convention, the curvature at each of the punctured points, is 2π. Observe that the Gauss–Bonnet formula for the sphere holds.

2.3. Cone metrics on disk

Definition 9. *A (flat) cone metric on a DL surface S^L is a metric on $S_{\mathfrak{p},\mathfrak{p}'}$ so that each point x in $S_{\mathfrak{p},\mathfrak{p}'}$ has a neighborhood isometric to a neighborhood of the apex of cone $C_\theta = C_{\theta_x}$ or a section of a cone $V_\theta = V_{\theta_x}$, and*

1. $\mathfrak{l} = \{y \in S_{\mathfrak{p},B} : \ \theta_y \neq 2\pi\}$,
2. $\mathfrak{l}' = \{y \in B - \mathfrak{p}' : \ \theta_y \neq \pi\}$.

Angle at x, $\theta(x)$ is defined to be θ_x. If $x \in S_{\mathfrak{p},B}$, then the curvature at x, $\kappa(x)$, is defined as $2\pi - \theta(x)$. If $x \in B - \mathfrak{p}'$, then the curvature is $\kappa(x) = \pi - \theta(x)$. x is called singular if $\kappa(x) \neq 0$, otherwise it is called nonsingular.

Observe that the two conditions above guarantee that set of singular points of S^L is $\mathfrak{l} \cup \mathfrak{l}'$.

Cones, cylinders, sections of cones, and sections of cylinders are examples of cone metrics. Observe that each FDL surface can be regarded as a cone metric on the underlying DL surface. Note that by an isometry of cone metrics on DL surfaces S^L and $\bar{S}^L$, we mean an isometry of underlying metric spaces $S_{\mathfrak{p},\mathfrak{p}'}$ and $\bar{S}_{\bar{\mathfrak{p}},\bar{\mathfrak{p}}'}$.

Now we state some elementary facts about cones, cylinders, sections of cones, and sections of cylinders without proof.

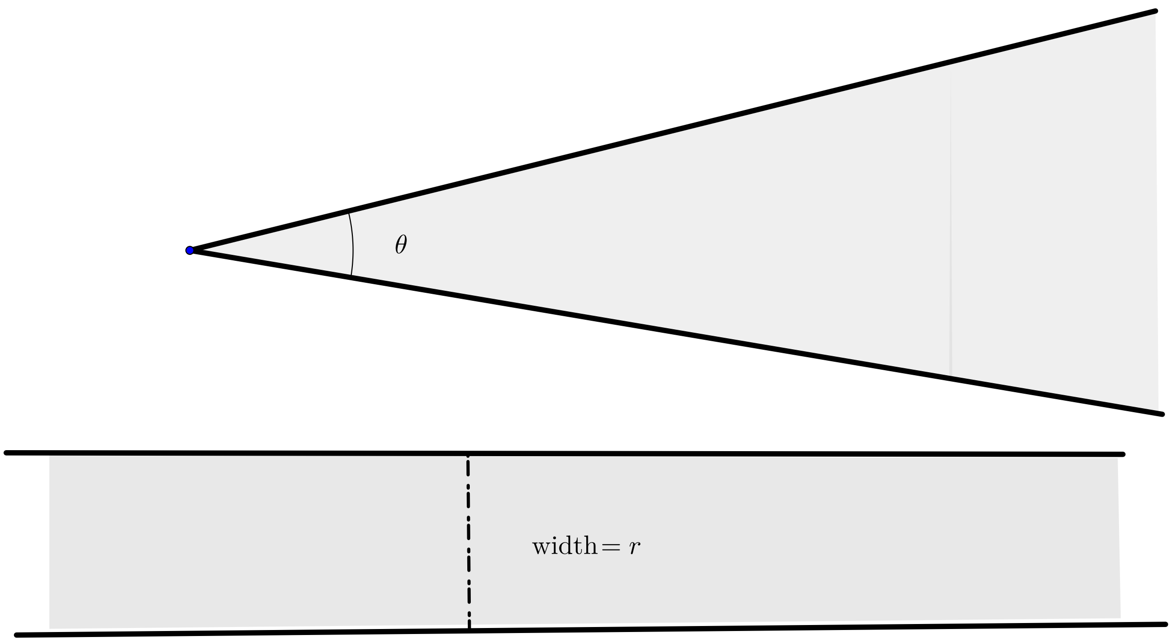

Figure 3. A section of a cone of angle θ and a section of a cylinder of width r.

Proposition 4. *The following statements hold:*

1. *Let d be a complete cone metric on a 1-punctured and 1-labeled DL sphere S^L. S^L is isometric to C_θ, for some $\theta > 0$.*

2. *Let d be a complete cone metric on the 2-punctured DL sphere S^L. S^L is isometric to $C_{0,r}$, for some $r > 0$.*

3. *Let d be a complete cone metric on the 2-punctured DL disk S^L, where the punctures are on the boundary. S^L is isometric to $I(r)$, for some $r > 0$.*

4. *Let d be a complete cone metric on the a DL disk S^L with one punctured and one labeled boundary points. S^L is isometric to a cut of a cylinder V_θ, for some $\theta > 0$.*

Proposition 5. *The following statements hold:*

1. *Two cones C_θ and $C_{\theta'}$ are isometric if and only if $\theta = \theta'$,*

2. *Two sections of cones V_θ and $V_{\theta'}$ are isometric if and only if $\theta = \theta'$,*

3. *Two sections of cylinders $I(r)$ and $I(r')$ are isometric if and only if $r = r'$,*

4. *Two cylinders $C_{0,r}$ and $C_{0,r'}$ are isometric if and only if $r = r'$.*

2.3.1. Cone metrics on disk with one punctured and two labeled boundary points

Our next objective is to classify cone metrics on a DL disk S^L with 1 punctured and 2 labeled boundary points. See Figure 4. Let x, y be the labeled points of the boundary and g be the part of the boundary which connects x and y. Note that we want to classify cone metrics up to isometries which fix x and y.

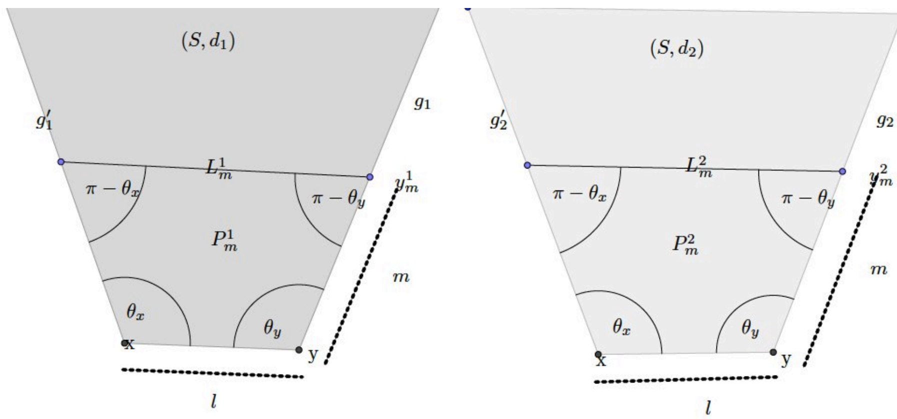

Figure 4.

Proposition 6. *1. For each triple of positive numbers (θ_x, θ_y, l), $\theta_x + \theta_y \geq \pi$, there exists a complete cone metric on S^L so that the angle at x is θ_x, the angle at y is θ_y and length of g is l.*

2. Each complete cone metric on S^L is uniquely determined by its angles and length of g.

Proof

1. There are two cases to be considered separately.

   (a) Assume one of θ_x and θ_y is less than or equal to $\frac{\pi}{2}$. Without loss of generality, let this be θ_x. Therefore, $\theta_y \geq \frac{\pi}{2}$. If $\theta_x = \theta_y = \frac{\pi}{2}$, we know that there exists such a region in the Euclidean plane. If not, form a complete cone metric on a 2-labeled 1-punctured disk with cone angles at θ_x, $\pi - \theta_x$ and length of the segment joining labeled points is l. Indeed, such a surface can be drawn in plane. Let us denote the vertices of the surface with angles θ_x and $\pi - \theta_y$ by x and y, respectively. Let us denote the half line on the surface originating from y as L_y. Now, take a section of a cone of angle $\theta_y - \pi + \theta_x$. Glue one of the boundaries of the section of the cone with L_y. The resulting surface has the properties that we want.

   (b) Assume $\theta_x, \theta_y > \frac{\pi}{2}$. By the first part, there exists a complete cone metric on a 2-labeled and 1-punctured disk with cone angles $\frac{\pi}{2}$, and θ_y and the length of the segment joining these labeled points is l. Call the vertices on this surface with angles $\frac{\pi}{2}$ and θ_y as x and y, respectively. Let us denote the half line on this surface originating from x by L_x. Glue one of the boundary geodesics of the cut of a cone of angle $\theta_x - \frac{\pi}{2}$ with the L_x. The resulting metric has the desired properties.

2. Take two complete cone metrics d_1, d_2 on S^L with the same angle and length data, (θ_x, θ_y, l). We will consider the following two cases separately:

   (a) For each $i = 1, 2$, let g_i be the half line on the boundary of surface which is based at y, with respect to d_i. For each $i = 1, 2$, let g_i' be the half line on the boundary of surface which is based at y, with respect to d_i. Let $y_m^i, m \in \mathbb{N}$, be the point on g_i whose distance with y is m, with respect to d_i. Let L_m^i be the line segment joining y_m^i with g_i' so that the angle between L_m^i and the line segment $[y, y_m^i]$ is $\pi - \theta_y$. If we cut (S, d_i) through L_m^i, we will get convex polygons, P_m^i, for each $i = 1, 2$

and for each $m \in \mathbb{N}$, which are evidently isometric. For each $i = 1, 2$, $\cup_{m\in\mathbb{N}} P^i_m = S_{\mathfrak{p},\mathfrak{p}'}$; therefore, (S, d_1) and (S, d_2) are isometric. See Figure 4.

(b) If one of θ_x and θ_y is greater than or equal to π, one can cut both of the cone metrics through half-lines originating from x and y to reduce the problem to the previous case. We omit the details.

S^L together with such a cone metric will be denoted by $D(\theta_1, \theta_2, l)$.

Remark 3. *There is no complete cone metric on S^L having angle data (θ_1, θ_2) so that $\theta_1 + \theta_2 < \pi$.*

Remark 4. *Assume that $\theta_1 + \theta_2 \geq \pi$. For each positive real number r, $D_{\bar{\kappa}}(l)$ can be triangulated so that*

1. *The length of edges of triangles lying in half-lines of the boundary is r.*
2. *The triangulation satisfies properties in Definition 2.*
3. *The metric obtained by triangulation is the exactly that of $D_{\bar{\kappa}}(l)$.*

One can manage to do this by decomposing $D_{\bar{\kappa}}(l)$ as in the proof of Proposition 6.

2.4. Cone metrics on the closed disk with one punctured interior or one punctured boundary point

Let $D^L_{1,n}$ be a DL disk with one punctured point at its interior and n labeled points on its boundary so that $\mathfrak{p}'$ and $\mathfrak{l}$ are empty. Similarly, let $\bar{D}^L_{1,n}$ be a DL disk with one punctured and n labeled boundary points so that $\mathfrak{p}, \mathfrak{l}$ are empty. The aim of this section is to give a complete classification of cone metrics of nonpositive curvature on $D^L_{1,n}$. It turns out that the length and the curvature data on the boundary of such a disk explicitly describe the cone metric. We also give a similar result for the case of $\bar{D}^L_{1,n}$.

Lemma 4. *Consider a complete cone metric on $D^L_{1,n}$ and a boundary point x. Assume that curvature at each boundary point is nonpositive. Let g be a geodesic starting at x and pointing the interior of $D^L_{1,n}$. g does not hit the boundary and it is not self intersecting.*

Proof If curvature at each boundary point is 0, then $D^L_{1,n}$ is isometric to half of a cylinder and the statement is true. Assume that this is not the case. Observe that a geodesic with above properties cannot intersect itself without winding once around the puncture. Otherwise, we get a disk with only one singular point, and this singular point is on the boundary. Clearly, such a disk cannot exist. Assume that it intersects the boundary or itself.

There are two cases to be considered. First, consider the case in which the geodesic intersects the boundary *before* it intersects itself. In that case, some part of the geodesic and the boundary form a polygon which has at most two vertices, having angles less than π. Vertices at the intersection of the geodesic with the boundary, and all the other vertices have angles bigger than or equal to π. By the Gauss–Bonnet theorem for compact surfaces with boundary, such a polygon does not exist [14].

Second, assume that the geodesic *first* intersects itself. We can cut S^L through the loop formed by the geodesic and obtain a cone metric on a closed annulus. Total curvature for the boundary component of the annulus, which results from the boundary of $D^L_{1,n}$, is negative. Total curvature for the other boundary

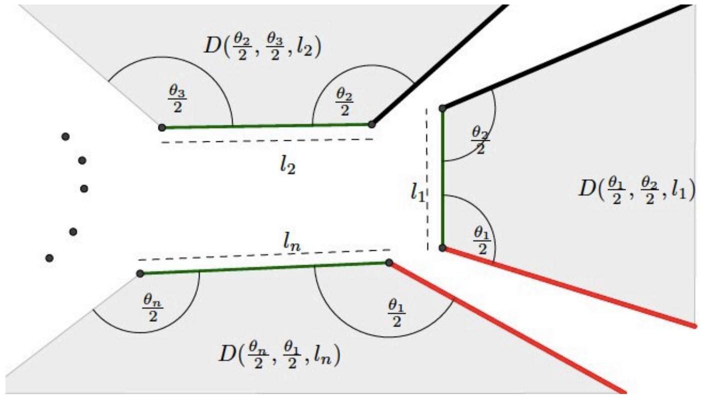

Figure 5. Complete cone metrics on once punctured disk can be obtained from cone metrics on the disk with one puncture on its boundary.

component is nonpositive. Indeed, it contains at most one singular point which has nonpositive curvature. This contradicts with the Gauss–Bonnet theorem since such an annulus should have *zero* total curvature [14].

We point out that geodesics on disks above tend to the punctured point, or the *point at infinity of the disk*. Let $b_1, b_2, \dots b_n$ be the labeled points given in a cyclic order on the boundary. Let κ_i and l_i, $i = 1, \dots, n$, be real numbers so that $\kappa_i < 0$ and $l_i > 0$ for each i.

Lemma 5. *There exists a complete cone metric on $D^L_{1,n}$ so that for each $i = 1, 2 \dots, n$ the curvature at b_i is κ_i and the length of $[b_i, b_{i+1}]$ is l_i.*

Proof Let $\theta_i = \pi - \kappa_i$. Consider $D(\frac{\theta_1}{2}, \frac{\theta_2}{2}, l_1), \dots, D(\frac{\theta_n}{2}, \frac{\theta_1}{2}, l_n)$. For $i < n-1$, glue $D(\frac{\theta_i}{2}, \frac{\theta_{i+1}}{2}, l_i)$, along the geodesic originating from the vertex having angle $\frac{\theta_{i+1}}{2}$, with $D(\frac{\theta_{i+1}}{2}, \frac{\theta_{i+2}}{2}, l_{i+1})$, along the geodesic originating from the vertex having angle $\frac{\theta_{i+1}}{2}$. Do the same for $D(\frac{\theta_n}{2}, \frac{\theta_1}{2}, l_n)$ and $D(\frac{\theta_1}{2}, \frac{\theta_2}{2}, l_1)$. One will get a metric of the desired type. See Figure 5

Remark 5. *Assume that $D^L_{1,n}$ has a complete metric with curvature at each boundary point less than or equal to 0. Let x and y be two distinct boundary points so that there is a straight boundary segment joining them. Take two half-lines originating from x and y which are perpendicular to the segment considered. The Gauss–Bonnet theorem implies that these two half-lines do not intersect.*

Lemma 6. *Consider $D^L_{1,1}$. Let κ_1 and l_1 be as above. Then there is a unique cone metric on $D^L_{1,1}$ having curvature κ_1 at b_1 and the length of the boundary is l_1.*

Proof We proved the existence of the metric. See Lemma 5. Take such a metric on $D^L_{1,1}$. Let $\theta = \pi - \kappa_1$. If we cut $D^L_{1,1}$ through the geodesic making an angle $\frac{\theta}{2}$ with the boundary, then the resulting surface is a disk with a punctured and two labeled boundary points. Therefore, it is isometric to $D(\frac{\theta}{2}, \frac{\theta}{2}, l_1)$. If we glue back, we get the metric we started. Thus, any metric with these properties obtained by gluing the half lines of the boundary of $D(\frac{\theta}{2}, \frac{\theta}{2}, l)$. Hence, there exists a unique metric having properties stated in the present lemma.

Theorem 1. *Let $\kappa_i < 0$ and $l_i > 0$, $i = 1, \ldots, n$, be real numbers. There is a unique complete cone metric on $D^L_{1,n}$, up to isometries respecting labeling, so that curvature at b_i is κ_i and length of the segment $[b_i, b_{i+1}]$ is l_i for each $i = 1, \ldots, n$.*

Proof We proved the existence of such a metric. See Lemma 5.

Uniqueness

We use induction on number of labeled points to prove the statement. Lemma 6 asserts that the statement is true if number of labeled points is one. Assume that the statement is true for the case that there are n or less labeled points. Let d_1 and d_2 be metrics on $D^L_{1,n+1}$ having same curvature data. Consider the the segment joining b_n and b_{n+1}, call it g. By assumption, g has the same length with respect to two metrics. For each $i = 1, 2$, let g_i and h_i be the half-lines originating from b_n and b_{n+1}, with respect to d_i, so that g_i and h_i are perpendicular to g. Cut $(D^L_{1,n}, d_i)$ through g_i and h_i. For each i, we get two DL surfaces S_i and $D(\frac{\pi}{2}, \frac{\pi}{2}, l)$ where l is the length of the segment g. Glue S_i through the half-lines on the boundary to get complete cone metrics on the disk with n labeled points and one puncture. Call these surfaces, together with induced metrics, (S'_i, d'_i). By induction hypothesis (S'_1, d'_1) and (S'_2, d'_2) are isometric. Thus, metrics on $D^L_{1,n}$ obtained from d'_1 and d'_2 by reversing the cutting and gluing operation above are same. Therefore, these induced metrics should coincide with d_1 and d_2. Hence, $(D^L_{1,n}, d_1)$ and $(D^L_{1,n}, d_2)$ are isometric.

Remark 6. *If a cone metric on $D^L_{1,n}$ is complete, then $\sum_{x \in \mathfrak{l}'} \kappa(x) \leq 0$.*

A DL surface together with the metric having curvature data $\bar{\kappa} = (\kappa_1, \ldots, \kappa_n)$ and length data $\bar{l} = (l_1, \ldots, l_n)$ will be denoted by $D_{\bar{\kappa}}(\bar{l})$, where $\kappa_i < 0, l_i > 0$.

Corollary 1. *$D_{\bar{\kappa}}(\bar{l})$ can be triangulated so that the triangulation has properties in Definition 2 and the induced metric coincides with the metric of $D_{\bar{\kappa}}(\bar{l})$.*

Proof By Theorem 1, $D^L_{1,n}$ can be decomposed into finite numbers of disks of the form $D(\theta_1, \theta_2, l)$. Hence, the result follows from Remark 4.

Corollary 2. *Assume that κ_i and l_i satisfy the above conditions, and also $\kappa_i = \kappa_j = \kappa > -\pi$ and $_i = _j = l$ for all $i, j = 1, \ldots, n$. $D_{\bar{\kappa}}(\bar{l})$ can be embedded in a cone.*

Proof Consider the cone with angle $-n\kappa$. Obviously, there is a compact polygonal part of the cone, homeomorphic to a disk, having the apex as an interior point and n boundary edges of length l, n boundary points of angle $\pi + \kappa$. Closure of the complement of this disk has the same length and curvature data with that of $D_{\bar{\kappa}}(\bar{l})$. The result follows from the uniqueness part of the above theorem.

Example. *Consider the cone metric on $D^L_{1,3}$ obtained by gluing two copies of $D(\frac{5\pi}{6}, \frac{5\pi}{6}, 1)$ and one copy of $D(\frac{5\pi}{6}, \frac{5\pi}{6}, 2)$. $D^L_{1,3}$, together with this metric, cannot be embedded into a cone. Otherwise, by the Gauss–Bonnet formula, this cone would have the angle at its apex equal to 2π. Hence, it would be the Euclidean plane. This embedding produces a triangle in the plane having edge lengths $1, 1, 2$ and angles $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$ on the plane, which does not exist. Also observe that one cannot embed $D^L_{1,3}$ into a cone even after removing any compact set. This means that $D^L_{1,3}$ has an irregular puncture. See Figure 6.*

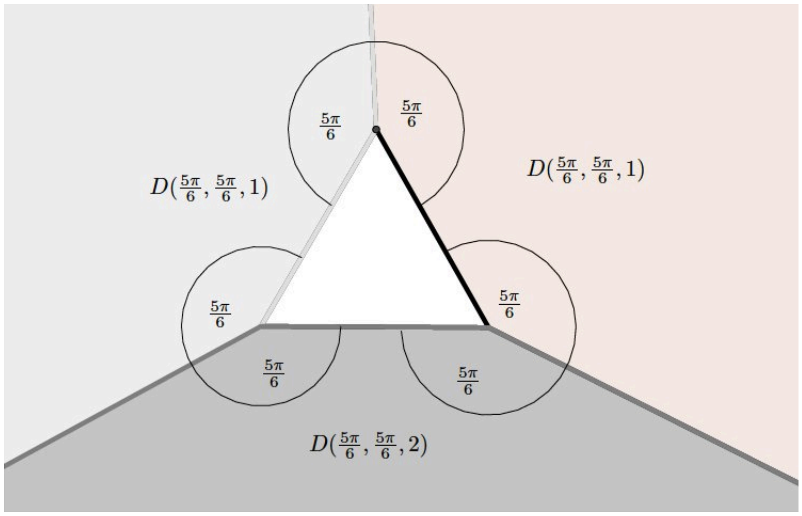

Figure 6. A nonplanar cone metric on once punctured disk. This disk also has an irregular puncture.

Now we state the analogous results for $\bar{D}^L_{1,n}$. We omit the proofs since they are entirely analogous to the proofs of the facts we obtained for complete cone metrics on $D^L_{1,n}$. Assume that $b_1, \dots b_n$ are labeled boundary points so that $b_1, \dots, b_n$ and the punctured point are in a cyclic order in the boundary. Note that this labeling implies that b_1 and b_n share the same edges with the puncture.

Theorem 2. *Assume that we are given numbers $\kappa_1, \dots, \kappa_n$, $n \geq 1$, so that*

- $\kappa_1, \kappa_n < \pi$,
- $\kappa_2, \dots, \kappa_{n-1} < 0$
- $\sum_{i=1}^{n} \kappa_i \leq \pi$

and $_1, \dots, _{n-1}$ so that $l_i > 0$ for each i. There exists a unique complete cone metric on $\bar{D}^L_{1,n}$ so that curvature at b_i is κ_i and length of the segment $[b_i, b_{i+1}]$ is l_i. Also, two cone metrics having the same length and curvature data are isometric.

Remark 7. *If a cone metric on $\bar{D}^L_{1,n}$ is complete, then $\sum_{x \in l'} \kappa(x) \leq \pi$.*

We will denote $\bar{D}^L_{1,n}$ together with such a metric by $\bar{D}_{\bar{\kappa}}, (\bar{l})$, where $\bar{\kappa} = (\kappa_1, \dots, \kappa_n)$, $\bar{l} = (l_1, \dots, l_{n-1})$. Note $\bar{D}^L_{1,1}$ is nothing else than a cut of cone V_θ. See Proposition 4.

Corollary 3. *Let $\bar{\kappa}$, $\bar{l}$ be the curvature and the length data satisfying properties in Theorem 2. $\bar{D}^L_{1,n}$ can be triangulated so that the triangulation satisfies the properties of Definition 2 and the induced cone metric coincides with the metric of $D_{\bar{\kappa}}(\bar{l})$.*

2.5. Modification

We conclude this section by some results about cone metrics on DL disks without labeled interior points.

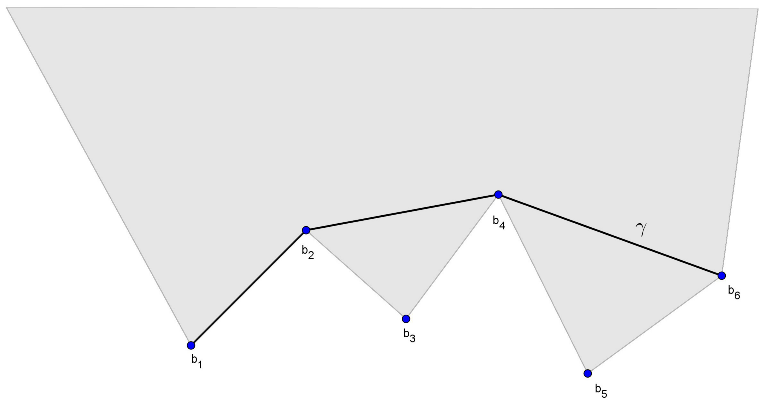

Figure 7. $\bar{D}^L_{1,m}$ and γ. γ is length-minimizing curve joining b_1 with b_m.

Modification: By a modification of a cone metric on a DL closed disk without labeled interior points, we mean the resulting surface (with the induced metric) obtained after recursively cutting finitely many Euclidean triangles which are incident to the boundary at least at one edge. Note that we require the triangles to be incident with the boundary at most at one edge.

Proposition 7. *Every complete cone metric on $\bar{D}^L_{1,n}$ can be modified as follows:*

1. *If $x \in B$, then $\theta(x) < 2\pi$.*
2. *If $x \in$ ľ′ and does not share an edge with the puncture, then $\kappa(x) < 0$.*

Proof First of all, consider $\bar{D}^L_{1,1}$. A complete cone metric on it is nothing else than a cut of cone, and the statement is true if its angle $\theta < 2\pi$. The statement is also is true for $\bar{D}^L_{1,0}$ which is half plane.

Consider a complete flat metric on $\bar{D}^L_{1,n}$, $n \geq 2$, or on $\bar{D}^L_{1,1}$ where the angle at the singular vertex is greater than or equal to 2π.

- If there is a boundary point having an angle greater than or equal to 2π, then we can remove a polygon about it so that resulting singular points have angle less than 2π. Therefore, by removing a polygon about each singular point x such that $\theta(x) \geq 2\pi$, we get a complete DL disk for which boundary points have angle less than 2π.
- If, after the operation above, we get $\bar{D}^L_{1,0}$, $\bar{D}^L_{1,1}$ or $\bar{D}^L_{1,2}$, then we are done. Thus, assume that we get a complete cone metric on $\bar{D}^L_{1,m}$, $m \geq 3$. Let us label its singular points as $b_1, \dots b_m$. See Figure 7. Take a loop joining b_1 with b_n. There is a length-minimizing curve in its homotopy class. See [8]. Call this γ. If we cut $D^L_{1,m}$, then we get a surface of the type we want. If this is not the case, then γ has two edges making the angle less than π. This implies that γ is not length-minimizing. See Figure 8.

There is a similar result for the cone metrics on the closed disk having one punctured interior point. The proof is also similar. We state it and outline its proof.

Lemma 7. *Each complete cone metric on a $D^L_{1,n}, n \geq 2$ can be modified so that the resulting surface has the following properties:*

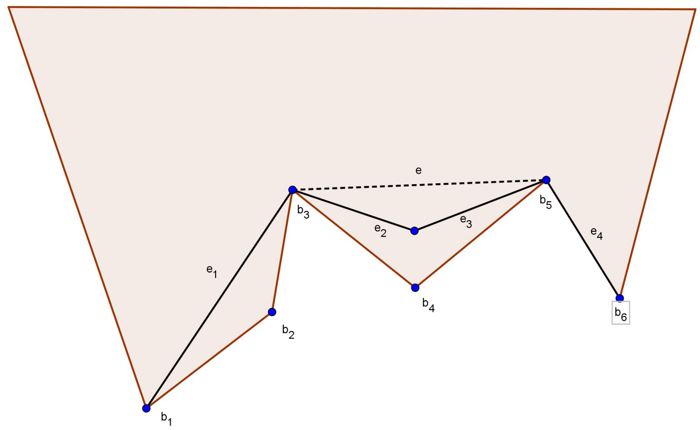

Figure 8. The curve with edges e_1, e_2, e_3, e_4 cannot be length-minimizing since $|e_2| + |e_3| > |e|$.

1. *Each point on the boundary has and angle less than* 2π*.*

2. *There is at most one singular point of positive curvature.*

Proof First, modify the cone metric so that there are no singular points having an angle greater than and equal to 2π. One can do this as in the proof of Proposition 7. Then take a boundary point x and consider a loop based at x and winding once around the puncture. Take a length-minimizing path in the homotopy class of the loop and cut the surface through this path. The resulting cone metric has at most one singular point of positive curvature, x.

Proposition 8. *Let* $D^L_{1,n}$ *be an FDL surface satisfying the following conditions:*

1. *It has one singular point of positive curvature,*

2. *Each boundary point has an angle less than* 2π*,*

then $D^L_{1,n}$ *has a modification so that for each boundary point* x*,* $\pi \leq \theta(x) < 2\pi$*.*

Proof Since total curvature at the boundary of $D^L_{1,n}$ is nonpositive, $n \neq 0, 1$. Assume that $n = 2$. Let p be the singular point with positive curvature. Take a length-minimizing loop which is based at p and winds once around boundary. If we cut the disk through this loop, we get a disk at most one singular point, p. The curvature at p is not positive, since modification does no change the total curvature. Also, it is clear that the angle at p is less than 2π.

We do induction on number of singular points. Consider a flat metric on $D^L_{1,n}$, $n \geq 3$. We denote the singular point with positive curvature by p, and singular points which share an *edge* with p by q and r. See Figure 9. In this figure, θ is the angle at p, $\alpha = \pi + \kappa(q)$, $\beta = \pi + \kappa(r)$. There are two cases to be considered:

a) $\theta < \alpha + \beta$. In this case, we can extend the edges E and F to form the quadrangle Q. See left of Figure 9. If we remove Q, we get a surface with at most one singular point of positive curvature, and it is clear that the number of singular points of this surface is less than n.

b) $\theta \geq \alpha + \beta$. Draw a line segment joining q and r to form a triangle. Call the segment G and the triangle T.

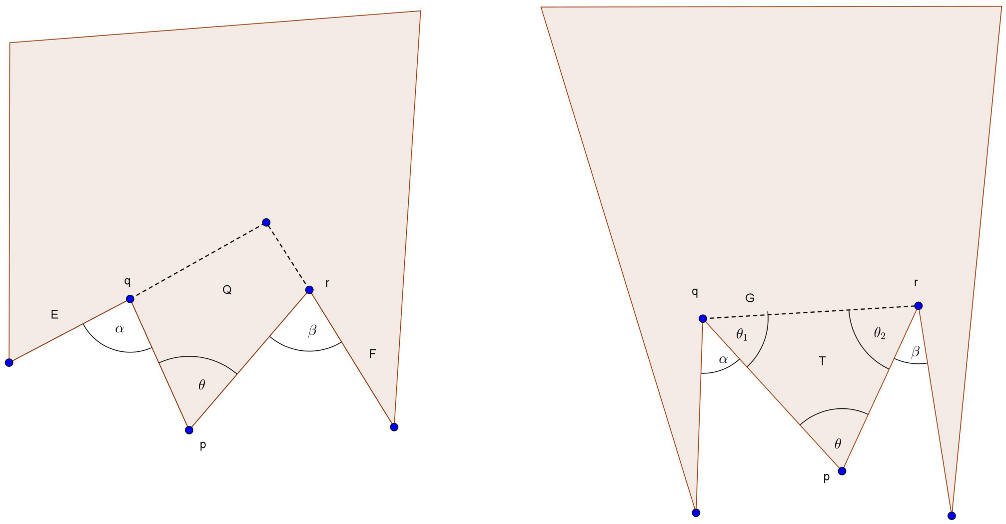

Figure 9. If $\theta < \alpha + \beta$, then we remove the quadrilateral Q to obtain the desired modification. See left of the figure. If $\theta \geq \alpha + \beta$, then we remove the triangle T to obtain the desired modification. See right of the figure.

See right of Figure 9. Let θ_1 and θ_2 be the angles at q and r, respectively. Since $\pi - \theta_1 - \theta_2 = \theta$, we have

$$\pi - \theta_1 - \theta_2 \geq \alpha + \beta,$$

$$\pi \geq \alpha + \theta_1 + \beta + \theta_2.$$

Therefore, one of $\alpha + \theta_1$ and $\beta + \theta_2$ is less than π. This means that when we remove the triangle, we reduce the number of singular points and the resulting surface has at most one singular point of positive curvature.

Corollary 4. *Each complete cone metric on $D^L_{1,n}$, $n \geq 0$, can be modified so that the resulting disk does not have points with positive curvature on its boundary.*

Proof The statement immediately follows from Lemma 7 and Proposition 8

3. DL surfaces with complete cone metrics can be triangulated

In this section, we prove that DL surfaces together with complete cone metrics can be triangulated so that the resulting metric coincides with the given one. This theorem is well-known for compact surfaces [14]. Our strategy is to cut such a surface around its punctures and reduce the problem to the cases for compact surfaces and the surfaces $\bar{D}(\bar{\kappa}, \bar{l}), D(\bar{\kappa}, \bar{l})$.

Theorem 3. *Every DL surface S^L together with a complete metric d can be triangulated as in Definition 2 so that the resulting metric coincides with d.*

Proof For each point $p \in \mathfrak{p}$, take a nonself intersecting polygonal loop around p such that the punctured disk bounded by p and the loop has no labeled point on its interior and no punctured point on it except p. For each point $p' \in \mathfrak{p}'$, take a polygonal path joining to half-lines *incident* p' such that the punctured disk bounded by p' and the path has no labeled points on its interior and no punctured point on it except p'. Also observe that we can choose these loops and paths so that the resulting disks are pairwise disjoint. Note that we may assume that these disks satisfy properties in Proposition 7 and Corollary 4. Now we know that these disks can be triangulated nicely. See Definition2 and Corollaries 1, 3. If we remove interiors of these disks, what we get is a compact surface together with a cone metric. It is well known that such a surface can be triangulated with only finitely many triangles. Therefore, we can use triangulations on these pieces to obtain a triangulation on S^L. This triangulation has the properties in Definition 2 and metric d coincides with the metric induced by the triangulation at each triangle. Hence, these metrics coincide globally.

4. The Gauss–Bonnet formula

The Gauss–Bonnet formula for compact flat surface is well-known. There is a variant of the formula for the noncompact case. However, it preassumes that each punctured interior point has a neighborhood isometric to a neighborhood of point at infinity of a cone. See [14, 16].

We start with defining curvature at the punctures of a DL surface with a complete cone metric. Then we will state and prove the Gauss–Bonnet formula.

Remark 8. *A modification of a complete cone metric on $D^L_{1,n}$ does not change the total curvature of the boundary of $D^L_{1,n}$. Similarly, a modification of a complete cone metric on $\bar{D}^L_{1,n}$ does not change the total curvature of the boundary of $\bar{D}^L_{1,n}$.*

Definition 10. *1. If $\bar{D}^L_{1,n}$ has a complete flat metric, then the curvature at its puncture p' is defined as*

$$\kappa(p') = 2\pi - \sum_{x \in \mathfrak{l}'} \kappa(x).$$

The angle at p' is $\theta(p') = \pi - \kappa(p')$.

2. If $D^L_{1,n}$ has a complete flat metric, then the curvature at its puncture p is defined as

$$\kappa(p) = 2\pi - \sum_{x \in \mathfrak{l}'} \kappa(x).$$

The angle at p is $\theta(p) = 2\pi - \kappa(p)$.

3. Let S^L be a DL surface (together with a complete cone metric) and $p \in \mathfrak{p}$. The curvature at p, $\kappa(p)$, is the curvature of p as a punctured point of a disk in S^L containing p and having no singular points on its interior. The angle at p is $\theta(p) = 2\pi - \kappa(p)$.

4. Let S'^L be a DL surface (together with a complete cone metric) and $p' \in \mathfrak{p}'$. The curvature at p', $\kappa(p')$, is the curvature of p' as a punctured point of a disk in S'^L containing p' and having no singular points on its interior. The angle at p' is $\theta(p') = \pi - \kappa(p')$.

Remark 9. *By Remark 8, the last two items of the above definition make sense. Any two such disks containing p can be modified to a common disk and hence have the same total curvature at their boundaries.*

Theorem 4 (the Gauss–Bonnet formula)**.** *Let S^L be a DL surface together with a complete cone metric. The following formula holds:*

$$\sum_{x \in S} \kappa(x) = 2\pi\chi(S). \tag{8}$$

Proof Assume that S^L has n punctured points on its interior and m punctured points on its boundary. As in the proof of Theorem 3, choose *disks* around the punctures. Let S' be the compact surface, with induced metric, obtained by removing these disks. Observe that

- $\chi(S') = \chi(S) - n$, and
- $\sum_{y \in S'} \kappa(y) = 2\pi\chi(S')$

by the Gauss-Bonnet Formula for compact surfaces. Now, observe that removing one appropriate disk around a punctured interior point decreases the total curvature 2π. Therefore, if we remove n such disks, the total curvature decreases $2n\pi$. Also, observe that removing an appropriate disk around a punctured boundary point does not change the total curvature. Therefore, we have

$$\sum_{x \in S} \kappa(x) = 2n\pi + \sum_{y \in S'} \kappa(y) = 2n\pi + 2\pi\chi(S') = 2\pi\chi(S).$$

5. Existence of geodesic representatives in free homotopy classes of loops

It is well known that any loop in any compact flat surface has a length-minimizing closed geodesic representative in its *free* homotopy class [8]. Recall that by a length-minimizing closed geodesic, we mean a closed geodesic which has a length less than or equal to the length of each curve in its free homotopy class. We prove that this property is valid for any FDL surfaces. The idea of our proof is to cut the surface through the punctures and reduce the problem to the case of compact surfaces. We start with some observations.

Remark 10. *Let S^L be an FDL surface so that $\mathfrak{p}, \mathfrak{p}'$ are empty. Assume that it has a boundary component of nonnegative curvature. Any geodesic loop in S^L either lies in this boundary component or does not intersect with this component. See Figure 10. If the loop lies in this boundary component, then this component is nonsingular; each point at this component has zero curvature.*

Remark 11. *Let S^L be an FDL surface. As in the proof of Proposition 7 and Corollary 4, cut S^L through disks around its punctures so that at each point of each resulting boundary component, curvature is nonnegative. Let $\mathfrak{D}$ be the resulting compact surface. For each loop L in S^L which intersects such a component, there exists a loop in its homotopy class which has a length less than or equal to the length of L and lies in $\mathfrak{D}$. See Figure 11. The part of the loop L which does not lie in $\mathfrak{D}$ has a length greater than $|[a,v]| + |[v,b]|$. This happens since the boundary points of $\mathfrak{D}$ has nonnegative curvature.*

Theorem 5. *Given an FDL surface S^L and a loop L on it, there exists a length- minimizing geodesic on its free homotopy class.*

Proof We cut the surface around its punctures as in Lemma 7 and Corollary 4 so that the resulting disks do not intersect L. Thus, the part left is a compact surface $\mathfrak{D}$ containing L, and there exists a length-minimizing geodesic g in the homotopy class of L in $\mathfrak{D}$. Since the curvature at each point of a *resulting*

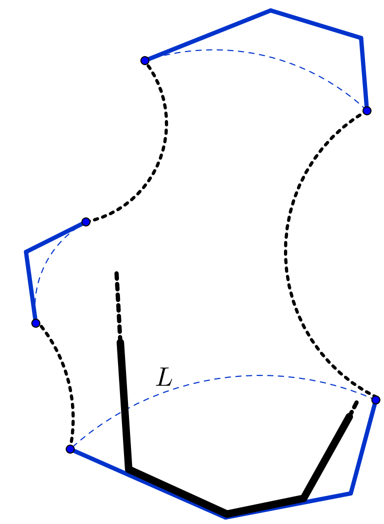

Figure 10. A sphere with 3 boundary components where the boundary components are in blue. The loop L, thick black one, is not a geodesic since it is not length-minimizing.

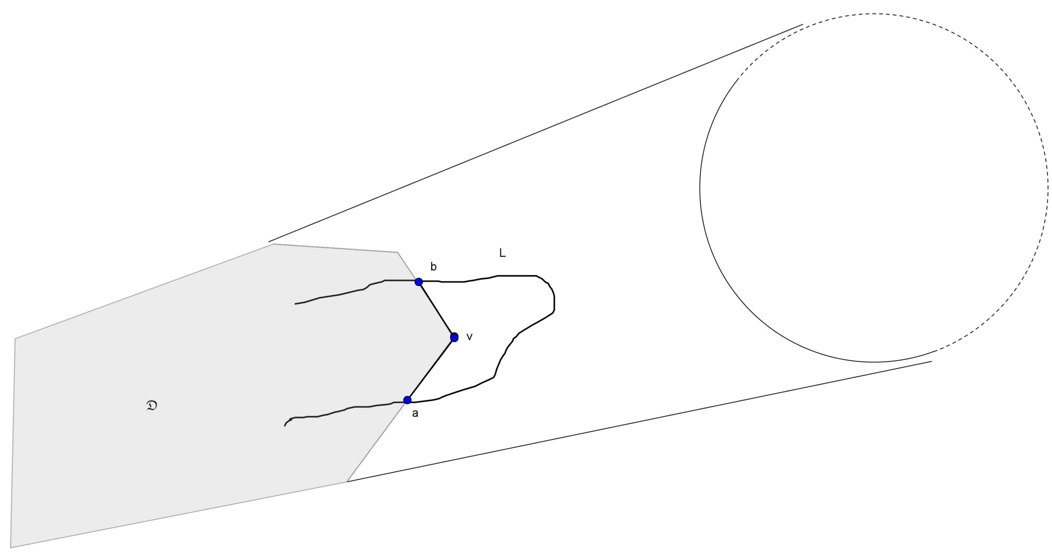

Figure 11. Since boundary points of $\mathfrak{D}$ have nonnegative curvature, a loop which intersects with D and its complement cannot be length-minimizing.

boundary component is nonnegative, we see that either g lies in such a boundary component and this boundary component is nonsingular, or it does not intersect such a boundary component. See Remark 10. This shows that g is indeed a geodesic in S^L and it is in homotopy class of L.

Assume that there is a geodesic g' in homotopy class of g whose length is less than the length of g. It follows that g' does not lie completely in $\mathfrak{D}$, and Remark 11 implies that there is a shorter loop g'' in the same homotopy class which lies in $\mathfrak{D}$. This implies that the length of g'' is greater than or equal to the length of g, which is a contradiction.

Acknowledgments

I am really grateful to Muhammed Uludağ for his suggestions. Also, I thank Deniz Kutluay for reading the first version of the present manuscript. I am also grateful to Ayberk Zeytin and Susumu Tanabe for their comments.

References

[1] Allcock D. The Leech lattice and complex hyperbolic reflections. Inventiones Mathematicae 2000; 140 (2): 283-301.

[2] Bavard C, Ghys É. Polygones du plan et polyedres hyperboliques. Geometriae Dedicata 1992; 43 (2): 207-224.

[3] Burago D, Burago Y, Ivanov S. A Course in Metric Geometry. Providence, RI, USA: American Mathematical Society, 2001.

[4] Delign P, Mostow GD., Monodromy of hypergeometric functions and non-lattice integral monodromy. Publications Mathématiques de l'IHÉS, 1986; 63 (1): 5-89.

[5] Farb B, Margalit D. A Primer on Mapping Class Groups. Princeton, NJ, USA: Princeton University Press, 2012.

[6] Fillastre F. From spaces of polygons to spaces of polyhedra following Bavard, Ghys and Thurston. Enseign Math 2011; 57 (2): 23-56.

[7] Fillastre F, Izmestiev I. Shapes of polyhedra, mixed volumes, and hyperbolic geometry. Mathematika 2017; 63 (1): 124-183.

[8] Gromov M. Metric structures for Riemannian and non-Riemannian spaces, Springer Science and Business Media, 2007.

[9] Hulin D, Troyanov M. Prescribing curvature on open surfaces. Mathematische Annalen 1992; 293(1): 277-315.

[10] Masur H, Tabachnikov S. Rational billiards and flat structures. In: Hasselblatt B, Katok A,editors. Handbook of dynamical systems 1. Amsterdam, Netherlands: Elsevier Science, 2002, pp. 1015-1089.

[11] Rivin I. Euclidean structures on simplicial surfaces and hyperbolic volume. Annals of Mathematics 1994; 139(3): 553-580.

[12] Thurston WP. Shapes of polyhedra and triangulations of the sphere, Geometry and Topology monographs, 1998; 1: 511-549.

[13] Troyanov M. Les surfaces euclidiennes à singularités coniques. Ens. Math. 1986; 32: 79-94.

[14] Troyanov M. Prescribing curvature on compact surfaces with conical singularities. Transactions of the American Mathematical Society 1991; 324(2): 793-821.

[15] Troyanov M. On the moduli space of singular Euclidean surfaces. In: Athanase Papadopoulos, editor. Handbook of Teichmüller Theory 1. Zurich, Switzerland: European Mathematical Society 2007, pp. 507-540.

[16] Hulin D, Troyanov M. Prescribing curvature on open surfaces. Mathematische Annalen 1992; 293.1: 277-315.

[17] Uludağ AM and Sağlam İ. Hypergeometric Galois Actions.In: Athanase Papadopoulos, editor. Handbook of Teichmuller Theory 6. Zurich, Switzerland: European Mathematical Society, 2016, pp 467-500.

[18] Uludağ AM, Zeytin A. A panaroma of the fundamental group of the modular orbifold. In: Athanase Papadopoulos, editor. Handbook of Teichmüller Theory 6. Zurich, Switzerland: European Mathematical Society, 2016, pp. 501-519.

[19] Zeytin A, Uludağ AM. Quadrangulations of sphere and ball quotients, Mathematische Nachrichten, 2014; 287(1) 105-121.