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Jörgensen's inequality and purely loxodromic two-generator free Kleinian groups

İlker Savaş YÜCE*
Department of Mathematics, Faculty of Art and Sciences, Yeditepe University, İstanbul, Turkey

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Abstract: Let ξ and η be two noncommuting isometries of the hyperbolic 3-space $\mathbb{H}^3$ so that $\Gamma = \langle \xi, \eta \rangle$ is a purely loxodromic free Kleinian group. For $\gamma \in \Gamma$ and $z \in \mathbb{H}^3$, let $d_\gamma z$ denote the hyperbolic distance between z and $\gamma(z)$. Let z_1 and z_2 be the midpoints of the shortest geodesic segments connecting the axis of ξ to the axes of $\eta\xi\eta^{-1}$ and $\eta^{-1}\xi\eta$, respectively. In this manuscript, it is proved that if $d_\gamma z_2 < 1.6068...$ for every $\gamma \in \{\eta, \xi^{-1}\eta\xi, \xi\eta\xi^{-1}\}$ and $d_{\eta\xi\eta^{-1}} z_2 \leq d_{\eta\xi\eta^{-1}} z_1$, then $|\text{trace}^2(\xi) - 4| + |\text{trace}(\xi\eta\xi^{-1}\eta^{-1}) - 2| \geq 2\sinh^2\left(\frac{1}{4}\log\alpha\right) = 1.5937....$ Above $\alpha = 24.8692...$ is the unique real root of the polynomial $21x^4 - 496x^3 - 654x^2 + 24x + 81$ that is greater than 9. Generalizations of this inequality for finitely generated purely loxodromic free Kleinian groups are also proposed.

Key words: Free Kleinian groups, Jörgensen's inequality, the $\log 3$ theorem, loxodromic isometries, hyperbolic displacements

1. Introduction

A Kleinian group Γ is a nonelementary discrete subgroup of the group $\text{PSL}(2,\mathbb{C})$ of orientation-preserving isometries of the hyperbolic 3-space $\mathbb{H}^3$. Any orientable hyperbolic 3-manifold M can be viewed as a quotient $\mathbb{H}^3/\Gamma$ for a Kleinian group Γ. By Mostow's rigidity [16], this reduces the study of hyperbolic 3-manifolds to the study of Kleinian groups. This, in turn, makes the investigation of criteria for discreteness of the subgroups of $\text{PSL}(2,\mathbb{C})$ one of the main topics of interest in the theory of 3-dimensional hyperbolic manifolds.

It was proved by Jørgensen [12] that $\Gamma \leq \text{PSL}(2,\mathbb{C})$ is discrete if and only if every nonelementary two-generator subgroup of Γ is discrete. Accordingly, significant progress in the literature has occurred since then towards a resolution of the discreteness problem for subgroups of $\text{PSL}(2,\mathbb{C})$ through the examination of two-generator subgroups (see [7], [11], [10], [14], [17] and the references therein). A particularly remarkable result was presented by Gilman in [8] with an algorithm for deciding the discreteness of the subgroups of $\text{PSL}(2,\mathbb{R})$. In this paper, we will concentrate on two-generator purely loxodromic free subgroups of $\text{PSL}(2,\mathbb{C})$ and provide some necessary discreteness criteria for these groups satisfying certain conditions. Furthermore, we will suggest discreteness criteria for finitely generated such groups.

There is a large class of subgroups of $\text{PSL}(2,\mathbb{C})$ in the aforementioned category. In fact, all finitely generated Schottky groups are purely loxodromic and free [13, H.2.Proposition]. However, the main motivation behind this text for focusing on these particular subgroups of $\text{PSL}(2,\mathbb{C})$ is that every two-generator subgroup

*Correspondence: ilker.yuce@yeditepe.edu.tr
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of the fundamental group $\pi_1(M)$ of an orientable closed hyperbolic 3-manifold M is purely loxodromic and free provided that the first Betti number of M is at least 3 (see [4, Propositions 9.2 and 10.2]). Culler and Shalen used this fact to show that the volume of M is at least 0.92 (see [4, Theorem 10.3]), connecting the geometry of such hyperbolic 3-manifolds to their topology. This volume bound, later superseded by Gabai et al. [6] and Milley [15] by the introduction of Mom technology, is calculated by computing the lower bound $\log 3$ for the maximum of the hyperbolic displacements given by the generators of two-generator subgroups of $\pi_1(M)$ [4]. The statement in [4, Theorem 9.1] in which the lower bound $\log 3$ is computed is known in the literature as the $\log 3$ theorem.

Due to an extension introduced in [18, 19] by the author, the techniques developed by Culler and Shalen in the proof of the $\log 3$ theorem can be used to calculate a lower bound for the maximum of the hyperbolic displacements under any finite set of isometries in a purely loxodromic finitely generated free Kleinian group Γ. In particular, in the case of two-generator, e.g., if $\Gamma = \langle \xi, \eta \rangle$, it is possible to compute a lower bound for the maximum of the hyperbolic displacements given by the set Γ_* of isometries

$$\{1\} \cup \Gamma_1 \cup \{\xi\eta\xi^{-1}, \xi^{-1}\eta\xi, \eta\xi\eta^{-1}, \eta^{-1}\xi\eta, \xi\eta^{-1}\xi^{-1}, \xi^{-1}\eta^{-1}\xi, \eta\xi^{-1}\eta^{-1}, \eta^{-1}\xi^{-1}\eta\}, \tag{1.1}$$

where $\Gamma_1 = \{\xi, \eta, \eta^{-1}, \xi^{-1}\}$. Explicitly, in Section 4, we shall prove the statement below:

Theorem 1.1 *Suppose that $\Gamma = \langle \xi, \eta \rangle$ is a purely loxodromic free Kleinian group. Then, for Γ_* in (1.1), we have $\max_{\gamma \in \Gamma_*}\{d_\gamma z\} \geq 1.6068...$ for any $z \in \mathbb{H}^3$.*

This theorem leads to a reversal of the roles of trace and hyperbolic displacements in the statement of the following theorem of Beardon [1, Theorem 5.4.5]:

Theorem 1.2 *If $\langle \xi, \eta \rangle$ is a Kleinian group so that ξ is elliptic or strictly loxodromic and $|\mathrm{trace}^2(\xi) - 4| < \frac{1}{4}$, then for any z in $\mathbb{H}^3$ we have $\max\{\sinh(\frac{1}{2}d_\xi z),\ \sinh(\frac{1}{2}d_{\eta\xi\eta^{-1}} z)\} \geq \frac{1}{4}$.*

In other words, we will show in Section 4 that Theorem 1.1 implies the main result of this paper, which can be stated as follows:

Theorem 1.3 *If $d_\gamma z_2 < 1.6068...$ for $\gamma \in \Phi_1 = \{\eta, \xi^{-1}\eta\xi, \xi\eta\xi^{-1}\}$ and $d_{\eta\xi\eta^{-1}} z_2 \leq d_{\eta\xi\eta^{-1}} z_1$, then we have $|\mathrm{trace}^2(\xi) - 4| + |\mathrm{trace}(\xi\eta\xi^{-1}\eta^{-1}) - 2| \geq 1.5937...$.*

Above z_1 and z_2 denote the midpoints of the shortest geodesic segments connecting the axis of ξ to the axes of $\eta\xi\eta^{-1}$ and $\eta^{-1}\xi\eta$, respectively. Theorems 1.1 and 1.3 are restated as Theorems 4.2 and 4.3, respectively, in Section 4. The expressions $\mathrm{trace}^2(\xi)$ and $\mathrm{trace}(\xi\eta\xi^{-1}\eta^{-1})$ are used in place of $\mathrm{trace}^2(A)$ and $\mathrm{trace}(ABA^{-1}B^{-1})$, where A represents the loxodromic isometry ξ and B represents the loxodromic isometry η in $\mathrm{PSL}(2, \mathbb{C})$.

Theorem 1.3 can be considered as a refinement of the best general discreteness criterion for the subgroups of $\mathrm{PSL}(2, \mathbb{C})$ for the groups under consideration in this paper. This criterion is due to Jørgensen [12], called the Jørgensen's inequality, given below.

Theorem 1.4 *If $\langle \xi, \eta \rangle$ is a Kleinian group, then $|\mathrm{trace}^2(\xi) - 4| + |\mathrm{trace}(\xi\eta\xi^{-1}\eta^{-1}) - 2| \geq 1$, where the lower bound is the best possible.*

Theorem 1.2 is an implication of Theorem 1.4 (see [1, Theorem 5.4.5]).

In the rest of this section, we will summarize the proofs of Theorems 1.1 and 1.3. In particular, we will introduce some notation and review the Culler–Shalen machinery introduced in [4], which will be used to calculate a lower bound for the maximum of the hyperbolic displacements needed here. The proof of Theorem 1.3 will involve the computations given in the proof of Theorem 5.4.5 in [1], which uses the geometry of the action of loxodromic isometries together with some elementary inequalities involving hyperbolic trigonometric functions. However, most of the technical work in this paper will be required to prove Theorem 1.1.

Let us define Ψ as the set of isometries in $\Gamma = \langle \xi, \eta \rangle$ whose elements are listed and enumerated below:

$$\begin{array}{llllllll}
\xi\eta^{-1}\xi^{-1} & \mapsto 1, & \eta^{-1}\xi^{-1}\eta^{-1} & \mapsto 8, & \eta\xi^{-1}\eta^{-1} & \mapsto 15, & \xi^{-1}\eta^{-1}\xi^{-1} & \mapsto 22,\\
\xi\eta^{-1}\xi & \mapsto 2, & \eta^{-1}\xi^{-1}\eta & \mapsto 9, & \eta\xi^{-1}\eta & \mapsto 16, & \xi^{-1}\eta^{-1}\xi & \mapsto 23,\\
\xi\eta^{-2} & \mapsto 3, & \eta^{-1}\xi^{-2} & \mapsto 10, & \eta\xi^{-2} & \mapsto 17, & \xi^{-1}\eta^{-2} & \mapsto 24,\\
\xi\eta^{2} & \mapsto 4, & \eta^{-1}\xi^{2} & \mapsto 11, & \eta\xi^{2} & \mapsto 18, & \xi^{-1}\eta^{2} & \mapsto 25,\\
\xi\eta\xi^{-1} & \mapsto 5, & \eta^{-1}\xi\eta^{-1} & \mapsto 12, & \eta\xi\eta^{-1} & \mapsto 19, & \xi^{-1}\eta\xi^{-1} & \mapsto 26,\\
\xi\eta\xi & \mapsto 6, & \eta^{-1}\xi\eta & \mapsto 13, & \eta\xi\eta & \mapsto 20, & \xi^{-1}\eta\xi & \mapsto 27,\\
\xi^{2} & \mapsto 7, & \eta^{-2} & \mapsto 14, & \eta^{2} & \mapsto 21, & \xi^{-2} & \mapsto 28.
\end{array} \tag{1.2}$$

We shall denote this enumeration by $p : \Psi \to \{1, \ldots, 28\}$. Let $\Psi_r = \Gamma_1 = \{\xi, \eta^{-1}, \eta, \xi^{-1}\}$. Since it is assumed that $\Gamma = \langle \xi, \eta \rangle$ is free, it can be decomposed as follows:

$$\Gamma = \{1\} \cup \Psi_r \cup \bigcup_{\psi \in \Psi} J_\psi, \tag{1.3}$$

where J_ψ denotes the set of all words starting with the word $\psi \in \Psi$. We will name this decomposition $\Gamma_{\mathcal{D}^*}$. Let us define $J_\Phi = \cup_{\psi\in\Phi} J_\psi$ for $\Phi \subseteq \Psi$. A group-theoretical relation for a given decomposition of $\Gamma = \langle \xi, \eta \rangle$ is a relation among the sets J_ψ. As an example,

$$\xi\eta\xi^{-1} J_{\xi\eta^{-1}\xi^{-1}} = \Gamma - \left(\{\xi\} \cup J_{\{\xi^2, \xi\eta^{-1}\xi^{-1}, \xi\eta^{-1}\xi, \xi\eta^{-2}, \xi\eta^2, \xi\eta\xi^{-1}, \xi\eta\xi\}}\right) \tag{1.4}$$

is a group-theoretical relation of the decomposition in (1.3), which indicates that when multiplied on the left by $\xi\eta\xi^{-1}$ the set of words in $\Gamma = \langle \xi, \eta \rangle$ starting with $\xi\eta^{-1}\xi^{-1}$ translates into the set of words starting with the words whose initial letters are different than ξ. Isometries in Ψ_r that appear in the relations have no effect in the upcoming computations. Therefore, we shall denote a generic group-theoretical relation of $\Gamma_{\mathcal{D}^*}$ by $(\gamma, s(\gamma), S(\gamma))$, where $\gamma \in \Gamma_*$, $s(\gamma) \in \Psi$, and $S(\gamma) \subset \Psi$. In (1.4) we have

$$\gamma = \xi\eta\xi^{-1},\ s(\gamma) = \xi\eta^{-1}\xi^{-1},\ S(\gamma) = \{\xi^2, \xi\eta^{-1}\xi^{-1}, \xi\eta^{-1}\xi, \xi\eta^{-2}, \xi\eta^2, \xi\eta\xi^{-1}, \xi\eta\xi\}.$$

There are 128 group-theoretical relations for $\Gamma_{\mathcal{D}^*}$ in total, but we will be interested in 60 of them listed in Lemma 2.1 (see Tables 1, 2, 3, and 4) for which $\gamma \in \Gamma_* \subset \Psi_r \cup \Psi$ defined in (1.1). Then we consider the following cases:

(I) when $\Gamma = \langle \xi, \eta \rangle$ is geometrically infinite; that is, $\Lambda_{\Gamma\cdot z} = S_\infty$ for every $z \in \mathbb{H}^3$;

(II) when $\Gamma = \langle \xi, \eta \rangle$ is geometrically finite.

Above the expression S_∞ denotes the boundary of the canonical compactification $\overline{\mathbb{H}^3}$ of $\mathbb{H}^3$. Note that $S_\infty \cong S^2$. The notation $\Lambda_{\Gamma\cdot z}$ means the limit set of the Γ-orbit of $z \in \mathbb{H}^3$ on S_∞. In case (I), we first prove the statement below:

Theorem 1.5 *Let $\Gamma=\langle\xi,\eta\rangle$ be a purely loxodromic, free, and geometrically infinite Kleinian group. Let $\Gamma_{\mathcal{D}^*}$ be the decomposition of Γ in (1.3). If z denotes a point in $\mathbb{H}^3$, then there is a family of Borel measures $\{\nu_\psi\}_{\psi\in\Psi}$ defined on S_∞ such that we have (i) $A_z=\sum_{\psi\in\Psi}\nu_\psi$; (ii) $A_z(S_\infty)=1$; and for $\gamma\in\Gamma_*$*

$$(iii)\quad \int_{S_\infty}(\lambda_{\gamma,z})^2\,d\nu_{s(\gamma)}=1-\sum_{\psi\in S(\gamma)}\int_{S_\infty}d\nu_\psi$$

for all group-theoretical relations $(\gamma,s(\gamma),S(\gamma))$ of $\Gamma_{\mathcal{D}^*}$, where A_z is the area measure on S_∞ based at z.

This theorem basically states that the normalized area measure A_z on the sphere at infinity can be decomposed as a sum of Borel measures ν_ψ indexed by $\psi\in\Psi$ so that each group-theoretical relation of $\Gamma_{\mathcal{D}^*}$ translates into a measure-theoretical relation among the Borel measures $\{\nu_\psi\}_{\psi\in\Psi}$ as described in part (iii) of the theorem. In particular, each measure ν_ψ is transformed to the complement of certain measures in the set $\{\nu_\gamma:\gamma\in\Psi-\{\psi\}\}$. For example, Theorem 1.5 (iii) and the group-theoretical relation given in (1.4) imply that

$$\int_{S_\infty}\lambda^2_{\xi\eta\xi^{-1},z}\,d\nu_{\xi\eta^{-1}\xi^{-1}}=1-\sum\nolimits_{\psi\in\{\xi^2,\xi\eta^{-1}\xi^{-1},\xi\eta^{-1}\xi,\xi\eta^{-2},\xi\eta^2,\xi\eta\xi^{-1},\xi\eta\xi\}}\nu_\psi(S_\infty). \tag{1.5}$$

By a formula proved in [4] and improved in [5] by Culler and Shalen, each hyperbolic displacement $d_\gamma z$ for $\gamma\in\Gamma_*$ has a lower bound involving the Borel measures in $\{\nu_\psi\}_{\psi\in\Psi}$. This formula is given as follows:

Lemma 1.6 [4, Lemma 5.5] [5, Lemma 2.1] *Let a and b be numbers in $[0,1]$ that are not both equal to 0 and are not both equal to 1. Let γ be a loxodromic isometry of $\mathbb{H}^3$ and let z be a point in $\mathbb{H}^3$. Suppose that ν is a measure on S_∞ such that $\nu\le A_z$, $\nu(S_\infty)\le a$, and $\int_{S_\infty}(\lambda_{\gamma,z})^2d\nu\ge b$. Then $a>0$, $b<1$, and*

$$d_\gamma z\ge\tfrac{1}{2}\log\frac{\sigma(a)}{\sigma(b)},$$

where $\sigma(x)=1/x-1$ for $x\in(0,1)$.

Provided that $0<\nu_{s(\gamma)}(S_\infty)<1$ for every group-theoretical relation $(\gamma,s(\gamma),S(\gamma))$ of $\Gamma_{\mathcal{D}^*}$, when we let $\nu=\nu_{s(\gamma)}$, $a=\nu_{s(\gamma)}(S_\infty)$, and $b=\int_{S_\infty}(\lambda_{\gamma,z_0})^2d\nu_{s(\gamma)}$, Theorem 1.5 and Lemma 1.6 produce a set $\mathcal{G}=\{f_l\}_{l=1}^{60}$ of real-valued functions on Δ^{27} such that

$$e^{2d_\gamma z}\ge f_l(\mathbf{m})=\sigma\left(\sum_{\psi\in S(\gamma)}\int_{S_\infty}d\nu_\psi\right)\sigma\left(\int_{S_\infty}d\nu_{s(\gamma)}\right) \tag{1.6}$$

for every $\gamma\in\Gamma_*$ for some $l=1,\dots,60$. This is established in Proposition 2.3, in which formulas of the functions in $\mathcal{G}$ are explicitly stated. In the equation in (1.6) above, $\mathbf{m}=(\nu_{\xi\eta^{-1}\xi^{-1}}(S_\infty),\dots,\nu_{\xi^{-2}}(S_\infty))$ is a point of the set

$$\Delta^{27}=\left\{\mathbf{x}=(x_1,x_2,\dots,x_{28})\in\mathbb{R}^{28}_+:\sum_{l=1}^{28}x_i=1\right\},$$

whose entries are ordered by p in (1.2). As a particular example, by the group-theoretical relation in (1.4), the equality in (1.5), Lemma 1.6, and Proposition 2.3, for $z \in \mathbb{H}^3$, we have $d_{\xi\eta\xi^{-1}}z \geq \frac{1}{2}\log f_1(\mathbf{m})$, where

$$f_1(\mathbf{x}) = \frac{1-x_1-x_2-x_3-x_4-x_5-x_6-x_7}{x_1+x_2+x_3+x_4+x_5+x_6+x_7}\cdot\frac{1-x_1}{x_1}.$$

As a consequence of Theorem 1.5, Lemma 1.6, and Proposition 2.3, in case (I), Theorem 1.1 follows from the statement below and the inequality following;

Theorem 1.7 *If $G : \Delta^{27} \to \mathbb{R}$ is the function defined by $\boldsymbol{x} \mapsto \max\{f(\boldsymbol{x}) : f \in \mathcal{G}\}$, then we have* $\inf_{\boldsymbol{x}\in\Delta^{27}} G(\boldsymbol{x}) =$ 24.8692...,

$$\max_{\gamma\in\Gamma_*}\{d_\gamma z\} \geq \tfrac{1}{2}\log G(\boldsymbol{m}) \geq \tfrac{1}{2}\log\left(\inf_{\boldsymbol{x}\in\Delta^{27}} G(\boldsymbol{x})\right). \tag{1.7}$$

To prove Theorem 1.7, we shall show that there exists a subset $\mathcal{F} = \{f_1,\ldots,f_{28}\}$ of $\mathcal{G}$ such that the equality $\inf_{\mathbf{x}\in\Delta^{27}} G(\mathbf{x}) = \inf_{\mathbf{x}\in\Delta^{27}} F(\mathbf{x})$ holds for $F(\mathbf{x}) = \max\{f(\mathbf{x}) : f \in \mathcal{F}\}$. We will compute $\inf_{\mathbf{x}\in\Delta^{27}} F(\mathbf{x})$ by using the following properties of F:

(A) $\inf\limits_{\mathbf{x}\in\Delta^{27}} F(\mathbf{x}) = \min\limits_{\mathbf{x}\in\Delta^{27}} F(\mathbf{x}) = \alpha_*$ at some $\mathbf{x}^* \in \Delta^{27}$,

(B) $\mathbf{x}^*$ is unique and $\mathbf{x}^* \in \Delta_{27} = \{\mathbf{x} \in \Delta^{27} : f_i(\mathbf{x}) = f_j(\mathbf{x})$ for every $f_i, f_j \in \mathcal{F}\}$.

Property A is proved in Lemma 3.1, which exploits the fact that on any sequence $\{\mathbf{x}_n\} \subset \Delta^{27}$ that limits on the boundary of the simplex Δ^{27} some of the displacement functions $f_i \in \mathcal{F}$ approach infinity.

Each statement in Property B is proved in Proposition 3.11 and Proposition 3.14, respectively. We shall first prove Proposition 3.11. We will see that the functions in $\mathcal{F}' = \{f_1, f_5, f_9, f_{13}, f_{15}, f_{19}, f_{23}, f_{27}\}$ in $\mathcal{F}$ play a more important role in computing α_*. At least one of the functions in $\mathcal{F}'$ takes the value α_*. This is showed in Lemma 3.2. Each function f_l in $\mathcal{F}'$ is a strictly convex function on an open convex subset C_{f_l}, defined in (3.3), of Δ^{27} for $l \in J = \{1,5,9,13,15,19,23,27\}$. Moreover, by Lemma 3.4 and Lemma 3.5 we shall show that $\mathbf{x}^* \in C = \bigcap_{l\in J} C_{f_l}$, which is itself convex. The minimum of the maximum of the functions in $\mathcal{F}'$ on C is calculated as α_* in Lemma 3.7. Then, by standard facts from convex analysis, Proposition 3.11 will follow.

Proposition 3.11 reduces the computation of α_* to the comparison of only four values, $f_1(\mathbf{x}^*) = \alpha_*$, $f_2(\mathbf{x}^*) \leq \alpha_*$, $f_3(\mathbf{x}^*) \leq \alpha_*$, and $f_7(\mathbf{x}^*) \leq \alpha_*$, which is proved in Lemma 3.12. Considering Δ^{27} as a submanifold of $\mathbb{R}^{28}$, if $f_l(\mathbf{x}^*) < \alpha_*$ for some $l \in \{2,3,7\}$, the fact that there are directions in the tangent space $T_{\mathbf{x}^*}\Delta^{27}$ of Δ^{27} at $\mathbf{x}^*$ so that all of the displacement functions in $\mathcal{F}$ take values strictly less than α_* on the line segments extending in these directions will prove Proposition 3.14. Existence of these directions will be showed either by a direct calculation or by Lemma 3.13.

Since the coordinate sum of $\mathbf{x}^*$ is 1, Proposition 3.11 and Proposition 3.14 together give a method to calculate the coordinates of $\mathbf{x}^*$ explicitly. By evaluating any of the displacement functions in $\mathcal{F}$ at $\mathbf{x}^*$, we find the value of α_*. Details of this method will be given in Theorem 3.15. Finally, we will show that $f(\mathbf{x}^*) < \alpha_*$ for every $f \in \mathcal{G} - \mathcal{F}$, which implies that $\alpha_* = \inf_{\mathbf{x}\in\Delta^{27}} G(\mathbf{x})$, completing the proof of Theorem 3.16.

Let $\mathfrak{X}$ denote the character variety $PSL(2,\mathbb{C}) \times PSL(2,\mathbb{C})$ and $\mathfrak{GF}$ be the set of pairs of isometries $(\xi,\eta) \in \mathfrak{X}$ such that $\langle\xi,\eta\rangle$ is free, geometrically finite, and without any parabolic. In case (II), when $\Gamma = \langle\xi,\eta\rangle$

is geometrically finite, for a fixed $z \in \mathbb{H}^3$, we define the function $f_z : \mathfrak{X} \to \mathbb{R}$ for Γ_*, described in (1.1), with the formula

$$f_z(\xi, \eta) = \max_{\psi \in \Gamma_*}\{\mathrm{dist}(z,\ \psi \cdot z)\}.$$

This function is continuous and proper. Moreover, by similar arguments given in [4, Theorem 9.1], [18, Theorem 5.1], and [19, Theorem 4.1], it can be shown that it takes its minimum value in the boundary $\overline{\mathfrak{GF}} - \mathfrak{GF}$ of the open set $\mathfrak{GF}$. It is known by [4, Propositions 9.3 and 8.2], [3, Main Theorem], and [2] that the set of (ξ, η) such that $\langle \xi, \eta \rangle$ is free, geometrically infinite, and without any parabolic is dense in $\overline{\mathfrak{GF}} - \mathfrak{GF}$ and every $(\xi, \eta) \in \mathfrak{X}$ with $\langle \xi, \eta \rangle$ that is free and without any parabolic is in $\overline{\mathfrak{GF}}$. This reduces the geometrically finite case to the geometrically infinite case, completing the proof of Theorem 1.1.

We shall use the geometry of the action of the loxodromic elements of $\mathrm{Isom}^+(\mathbb{H}^3)$ to prove Theorem 1.3. Let ξ and η be two noncommuting loxodromic isometries of $\mathbb{H}^3$ and $z \in \mathbb{H}^3$. Then the displacement $d_\xi z$ given by ξ can be expressed as

$$\sinh^2 \tfrac{1}{2} d_\xi z = \sinh^2\left(\tfrac{1}{2}T_\xi\right)\cosh^2 d_z\mathcal{A} + \sin^2\theta \sinh^2 d_z\mathcal{A},$$

where T_ξ, θ, and $\mathcal{A}$ are the translation length, rotational angle, and axis of ξ, respectively. Above, $d_z\mathcal{A}$ denotes the distance between z and $\mathcal{A}$. Let $\mathcal{B}$ be the axis of $\eta\xi\eta^{-1}$. Similarly, $d_{\eta\xi\eta^{-1}} z$ can be expressed as

$$\sinh^2 \tfrac{1}{2} d_{\eta\xi\eta^{-1}} z = \sinh^2\left(\tfrac{1}{2}T_\xi\right)\cosh^2 d_z\mathcal{B} + \sin^2\theta \sinh^2 d_z\mathcal{B}.$$

Because $d_\xi z_1 = d_{\eta\xi\eta^{-1}} z_1$, by reversing the inequalities used to prove [1, Theorem 5.4.5], it is possible to show that

$$|\mathrm{trace}^2(\xi) - 4| + |\mathrm{trace}(\xi\eta\xi^{-1}\eta^{-1}) - 2| \geq 2\sinh^2 \tfrac{1}{2} d_\xi z_1$$

for the midpoint z_1 of the shortest geodesic segment joining $\mathcal{A}$ and $\mathcal{B}$. Then the main result of this paper, Theorem 1.3, follows from the inequality above, Lemma 4.1, and Theorem 1.1.

All of the computations summarized above to prove Theorem 1.1 and Theorem 1.3 for purely loxodromic 2-generator free Kleinian groups can be generalized to prove analogous results for purely loxodromic finitely generated free Kleinian groups. We will finish this paper by phrasing these generalizations in Conjectures 4.4 and 4.5 and presenting their proof sketches.

2. Displacement functions for the isometries in Γ_*

In this section, we shall determine the displacement functions for the hyperbolic displacements given by the isometries in Γ_*. We introduce the following subsets of Ψ defined in (1.2): Let $\Gamma_1 = \{\xi, \eta^{-1}, \eta, \xi^{-1}\}$ and $\Psi = \{\xi^2, \eta^{-2}, \eta^2, \xi^{-2}\} \cup \bigcup_{l=1}^{8} \Psi_l$, where

$$\begin{array}{lll} \Psi_1 = \{\xi\eta^{-1}\xi^{-1}, \xi\eta^{-1}\xi, \xi\eta^{-2}\}, & \Psi_2 = \{\xi\eta^2, \xi\eta\xi^{-1}, \xi\eta\xi\}, & \Psi_3 = \{\eta^{-1}\xi^{-1}\eta^{-1}, \eta^{-1}\xi^{-1}\eta, \eta^{-1}\xi^{-2}\}, \\ \Psi_4 = \{\eta^{-1}\xi^2, \eta^{-1}\xi\eta^{-1}, \eta^{-1}\xi\eta\}, & \Psi_5 = \{\eta\xi^{-1}\eta^{-1}, \eta\xi^{-1}\eta, \eta\xi^{-2}\}, & \Psi_6 = \{\eta\xi^2, \eta\xi\eta^{-1}, \eta\xi\eta\}, \\ \Psi_7 = \{\xi^{-1}\eta^{-1}\xi^{-1}, \xi^{-1}\eta^{-1}\xi, \xi^{-1}\eta^{-2}\}, & \Psi_8 = \{\xi^{-1}\eta^2, \xi^{-1}\eta\xi^{-1}, \xi^{-1}\eta\xi\}. & \end{array}$$

First, we prove the statement below, which gives the relevant group-theoretical relations of the decomposition $\Gamma_{\mathcal{D}^*}$ for the isometries in Γ_*:

Lemma 2.1 *Let $\Gamma = \langle \xi, \eta \rangle$ be a 2-generator free group and $\Gamma_{\mathcal{D}^*}$ be the decomposition of Γ in (1.3). Then there are 60 group-theoretical relations $(\gamma, s(\gamma), S(\gamma))$ for $\gamma \in \Gamma_*$.*

Proof We list all of the group-theoretical relations of $\Gamma_{\mathcal{D}^*}$ for $\gamma \in \Gamma_*$, defined in (1.1), in the following tables.

Table 1. Group-theoretical relations of $\Gamma_{\mathcal{D}^*}$ with 3-cancellation.

l	γ	$s(\gamma)$	$S(\gamma)$	l	γ	$s(\gamma)$	$S(\gamma)$
1	$\xi\eta\xi^{-1}$	$\xi\eta^{-1}\xi^{-1}$	$\{\xi^2\} \cup \Psi_1 \cup \Psi_2$	5	$\eta\xi\eta^{-1}$	$\eta\xi^{-1}\eta^{-1}$	$\{\eta^2\} \cup \Psi_5 \cup \Psi_6$
2	$\xi\eta^{-1}\xi^{-1}$	$\xi\eta\xi^{-1}$	$\{\xi^2\} \cup \Psi_1 \cup \Psi_2$	6	$\eta\xi^{-1}\eta^{-1}$	$\eta\xi\eta^{-1}$	$\{\eta^2\} \cup \Psi_5 \cup \Psi_6$
3	$\eta^{-1}\xi\eta$	$\eta^{-1}\xi^{-1}\eta$	$\{\eta^{-2}\} \cup \Psi_3 \cup \Psi_4$	7	$\xi^{-1}\eta\xi$	$\xi^{-1}\eta^{-1}\xi$	$\{\xi^{-2}\} \cup \Psi_7 \cup \Psi_8$
4	$\eta^{-1}\xi^{-1}\eta$	$\eta^{-1}\xi\eta$	$\{\eta^{-2}\} \cup \Psi_3 \cup \Psi_4$	8	$\xi^{-1}\eta^{-1}\xi$	$\xi^{-1}\eta\xi$	$\{\xi^{-2}\} \cup \Psi_7 \cup \Psi_8$

Table 2. Group-theoretical relations of $\Gamma_{\mathcal{D}^*}$ with 2-cancellation.

l	γ	$s(\gamma)$	$S(\gamma)$	l	γ	$s(\gamma)$	$S(\gamma)$
9	$\xi\eta\xi^{-1}$	$\xi\eta^{-2}$	$\Psi - \Psi_1$	13	$\eta\xi\eta^{-1}$	$\eta\xi^{-2}$	$\Psi - \Psi_5$
10	$\xi\eta^{-1}\xi^{-1}$	$\xi\eta^2$	$\Psi - \Psi_2$	14	$\eta\xi^{-1}\eta^{-1}$	$\eta\xi^2$	$\Psi - \Psi_6$
11	$\eta^{-1}\xi\eta$	$\eta^{-1}\xi^{-2}$	$\Psi - \Psi_3$	15	$\xi^{-1}\eta\xi$	$\xi^{-1}\eta^{-2}$	$\Psi - \Psi_7$
12	$\eta^{-1}\xi^{-1}\eta$	$\eta^{-1}\xi^2$	$\Psi - \Psi_4$	16	$\xi^{-1}\eta^{-1}\xi$	$\xi^{-1}\eta^2$	$\Psi - \Psi_8$

Table 3. Group-theoretical relations of $\Gamma_{\mathcal{D}^*}$ with 1-cancellation.

l	γ	$s(\gamma)$	$S(\gamma)$	l	γ	$s(\gamma)$	$S(\gamma)$
17	ξ^{-1}	$\xi\eta^{-1}\xi^{-1}$	$\Psi - \Psi_3$	31	η^{-1}	$\eta\xi^{-1}\eta^{-1}$	$\Psi - \Psi_7$
18	ξ^{-1}	$\xi\eta^{-1}\xi$	$\Psi - \Psi_4$	32	η^{-1}	$\eta\xi^{-1}\eta$	$\Psi - \Psi_8$
19	ξ^{-1}	$\xi\eta^{-2}$	$\Psi - \{\eta^{-2}\}$	33	η^{-1}	$\eta\xi^{-2}$	$\Psi - \{\xi^{-2}\}$
20	ξ^{-1}	$\xi\eta^2$	$\Psi - \{\eta^2\}$	34	η^{-1}	$\eta\xi^2$	$\Psi - \{\xi^2\}$
21	ξ^{-1}	$\xi\eta\xi^{-1}$	$\Psi - \Psi_5$	35	η^{-1}	$\eta\xi\eta^{-1}$	$\Psi - \Psi_1$
22	ξ^{-1}	$\xi\eta\xi$	$\Psi - \Psi_6$	36	η^{-1}	$\eta\xi\eta$	$\Psi - \Psi_2$
23	ξ^{-1}	ξ^2	$\Psi - \{\xi^2\} \cup \Psi_1 \cup \Psi_2$	37	η^{-1}	η^2	$\Psi - \{\eta^2\} \cup \Psi_5 \cup \Psi_6$
24	η	$\eta^{-1}\xi^{-1}\eta^{-1}$	$\Psi - \Psi_7$	38	ξ	$\xi^{-1}\eta^{-1}\xi^{-1}$	$\Psi - \Psi_3$
25	η	$\eta^{-1}\xi^{-1}\eta$	$\Psi - \Psi_8$	39	ξ	$\xi^{-1}\eta^{-1}\xi$	$\Psi - \Psi_4$
26	η	$\eta^{-1}\xi^{-2}$	$\Psi - \{\xi^{-2}\}$	40	ξ	$\xi^{-1}\eta^{-2}$	$\Psi - \{\eta^2\}$
27	η	$\eta^{-1}\xi^2$	$\Psi - \{\xi^2\}$	41	ξ	$\xi^{-1}\eta^2$	$\Psi - \{\eta^{-2}\}$
28	η	$\eta^{-1}\xi\eta^{-1}$	$\Psi - \Psi_1$	42	ξ	$\xi^{-1}\eta\xi^{-1}$	$\Psi - \Psi_5$
29	η	$\eta^{-1}\xi\eta$	$\Psi - \Psi_2$	43	ξ	$\xi^{-1}\eta\xi$	$\Psi - \Psi_6$
30	η	η^{-2}	$\Psi - \{\eta^{-2}\} \cup \Psi_3 \cup \Psi_4$	44	ξ	ξ^{-2}	$\Psi - \{\eta^{-2}\} \cup \Psi_7 \cup \Psi_8$

Table 4. Group-theoretical relations of $\Gamma_{\mathcal{D}^*}$ with 2-cancellation.

l	γ	$s(\gamma)$	$S(\gamma)$	l	γ	$s(\gamma)$	$S(\gamma)$
45	$\xi\eta^{-1}\xi^{-1}$	$\xi\eta\xi$	$\Psi - \{\xi^2\}$	49	$\eta\xi^{-1}\eta^{-1}$	$\eta\xi\eta$	$\Psi - \{\eta^2\}$
46	$\xi\eta\xi^{-1}$	$\xi\eta^{-1}\xi$	$\Psi - \{\xi^2\}$	50	$\eta\xi\eta^{-1}$	$\eta\xi^{-1}\eta$	$\Psi - \{\eta^2\}$
47	$\eta^{-1}\xi^{-1}\eta$	$\eta^{-1}\xi\eta^{-1}$	$\Psi - \{\eta^{-2}\}$	51	$\xi^{-1}\eta^{-1}\xi$	$\xi^{-1}\eta\xi^{-1}$	$\Psi - \{\xi^{-2}\}$
48	$\eta^{-1}\xi\eta$	$\eta^{-1}\xi^{-1}\eta^{-1}$	$\Psi - \{\eta^{-2}\}$	52	$\xi^{-1}\eta\xi$	$\xi^{-1}\eta^{-1}\xi^{-1}$	$\Psi - \{\xi^{-2}\}$

Table 5. Group-theoretical relations of $\Gamma_{\mathcal{D}^*}$ with 1-cancellation.

l	γ	$s(\gamma)$	$S(\gamma)$	l	γ	$s(\gamma)$	$S(\gamma)$
53	$\xi\eta^{-1}\xi^{-1}$	ξ^2	$\Psi - \{\xi\eta^{-1}\xi\}$	57	$\xi\eta\xi^{-1}$	ξ^2	$\Psi - \{\xi\eta\xi\}$
54	$\eta^{-1}\xi^{-1}\eta$	η^{-2}	$\Psi - \{\eta^{-1}\xi^{-1}\eta^{-1}\}$	58	$\eta^{-1}\xi\eta$	η^{-2}	$\Psi - \{\eta^{-1}\xi\eta^{-1}\}$
55	$\eta\xi^{-1}\eta^{-1}$	η^2	$\Psi - \{\eta\xi^{-1}\eta\}$	59	$\eta\xi\eta^{-1}$	η^2	$\Psi - \{\eta\xi\eta\}$
56	$\xi^{-1}\eta^{-1}\xi$	ξ^{-2}	$\Psi - \{\xi^{-1}\eta^{-1}\xi^{-1}\}$	60	$\xi^{-1}\eta\xi$	ξ^{-2}	$\Psi - \{\xi^{-1}\eta\xi^{-1}\}$

In Tables 1–5 all of the group-theoretical relations $(\gamma, s(\gamma), S(\gamma))$ of $\Gamma_{\mathcal{D}^*}$ for $\gamma \in \Gamma_*$ are counted. This completes the proof. □

Given the group-theoretical relations in Lemma 2.1, we decompose the area measure on S_∞ accordingly. This is stated in the following theorem. To save space, we will not give a proof of this theorem, which uses analogous arguments presented in the proofs of [4, Lemma 5.3], [18, Lemma 3.3, Theorem 3.4], and [19, Theorem 2.1].

Theorem 2.2 *Let $\Gamma = \langle \xi, \eta \rangle$ be a free, purely loxodromic, and geometrically infinite Kleinian group. Let $\Gamma_{\mathcal{D}^*}$ be the decomposition of Γ given in (1.3). If z denotes a point in $\mathbb{H}^3$, then there is a family of Borel measures $\{\nu_\psi\}_{\psi\in\Psi}$ defined on S_∞ such that (i) $A_z = \sum_{\psi\in\Psi} \nu_\psi$; (ii) $A_z(S_\infty) = 1$; and*

$$(iii) \quad \int_{S_\infty} (\lambda_{\gamma,z})^2 \, d\nu_{s(\gamma)} = 1 - \sum_{\psi\in S(\gamma)} \int_{S_\infty} d\nu_\psi$$

for each group-theoretical relation $(\gamma, s(\gamma), S(\gamma))$ of $\Gamma_{\mathcal{D}^*}$, where A_z is the area measure on S_∞ based at z.

Let $I = J_1 \cup J_2 \cup J_3 \cup J_4 = \{1, 2, \ldots, 28\}$ and I_l for $l \in \{1, \ldots, 8\}$ be the following index sets:

$$\begin{array}{lll} I_1 = \{1,2,3\}, & I_5 = \{15,16,17\}, & J_1 = \{1,2,3,4,5,6,7\}, \\ I_2 = \{4,5,6\}, & I_6 = \{18,19,20\}, & J_2 = \{8,9,10,11,12,13,14\}, \\ I_3 = \{8,9,10\}, & I_7 = \{22,23,24\}, & J_3 = \{15,16,17,18,19,20,21\}, \\ I_4 = \{11,12,13\}, & I_8 = \{25,26,27\}, & J_4 = \{22,23,24,25,26,27,28\}. \end{array} \tag{2.1}$$

We shall use the functions $\sigma : (0,1) \to (0,\infty)$, $\Sigma^i_J : \Delta^{27} \to (0,1)$, $\Sigma^J_i : \Delta^{27} \to (0,1)$, $\Sigma^j_I : \Delta^{27} \to (0,1)$, and $\Sigma^n : \Delta^{27} \to (0,1)$ with formulas $\sigma(x) = 1/x - 1$,

$$\Sigma^J_i(\mathbf{x}) = \sum_{l\in I-J_i} x_l, \quad \Sigma^i_J(\mathbf{x}) = \sum_{l\in J_i} x_l, \quad \Sigma^j_I(\mathbf{x}) = \sum_{l\in I-I_j} x_l, \quad \Sigma^n(\mathbf{x}) = \sum_{l\in I-\{n\}} x_l, \tag{2.2}$$

for $i \in \{1,2,3,4\}$, $j \in \{1,2,3,4,5,6,7,8\}$, and $n \in \{1,2,\ldots,28\}$, respectively, to express the displacement functions compactly. In particular, we prove the following:

Proposition 2.3 *Let $\Gamma = \langle \xi, \eta \rangle$ be a purely loxodromic, free, and geometrically infinite Kleinian group. Let $\Gamma_{\mathcal{D}^*}$ be the decomposition of Γ defined in (1.3). For any $z \in \mathbb{H}^3$ and for each $\gamma \in \Gamma_*$, the value $e^{2d_\gamma z}$ is bounded below by $f_l(\boldsymbol{x})$, $g_i(\boldsymbol{x})$, $h_j(\boldsymbol{x})$, or $u_n(\boldsymbol{x})$ for $\boldsymbol{x} \in \Delta^{27}$ for at least one of the displacement functions f_l, g_i, h_j, or u_n whose formulas are listed in the tables below*

Table 6. Displacement functions obtained from the group-theoretical relations in Table 1.

l		l	
1	$f_1(\mathbf{x}) = \sigma\left(\Sigma^1_J(\mathbf{x})\right)\sigma(x_1)$	5	$f_{15}(\mathbf{x}) = \sigma\left(\Sigma^3_J(\mathbf{x})\right)\sigma(x_{15})$
2	$f_5(\mathbf{x}) = \sigma\left(\Sigma^1_J(\mathbf{x})\right)\sigma(x_5)$	6	$f_{19}(\mathbf{x}) = \sigma\left(\Sigma^3_J(\mathbf{x})\right)\sigma(x_{19})$
3	$f_9(\mathbf{x}) = \sigma\left(\Sigma^2_J(\mathbf{x})\right)\sigma(x_9)$	7	$f_{23}(\mathbf{x}) = \sigma\left(\Sigma^4_J(\mathbf{x})\right)\sigma(x_{23})$
4	$f_{13}(\mathbf{x}) = \sigma\left(\Sigma^2_J(\mathbf{x})\right)\sigma(x_{13})$	8	$f_{27}(\mathbf{x}) = \sigma\left(\Sigma^4_J(\mathbf{x})\right)\sigma(x_{27})$

Table 7. Displacement functions obtained from the group-theoretical relations in Table 2.

l		l	
9	$f_3(\mathbf{x}) = \sigma\left(\Sigma_I^1(\mathbf{x})\right)\sigma(x_3)$	13	$f_{17}(\mathbf{x}) = \sigma\left(\Sigma_I^5(\mathbf{x})\right)\sigma(x_{17})$
10	$f_4(\mathbf{x}) = \sigma\left(\Sigma_I^2(\mathbf{x})\right)\sigma(x_4)$	14	$f_{18}(\mathbf{x}) = \sigma\left(\Sigma_I^6(\mathbf{x})\right)\sigma(x_{18})$
11	$f_{10}(\mathbf{x}) = \sigma\left(\Sigma_I^3(\mathbf{x})\right)\sigma(x_{10})$	15	$f_{24}(\mathbf{x}) = \sigma\left(\Sigma_I^7(\mathbf{x})\right)\sigma(x_{24})$
12	$f_{11}(\mathbf{x}) = \sigma\left(\Sigma_I^4(\mathbf{x})\right)\sigma(x_{11})$	16	$f_{25}(\mathbf{x}) = \sigma\left(\Sigma_I^8(\mathbf{x})\right)\sigma(x_{25})$

Table 8. Displacement functions obtained from the group-theoretical relations in Table 3.

l		l	
17	$g_1(\mathbf{x}) = \sigma\left(\Sigma_I^3(\mathbf{x})\right)\sigma(x_1)$	31	$g_{15}(\mathbf{x}) = \sigma\left(\Sigma_I^7(\mathbf{x})\right)\sigma(x_{15})$
18	$f_2(\mathbf{x}) = \sigma\left(\Sigma_I^4(\mathbf{x})\right)\sigma(x_2)$	32	$f_{16}(\mathbf{x}) = \sigma\left(\Sigma_I^8(\mathbf{x})\right)\sigma(x_{16})$
19	$g_3(\mathbf{x}) = \sigma\left(\Sigma^{14}(\mathbf{x})\right)\sigma(x_3)$	33	$g_{17}(\mathbf{x}) = \sigma\left(\Sigma^{28}(\mathbf{x})\right)\sigma(x_{17})$
20	$g_4(\mathbf{x}) = \sigma\left(\Sigma^{21}(\mathbf{x})\right)\sigma(x_4)$	34	$g_{18}(\mathbf{x}) = \sigma\left(\Sigma^{7}(\mathbf{x})\right)\sigma(x_{18})$
21	$g_5(\mathbf{x}) = \sigma\left(\Sigma_I^5(\mathbf{x})\right)\sigma(x_5)$	35	$g_{19}(\mathbf{x}) = \sigma\left(\Sigma_I^1(\mathbf{x})\right)\sigma(x_{19})$
22	$f_6(\mathbf{x}) = \sigma\left(\Sigma_I^6(\mathbf{x})\right)\sigma(x_6)$	36	$f_{20}(\mathbf{x}) = \sigma\left(\Sigma_I^2(\mathbf{x})\right)\sigma(x_{20})$
23	$f_7(\mathbf{x}) = \sigma\left(\Sigma_1^J(\mathbf{x})\right)\sigma(x_7)$	37	$f_{21}(\mathbf{x}) = \sigma\left(\Sigma_3^J(\mathbf{x})\right)\sigma(x_{21})$
24	$f_8(\mathbf{x}) = \sigma\left(\Sigma_I^7(\mathbf{x})\right)\sigma(x_8)$	38	$f_{22}(\mathbf{x}) = \sigma\left(\Sigma_I^3(\mathbf{x})\right)\sigma(x_{22})$
25	$g_9(\mathbf{x}) = \sigma\left(\Sigma_I^8(\mathbf{x})\right)\sigma(x_9)$	39	$g_{23}(\mathbf{x}) = \sigma\left(\Sigma_I^4(\mathbf{x})\right)\sigma(x_{23})$
26	$g_{10}(\mathbf{x}) = \sigma\left(\Sigma^{28}(\mathbf{x})\right)\sigma(x_{10})$	40	$g_{24}(\mathbf{x}) = \sigma\left(\Sigma^{14}(\mathbf{x})\right)\sigma(x_{24})$
27	$g_{11}(\mathbf{x}) = \sigma\left(\Sigma^{7}(\mathbf{x})\right)\sigma(x_{11})$	41	$g_{25}(\mathbf{x}) = \sigma\left(\Sigma^{21}(\mathbf{x})\right)\sigma(x_{25})$
28	$f_{12}(\mathbf{x}) = \sigma\left(\Sigma_I^1(\mathbf{x})\right)\sigma(x_{12})$	42	$f_{26}(\mathbf{x}) = \sigma\left(\Sigma_I^5(\mathbf{x})\right)\sigma(x_{26})$
29	$g_{13}(\mathbf{x}) = \sigma\left(\Sigma_I^2(\mathbf{x})\right)\sigma(x_{13})$	43	$g_{27}(\mathbf{x}) = \sigma\left(\Sigma_I^6(\mathbf{x})\right)\sigma(x_{27})$
30	$f_{14}(\mathbf{x}) = \sigma\left(\Sigma_2^J(\mathbf{x})\right)\sigma(x_{14})$	44	$f_{28}(\mathbf{x}) = \sigma\left(\Sigma_4^J(\mathbf{x})\right)\sigma(x_{28})$

Table 9. Displacement functions obtained from the group-theoretical relations in Table 4.

l		l	
45	$h_1(\mathbf{x}) = \sigma\left(\Sigma^{28}(\mathbf{x})\right)\sigma(x_1)$	49	$h_{15}(\mathbf{x}) = \sigma\left(\Sigma^{14}(\mathbf{x})\right)\sigma(x_{15})$
46	$h_5(\mathbf{x}) = \sigma\left(\Sigma^{28}(\mathbf{x})\right)\sigma(x_5)$	50	$h_{19}(\mathbf{x}) = \sigma\left(\Sigma^{14}(\mathbf{x})\right)\sigma(x_{19})$
47	$h_9(\mathbf{x}) = \sigma\left(\Sigma^{21}(\mathbf{x})\right)\sigma(x_9)$	51	$h_{23}(\mathbf{x}) = \sigma\left(\Sigma^{7}(\mathbf{x})\right)\sigma(x_{23})$
48	$h_{13}(\mathbf{x}) = \sigma\left(\Sigma^{21}(\mathbf{x})\right)\sigma(x_{13})$	52	$h_{27}(\mathbf{x}) = \sigma\left(\Sigma^{7}(\mathbf{x})\right)\sigma(x_{27})$

Table 10. Displacement functions obtained from the group-theoretical relations in Table 5.

l		l	
53	$h_7(\mathbf{x}) = \sigma\left(\Sigma^{2}(\mathbf{x})\right)\sigma(x_7)$	57	$u_7(\mathbf{x}) = \sigma\left(\Sigma^{6}(\mathbf{x})\right)\sigma(x_7)$
54	$h_{14}(\mathbf{x}) = \sigma\left(\Sigma^{8}(\mathbf{x})\right)\sigma(x_{14})$	58	$u_{14}(\mathbf{x}) = \sigma\left(\Sigma^{12}(\mathbf{x})\right)\sigma(x_{14})$
55	$h_{21}(\mathbf{x}) = \sigma\left(\Sigma^{16}(\mathbf{x})\right)\sigma(x_{21})$	59	$u_{21}(\mathbf{x}) = \sigma\left(\Sigma^{20}(\mathbf{x})\right)\sigma(x_{21})$
56	$h_{28}(\mathbf{x}) = \sigma\left(\Sigma^{22}(\mathbf{x})\right)\sigma(x_{28})$	60	$u_{28}(\mathbf{x}) = \sigma\left(\Sigma^{26}(\mathbf{x})\right)\sigma(x_{28})$.

Proof Let $\{\nu_\psi\}_{\psi\in\Psi}$ be the family of Borel measures on S_∞ given by Theorem 2.2. Since every isometry $\psi \in \Psi$ other than $\xi\eta^{-2}$, $\xi\eta^2$, $\eta^{-1}\xi^{-2}$, $\eta^{-1}\xi^2$, $\eta\xi^{-2}$, $\eta\xi^2$, $\xi^{-1}\eta^{-2}$, and $\xi^{-1}\eta^2$ has an inverse in Ψ, an analogous argument used in [19, Proposition 2.1] shows that $0 < \nu_\psi(S_\infty) < 1$ for these isometries.

It is clear that $\nu_{\xi\eta^{-2}}(S_\infty) \neq 1$ because otherwise we get $\nu_\psi(S_\infty) = 0$ for every $\psi \in \Psi - \{\xi\eta^{-2}\}$ by Theorem 2.2 (i), a contradiction. Assume that $\nu_{\xi\eta^{-2}}(S_\infty) = 0$. By the group-theoretical relation in Table 2, (2), and Theorem 2.2 (iii), we derive that $\nu_\psi(S_\infty) = 0$ for every $\psi \in \Psi_1 = \{\xi\eta^{-1}\xi^{-1}, \xi\eta^{-1}\xi, \xi\eta^{-2}\}$. This is a contradiction. By using the group-theoretical relations in Table 2 together with similar arguments given above for $\xi\eta^{-2}$, we conclude that $0 < \nu_\psi(S_\infty) < 1$ for every $\psi \in \Psi$.

Let $m_{p(\psi)} = \int_{S_\infty} d\nu_\psi$ for the bijection p in (1.2). Also let $\mathbf{m} = (m_1, m_2, \ldots, m_{28}) \in \Delta^{27}$. Since $0 < \nu_\psi(S_\infty) < 1$ for every $\psi \in \Psi$, we see by Theorem 2.2 (iii) and (ii) that $\nu_{s(\gamma)}(S_\infty)$ and $\int_{S_\infty} \lambda^2_{\gamma,z_0} d\mu_{V_{s(\gamma)}}$ satisfy the hypothesis of Lemma 1.6 for each group-theoretical relation $(\gamma, s(\gamma), S(\gamma))$ of $\Gamma_{\mathcal{D}^*}$ for $\gamma \in \Gamma_*$. By setting $\nu = \nu_{s(\gamma)}$, $a = \nu_{s(\gamma)}(S_\infty)$, and $b = \int_{S_\infty} \lambda^2_{\gamma,z_0} d\mu_{V_{s(\gamma)}}$ in Lemma 1.6, we obtain the lower bound

$$e^{2d_\gamma z} \geq \sigma\left(\sum_{\psi\in S(\gamma)} m_{p(\psi)}\right)\sigma\left(m_{p(s(\gamma))}\right) \tag{2.3}$$

for each group-theoretical relation $(\gamma, s(\gamma), S(\gamma))$ of $\Gamma_{\mathcal{D}^*}$ so that $\gamma \in \Gamma_*$. We replace each constant $m_{p(\psi)}$ appearing in (2.3) with the variable $x_{p(\psi)}$, which gives the functions listed in Tables 6, 7, 8, 9, and 10, proving the proposition. □

Let $\mathcal{G} = \{f_1, \ldots, f_{28}, g_1, g_3, \ldots, g_{27}, h_1, h_5, \ldots, h_{27}, u_7, u_{14}, \ldots, u_{28}\}$ be the set of all displacement functions given in the tables in the proposition above. Let $\mathcal{F} = \{f_1, \ldots, f_{28}\}$. Let G be the continuous function defined as

$$\begin{array}{rcccl} G & : & \Delta^{27} & \to & \mathbb{R} \\ & & \mathbf{x} & \mapsto & \max\{f(\mathbf{x}) : f \in \mathcal{G}\}. \end{array} \tag{2.4}$$

In the next section, we calculate $\inf_{\mathbf{x}\in\Delta^{27}} G(\mathbf{x})$ by using the subset $\mathcal{F}$ of functions in $\mathcal{G}$.

We finish Section 2 by listing explicit formulas of some of the displacement functions from each group $\{f_l\}$, $\{g_i\}$, $\{h_j\}$, and $\{u_k\}$ in $\mathcal{G}$ as examples to clarify the use of compact forms in these functions. For the index sets $J_1 = \{1,2,3,4,5,6,7\}$, $J_2 = \{8,9,10,11,12,13,14\}$, and $I_3 = \{8,9,10\}$ we have

$$f_9(\mathbf{x}) = \sigma(\Sigma^2_J(\mathbf{x}))\sigma(x_9) = \frac{1 - x_8 - x_9 - x_{10} - x_{11} - x_{12} - x_{13} - x_{14}}{x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14}} \cdot \frac{1 - x_9}{x_9},$$

$$f_7(\mathbf{x}) = \sigma(\Sigma^J_1(\mathbf{x}))\sigma(x_7) = \frac{1 - x_8 - x_9 - \cdots - x_{27} - x_{28}}{x_8 + x_9 + \cdots + x_{27} + x_{28}} \cdot \frac{1 - x_7}{x_7},$$

$$g_1(\mathbf{x}) = \sigma(\Sigma^3_I(\mathbf{x}))\sigma(x_1) = \frac{1 - x_1 - x_2 - \cdots - x_7 - x_{11} - \cdots - x_{28}}{x_1 + x_2 + \cdots + x_7 + x_{11} + \cdots + x_{28}} \cdot \frac{1 - x_1}{x_1},$$

$$g_{18}(\mathbf{x}) = \sigma(\Sigma^7(\mathbf{x}))\sigma(x_{18}) = \frac{1 - x_1 - x_2 - \cdots - x_6 - x_8 - \cdots - x_{28}}{x_1 + x_2 + \cdots + x_6 + x_8 + \cdots + x_{28}} \cdot \frac{1 - x_{18}}{x_{18}},$$

$$h_1(\mathbf{x}) = \sigma(\Sigma^{28}(\mathbf{x}))\sigma(x_1) = \frac{1 - x_1 - x_2 - x_3 - \cdots - x_{27}}{x_1 + x_2 + x_3 + \cdots + x_{27}} \cdot \frac{1 - x_1}{x_1},$$

$$u_7(\mathbf{x}) = \sigma(\Sigma^6(\mathbf{x}))\sigma(x_7) = \frac{1 - x_1 - \cdots - x_5 - x_7 - \cdots - x_{28}}{x_1 + \cdots + x_5 + x_7 + \cdots + x_{28}} \cdot \frac{1 - x_7}{x_7}.$$

Note that in the formula of f_9 only variables enumerated by the elements of J_2 appear in the first multiple. In the formula of f_7, variables enumerated by the elements of J_1 are missing in the first factor. Similarly, in the formula of g_1 variables enumerated by the elements of I_3 are missing. In the formulas of g_{18}, h_1, and u_7, variables x_7, x_{28}, and x_6 are missing, respectively, in the first quotients.

3. Infima of the maximum of the functions in $\mathcal{G}$ on Δ^{27}

In this section, we will mostly be dealing with the functions in $\mathcal{F} = \{f_l\}_{l\in I}$, where $I = \{1, 2, \ldots, 28\}$. We will show that $\inf_{\mathbf{x}\in\Delta^{27}} G(\mathbf{x}) = \inf_{\mathbf{x}\in\Delta^{27}} F(\mathbf{x})$ (see Theorems 3.15 and 3.16), such that F is the continuous function that has the formula

$$\begin{array}{rcccl} F & : & \Delta^{27} & \to & \mathbb{R} \\ & & \mathbf{x} & \mapsto & \max\left(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_{28}(\mathbf{x})\right). \end{array} \tag{3.1}$$

Therefore, it is enough to calculate $\inf_{\mathbf{x}\in\Delta^{27}} F(\mathbf{x})$. We start with the following lemma:

Lemma 3.1 *If F is the function defined in (3.1), then $\inf_{\boldsymbol{x}\in\Delta^{27}} F(\boldsymbol{x})$ is attained in Δ^{27} and contained in the interval $[1, \alpha]$, where $\alpha = 24.8692...$, the only real root of the polynomial $21x^4 - 496x^3 - 654x^2 + 24x + 81$ that is greater than 9.*

Proof To save space, we refer readers to [18, Lemma 4.2] and [19, Lemma 3.1] for details of the proof of the statement $\inf_{\mathbf{x}\in\Delta^{27}} F(\mathbf{x}) = \min_{\mathbf{x}\in\Delta^{27}} F(\mathbf{x})$. Briefly, the equality follows from the observation that on any sequence in Δ^{27} that limits on the boundary of Δ^{27} some of the functions in $\mathcal{F}$ approach infinity.

For some $l \in I = \{1, 2, \ldots, 28\}$, we have $f_l(\mathbf{x}) > 1$ for every $\mathbf{x} \in \Delta^{27}$, which shows $\min_{\mathbf{x}\in\Delta^{27}} F(\mathbf{x}) \geq 1$. Consider the point $\mathbf{y}^* = (y_1, y_2, \ldots, y_{28})$ in Δ^{27} such that $y_l = 1/(1 + 3\alpha) = 0.0132...$ for $l \in \{7, 14, 21, 28\}$, $y_l = 3/(3 + \alpha) = 0.1076...$ for $l \in \{1, 5, 9, 13, 15, 19, 23, 27\}$, and $y_l = 3(\alpha - 1)/(21\alpha^2 + 14\alpha - 3) = 0.0053...$ for indices $l \in \{2, 6, 8, 12, 16, 20, 22, 26\}$ and $l \in \{3, 4, 10, 11, 17, 18, 24, 25\}$. Then we see that $f_l(\mathbf{y}^*) = \alpha$ for every $l \in I$. This completes the proof. □

In the rest of this text, we will consider Δ^{27} as a submanifold of $\mathbb{R}^{28}$. The tangent space $T_{\mathbf{x}}\Delta^{27}$ at any $\mathbf{x} \in \Delta^{27}$ consists of vectors whose coordinates sum to 0. Note that each displacement function f_i for $i \in I$ is smooth in an open neighborhood of Δ^{27}. Therefore, the directional derivative of f_i in the direction of any $\vec{v} \in T_{\mathbf{x}}\Delta^{27}$ is given by $\nabla f_i(\mathbf{x}) \cdot \vec{v}$ for any $i \in I = \{1, 2, \ldots, 28\}$. The notation $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_{28}^*)$ will be used to denote a point at which the infimum of F is attained on Δ^{27}. We shall use α_* to denote the infimum of the maximum of the functions in $\mathcal{F}$ on Δ^{27}, i.e.

$$\alpha_* = \min_{\mathbf{x}\in\Delta^{27}} F(\mathbf{x}).$$

The displacement functions $\{f_l\}_{l\in J}$ for $J = \{1, 5, 9, 13, 15, 19, 23, 27\}$ in $\mathcal{F}$ play a special role in computing α_*. In particular, we have the following statement:

Lemma 3.2 *Let $\boldsymbol{x}^* \in \Delta^{27}$ so that $F(\boldsymbol{x}^*) = \alpha_*$. We have $f_l(\boldsymbol{x}^*) = \alpha_*$ for some $l \in J$.*

Proof Assume on the contrary that $f_l(\mathbf{x}^*) < \alpha_*$ for every $l \in J$. Let C_i^j denote the partial derivative of f_i with respect to x_j at $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_{28}^*)$. We form the 20×28 matrix below, whose rows are $\nabla f_l(\mathbf{x}^*)$ for

$l \in I - J$:

$$\left[\begin{array}{cccccccccccccccccccccccccccc}
C_2^1 & C_2^2 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & 0 & 0 & 0 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
0 & 0 & C_3^3 & C_3^4 \\
C_4^1 & C_4^1 & C_4^1 & C_4^4 & 0 & 0 & C_4^1 \\
C_6^1 & C_6^1 & C_6^1 & C_6^1 & C_6^1 & C_6^6 & C_6^1 & C_6^1 & C_6^1 & C_6^1 & C_6^1 & C_6^1 & C_6^1 & C_6^1 & C_6^1 & C_6^1 & C_6^1 & 0 & 0 & 0 & C_6^1 & C_6^1 & C_6^1 & C_6^1 & C_6^1 & C_6^1 & C_6^1 & C_6^1 \\
0 & 0 & 0 & 0 & 0 & 0 & C_7^7 & C_7^8 \\
C_8^1 & C_8^1 & C_8^1 & C_8^1 & C_8^1 & C_8^1 & C_8^1 & C_8^8 & C_8^1 & C_8^1 & C_8^1 & C_8^1 & C_8^1 & C_8^1 & C_8^1 & C_8^1 & C_8^1 & C_8^1 & C_8^1 & C_8^1 & C_8^1 & 0 & 0 & 0 & C_8^1 & C_8^1 & C_8^1 & C_8^1 \\
C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & 0 & 0 & C_{10}^{10} & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 & C_{10}^1 \\
C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^{11} & 0 & 0 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 & C_{11}^1 \\
0 & 0 & 0 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^{12} & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 & C_{12}^4 \\
C_{14}^1 & C_{14}^1 & C_{14}^1 & C_{14}^1 & C_{14}^1 & C_{14}^1 & C_{14}^1 & 0 & 0 & 0 & 0 & 0 & 0 & C_{14}^{14} & C_{14}^1 & C_{14}^1 & C_{14}^1 & C_{14}^1 & C_{14}^1 & C_{14}^1 & C_{14}^1 & C_{14}^1 & C_{14}^1 & C_{14}^1 & C_{14}^1 & C_{14}^1 & C_{14}^1 & C_{14}^1 \\
C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^{16} & C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^1 & C_{16}^1 & 0 & 0 & 0 & C_{16}^1 \\
C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & 0 & 0 & C_{17}^{17} & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 & C_{17}^1 \\
C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^{18} & 0 & 0 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 & C_{18}^1 \\
C_{20}^1 & C_{20}^1 & C_{20}^1 & 0 & 0 & 0 & C_{20}^1 & C_{20}^1 & C_{20}^1 & C_{20}^1 & C_{20}^1 & C_{20}^1 & C_{20}^1 & C_{20}^1 & C_{20}^1 & C_{20}^1 & C_{20}^1 & C_{20}^1 & C_{20}^1 & C_{20}^{20} & C_{20}^1 & C_{20}^1 & C_{20}^1 & C_{20}^1 & C_{20}^1 & C_{20}^1 & C_{20}^1 & C_{20}^1 \\
C_{21}^1 & C_{21}^1 & C_{21}^1 & C_{21}^1 & C_{21}^1 & C_{21}^1 & C_{21}^1 & C_{21}^1 & C_{21}^1 & C_{21}^1 & C_{21}^1 & C_{21}^1 & C_{21}^1 & C_{21}^1 & 0 & 0 & 0 & 0 & 0 & 0 & C_{21}^{21} & C_{21}^1 & C_{21}^1 & C_{21}^1 & C_{21}^1 & C_{21}^1 & C_{21}^1 & C_{21}^1 \\
C_{22}^1 & C_{22}^1 & C_{22}^1 & C_{22}^1 & C_{22}^1 & C_{22}^1 & C_{22}^1 & 0 & 0 & 0 & C_{22}^1 & C_{22}^1 & C_{22}^1 & C_{22}^1 & C_{22}^1 & C_{22}^1 & C_{22}^1 & C_{22}^1 & C_{22}^1 & C_{22}^1 & C_{22}^1 & C_{22}^{22} & C_{22}^1 & C_{22}^1 & C_{22}^1 & C_{22}^1 & C_{22}^1 & C_{22}^1 \\
C_{24}^1 & 0 & 0 & C_{24}^{24} & C_{24}^1 & C_{24}^1 & C_{24}^1 & C_{24}^1 \\
C_{25}^1 & C_{25}^{25} & 0 & 0 & C_{25}^1 \\
C_{26}^1 & C_{26}^1 & C_{26}^1 & C_{26}^1 & C_{26}^1 & C_{26}^1 & C_{26}^1 & C_{26}^1 & C_{26}^1 & C_{26}^1 & C_{26}^1 & C_{26}^1 & C_{26}^1 & C_{26}^1 & 0 & 0 & 0 & C_{26}^1 & C_{26}^1 & C_{26}^1 & C_{26}^1 & C_{26}^1 & C_{26}^1 & C_{26}^1 & C_{26}^1 & C_{26}^{26} & C_{26}^1 & C_{26}^1 \\
C_{28}^1 & 0 & 0 & 0 & 0 & 0 & 0 & C_{28}^{28}
\end{array}\right],$$

where the entries are given as follows:

$$C_2^1 = -\frac{\sigma(x_2^*)}{\left(\Sigma_I^4(\mathbf{x}^*)\right)^2}, \quad C_2^2 = -\frac{\sigma(x_2^*)}{\left(\Sigma_I^4(\mathbf{x}^*)\right)^2} - \frac{\sigma\left(\Sigma_I^4(\mathbf{x}^*)\right)}{(x_2^*)^2}, \quad C_3^3 = -\frac{\sigma\left(\Sigma_I^1(\mathbf{x}^*)\right)}{\left(x_3^*\right)^2}$$
$$C_3^4 = -\frac{\sigma(x_3^*)}{\left(\Sigma_I^1(\mathbf{x}^*)\right)^2}, \quad C_4^1 = -\frac{\sigma(x_4^*)}{\left(\Sigma_I^2(\mathbf{x}^*)\right)^2}, \quad C_4^4 = -\frac{\sigma\left(\Sigma_I^2(\mathbf{x}^*)\right)}{\left(x_4^*\right)^2},$$

$$C_6^1 = -\frac{\sigma(x_6^*)}{\left(\Sigma_I^6(\mathbf{x}^*)\right)^2}, \quad C_6^6 = -\frac{\sigma(x_6^*)}{\left(\Sigma_I^6(\mathbf{x}^*)\right)^2} - \frac{\sigma\left(\Sigma_I^6(\mathbf{x}^*)\right)}{(x_6^*)^2}, \quad C_7^7 = -\frac{\sigma\left(\Sigma_1^J(\mathbf{x}^*)\right)}{\left(x_7^*\right)^2},$$
$$C_7^8 = -\frac{\sigma(x_7^*)}{\left(\Sigma_1^J(\mathbf{x}^*)\right)^2}, \quad C_8^8 = -\frac{\sigma(x_8^*)}{\left(\Sigma_I^7(\mathbf{x}^*)\right)^2} - \frac{\sigma\left(\Sigma_I^7(\mathbf{x}^*)\right)}{(x_8^*)^2}, \quad C_8^1 = -\frac{\sigma(x_8^*)}{\left(\Sigma_I^7(\mathbf{x}^*)\right)^2},$$

$$C_{10}^1 = -\frac{\sigma(x_{10}^*)}{\left(\Sigma_I^3(\mathbf{x}^*)\right)^2}, \quad C_{10}^{10} = -\frac{\sigma\left(\Sigma_I^3(\mathbf{x}^*)\right)}{\left(x_{10}^*\right)^2}, \quad C_{11}^1 = -\frac{\sigma(x_{11}^*)}{\left(\Sigma_I^4(\mathbf{x}^*)\right)^2},$$
$$C_{11}^{11} = -\frac{\sigma\left(\Sigma_I^4(\mathbf{x}^*)\right)}{\left(x_{11}^*\right)^2}, \quad C_{12}^{12} = -\frac{\sigma(x_{12}^*)}{\left(\Sigma_I^1(\mathbf{x}^*)\right)^2} - \frac{\sigma\left(\Sigma_I^1(\mathbf{x}^*)\right)}{(x_{12}^*)^2}, \quad C_{12}^4 = -\frac{\sigma(x_{12}^*)}{\left(\Sigma_I^1(\mathbf{x}^*)\right)^2},$$
$$C_{14}^1 = -\frac{\sigma(x_{14}^*)}{\left(\Sigma_2^J(\mathbf{x}^*)\right)^2}, \quad C_{14}^{14} = -\frac{\sigma\left(\Sigma_2^J(\mathbf{x}^*)\right)}{\left(x_{14}^*\right)^2}, \quad C_{16}^1 = -\frac{\sigma(x_{16}^*)}{\left(\Sigma_I^8(\mathbf{x}^*)\right)^2},$$

$$C_{17}^1 = -\frac{\sigma(x_{17}^*)}{\left(\Sigma_I^5(\mathbf{x}^*)\right)^2}, \quad C_{16}^{16} = -\frac{\sigma(x_{16}^*)}{\left(\Sigma_I^8(\mathbf{x}^*)\right)^2} - \frac{\sigma\left(\Sigma_I^8(\mathbf{x}^*)\right)}{(x_{16}^*)^2}, \quad C_{17}^{17} = -\frac{\sigma\left(\Sigma_I^5(\mathbf{x}^*)\right)}{\left(x_{17}^*\right)^2},$$
$$C_{18}^1 = -\frac{\sigma(x_{18}^*)}{\left(\Sigma_I^6(\mathbf{x}^*)\right)^2}, \quad C_{18}^{18} = -\frac{\sigma\left(\Sigma_I^6(\mathbf{x}^*)\right)}{\left(x_{18}^*\right)^2}, \quad C_{20}^1 = -\frac{\sigma(x_{20}^*)}{\left(\Sigma_I^2(\mathbf{x}^*)\right)^2},$$

$$C_{21}^1 = -\frac{\sigma(x_{21}^*)}{\left(\Sigma_3^J(\mathbf{x}^*)\right)^2}, \quad C_{20}^{20} = -\frac{\sigma(x_{20}^*)}{\left(\Sigma_I^2(\mathbf{x}^*)\right)^2} - \frac{\sigma\left(\Sigma_I^2(\mathbf{x}^*)\right)}{(x_{20}^*)^2}, \quad C_{21}^{21} = -\frac{\sigma\left(\Sigma_3^J(\mathbf{x}^*)\right)}{\left(x_{21}^*\right)^2},$$
$$C_{22}^1 = -\frac{\sigma(x_{22}^*)}{\left(\Sigma_I^3(\mathbf{x}^*)\right)^2}, \quad C_{22}^{22} = -\frac{\sigma(x_{22}^*)}{\left(\Sigma_I^3(\mathbf{x}^*)\right)^2} - \frac{\sigma\left(\Sigma_I^3(\mathbf{x}^*)\right)}{(x_{22}^*)^2}, \quad C_{24}^1 = -\frac{\sigma(x_{24}^*)}{\left(\Sigma_I^7(\mathbf{x}^*)\right)^2},$$

$$C_{24}^{24} = -\frac{\sigma\left(\Sigma_I^7(\mathbf{x}^*)\right)}{(x_{24}^*)^2}, \qquad C_{25}^{1} = -\frac{\sigma(x_{25}^*)}{(\Sigma_I^8(\mathbf{x}^*))^2}, \qquad C_{25}^{25} = -\frac{\sigma\left(\Sigma_I^8(\mathbf{x}^*)\right)}{(x_{25}^*)^2},$$
$$C_{26}^{1} = -\frac{\sigma(x_{26}^*)}{(\Sigma_I^5(\mathbf{x}^*))^2}, \quad C_{26}^{26} = -\frac{\sigma(x_{26}^*)}{(\Sigma_I^5(\mathbf{x}^*))^2} - \frac{\sigma\left(\Sigma_I^5(\mathbf{x}^*)\right)}{(x_{26}^*)^2} \quad C_{28}^{1} = -\frac{\sigma(x_{28}^*)}{\left(\Sigma_4^J(\mathbf{x}^*)\right)^2},$$
$$C_{28}^{28} = -\frac{\sigma\left(\Sigma_4^J(\mathbf{x}^*)\right)}{(x_{28}^*)^2}.$$

Consider the vector $\vec{u} \in T_{\mathbf{x}^*}\Delta^{27}$ with the following coordinates:

$$(\vec{u})_i = \begin{cases} 1 & \text{if } i = 2,3,4,6,7,8,10,11,12,16,17,18,20,21,22,24,25,26, \\ -3 & \text{if } i = 5,9,13,19,23,27, \\ 2 & \text{if } i = 14,28, \\ -2 & \text{if } i = 1,15. \end{cases}$$

For $l \in \{2,3,6,7,8,16,17,20,21,22\}$, $i \in \{14,28\}$, $j \in \{4,10,11,18,24,25\}$, and $k \in \{12,26\}$, we compute that

$$\nabla f_l(\mathbf{x}^*) \cdot \vec{u} = C_l^l < 0, \quad \nabla f_i(\mathbf{x}^*) \cdot \vec{u} = 2C_i^i < 0, \quad \nabla f_j(\mathbf{x}^*) \cdot \vec{u} = C_j^1 + C_j^j < 0,$$
$$\nabla f_k(\mathbf{x}^*) \cdot \vec{u} = C_k^4 + C_k^k < 0.$$

This implies that the values of f_l for $l \in I - J$ decrease along a line segment in the direction of $\vec{u}$. For a sufficiently short distance along $\vec{u}$ the values of f_l for $l \in J$ are smaller than α_*. Thus, there exists a point $\mathbf{z} \in \Delta^{27}$ such that $f_l(\mathbf{z}) < \alpha_*$ for every $l \in I = \{1,2,\ldots,28\}$. This is a contradiction. Hence, $f_l(\mathbf{x}^*) = \alpha_*$ for some $l \in J = \{1,5,9,13,15,19,23,27\}$. □

Let $\Delta = \{(x,y) \in \mathbb{R}^2 : x+y < 1,\ 0 < x, 0 < y\}$. Introduce the function $g : \Delta \to (0,1)$ defined by

$$g(x,y) = \frac{1-x-y}{x+y} \cdot \frac{1-y}{y}. \tag{3.2}$$

Given a displacement function f_l in $\mathcal{F}$ for $l \in J = \{1,5,9,13,15,19,23,27\}$, it can be expressed as

$$f_l(\mathbf{x}) = g\left(\Sigma_J^i(\mathbf{x}) - x_l, x_l\right)$$

for some $i \in \{1,2,3,4\}$. The function g was also used in [19]. In fact, the following statement [19, Lemma 3.2] was proved for g:

Lemma 3.3 *Let $C_g = \{(x,y) \in \Delta : x + 2y - xy - y^2 < \frac{3}{4}\}$. Then C_g is an open convex set and $g(x,y)$ is a strictly convex function on C_g.*

Therefore, by this lemma, each displacement function f_l for $l \in J$ is a strictly convex function over the open convex subset

$$C_{f_l} = \{\mathbf{x} = (x_1,\ldots,x_{28}) \in \Delta^{27} : \Sigma(\mathbf{x}) + 2x_l - \Sigma(\mathbf{x})x_l - (x_l)^2 < \tfrac{3}{4}\} \tag{3.3}$$

of Δ^{27}, where we set $\Sigma(\mathbf{x}) = \Sigma_J^i(\mathbf{x}) - x_l$ for a chosen $i \in \{1,2,3,4\}$ depending on l.

If C_{f_l} for $l \in J$ are as described above, then the subset $C = \cap_{l\in J} C_{f_l}$ of Δ^{27} is nonempty. This is because, if we consider the point $\mathbf{y}^*$ given in the proof of Lemma 3.1, then

$$\Sigma_J^i(\mathbf{y}^*) - y_l = 0.1423...$$

for every $i \in \{1,2,3,4\}$. We find that $\Sigma(\mathbf{y}^*) + 2y_l - \Sigma(\mathbf{y}^*)y_l - (y_l)^2 = 0.3307... < \frac{3}{4}$ for every $l \in J$. Thus, $\mathbf{y}^*$ is in C. Additionally, we have $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_{28}^*) \in C$ implied by the following two lemmas:

Lemma 3.4 *Let $\boldsymbol{x}^* \in \Delta^{27}$ so that $\alpha_* = F(\boldsymbol{x}^*)$. Then $\boldsymbol{x}^* \in C_{f_1}$, defined in (3.3), where*

$$f_1(\boldsymbol{x}) = \sigma(\Sigma_J^1)\sigma(x_1) = \frac{1 - x_1 - x_2 - x_3 - x_4 - x_5 - x_6 - x_7}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7} \cdot \frac{1 - x_1}{x_1}.$$

Proof Assume on the contrary that $\mathbf{x}^* \notin C_{f_1}$. Then, by the definition of C_{f_1}, we have

$$\sum_{l=2}^{7} x_l^* + \left(2 - \sum_{l=2}^{7} x_l^*\right) x_1^* - (x_1^*)^2 \geq \frac{3}{4}. \tag{3.4}$$

Let us say $N = \frac{1}{4}(3 - \sqrt{3}) \approx 0.3170$. Also, let $\Sigma_1^* = \sum_{l=1}^{7} x_l^* = \Sigma_J^1(\mathbf{x}^*)$, $\Sigma_2^* = \sum_{l=8}^{14} x_l^* = \Sigma_J^2(\mathbf{x}^*)$, $\Sigma_3^* = \sum_{l=15}^{21} x_l^* = \Sigma_J^3(\mathbf{x}^*)$, and $\Sigma_4^* = \sum_{l=22}^{28} x_l^* = \Sigma_J^4(\mathbf{x}^*)$. Consider the following cases:

$$(A)\ \ \Sigma(\mathbf{x}^*) \geq N,\ x_1^* \geq N,\ (B)\ \ \Sigma(\mathbf{x}^*) \geq N > x_1^*,\ (C)\ \ x_1^* \geq N > \Sigma(\mathbf{x}^*), \tag{3.5}$$

where $\Sigma(\mathbf{x}^*) = \Sigma_J^1(\mathbf{x}^*) - x_1^* = \sum_{l=2}^{7} x_l^*$. Assume that (A) is the case. Note that $\Sigma_1^* \geq 2N$. Then we have

$$\Sigma_2^* + \Sigma_3^* + \Sigma_4^* \leq M = 1 - 2N \approx 0.3660. \tag{3.6}$$

If $\Sigma_2^* \leq M/3 \approx 0.1220$, using Lemma 3.1 and $\sigma(M/3)\sigma(x_l^*) \leq \sigma(\Sigma_2^*)\sigma(x_l^*) \leq \alpha$, we find for every $l \in \{9, 13\}$ that

$$x_l^* \geq \frac{\sigma(M/3)}{(\alpha - 1) + \sigma(M/3)} = \frac{3 - M}{(\alpha - 2)M + 3} \approx 0.2317.$$

Then we see that $x_9^* > \Sigma_2^*$, a contradiction. This implies that $\Sigma_2^* > M/3$. We can repeat this argument with Σ_3^* and Σ_4^* to show that $\Sigma_3^* > M/3$ and $\Sigma_4^* > M/3$. This is a contradiction, so (A) is not the case.

Assume that (B) holds. Since we have $\Sigma(\mathbf{x}^*) \geq N$, we obtain the following inequality:

$$x_1^* + \Sigma_2^* + \Sigma_3^* + \Sigma_4^* \leq M = 1 - N \approx 0.6830. \tag{3.7}$$

If $\Sigma_2^* \leq M/4 \approx 0.1707$, then by the inequality $\sigma(M/4)\sigma(x_l^*) \leq \sigma(\Sigma_2^*)\sigma(x_l^*) \leq \alpha$, we find for every $l \in \{9, 13\}$ that

$$x_l^* \geq \frac{\sigma(M/4)}{\alpha + \sigma(M/4)} = \frac{4 - M}{(\alpha - 2)M + 4} \approx 0.1691. \tag{3.8}$$

Note that $x_9^* + x_{13}^* > \Sigma_2^*$, a contradiction. Thus, we get $\Sigma_2^* > M/4$. Similar arguments for Σ_3^* and Σ_4^* show that $\Sigma_3^* > M/4$ and $\Sigma_4^* > M/4$. Then we compute from (3.7) that $x_1^* \leq M/4$. By (3.4), we calculate that

$$\Sigma(\mathbf{x}^*) \geq L = \frac{3 - 2M}{4 - M} \approx 0.4926. \tag{3.9}$$

This implies $\Sigma(\mathbf{x}^*) + \Sigma_2^* + \Sigma_3^* + \Sigma_4^* > L + 3M/4 \approx 1.0049 > 1$, a contradiction. Hence, (B) is also not the case.

Assume that (C) in (3.5) holds. Since $x_1^* \geq N$, we have

$$\Sigma(\mathbf{x}^*) + \Sigma_2^* + \Sigma_3^* + \Sigma_4^* \leq M = 1 - N \approx 0.6830. \tag{3.10}$$

If $\Sigma_2^* \leq M/4$, then by (3.8), we derive that $x_9^* + x_{13}^* > \Sigma_2^*$ as in case (B), a contradiction. Therefore, we must have $\Sigma_2^* > M/4$. Similar computations for Σ_3^* and Σ_4^* imply as in case (B) that $\Sigma_3^* > M/4$ and $\Sigma_4^* > M/4$. Then we find that $\Sigma(\mathbf{x}^*) \leq M/4$. Since $(2 - \Sigma(\mathbf{x}^*))x_1^* < 2x_1^*$, using the inequality in (3.4), we calculate that

$$x_1^* \geq L = \frac{1}{4}\left(4 - \sqrt{5 + \sqrt{3}}\right) \approx 0.3513. \tag{3.11}$$

Since $\Sigma_2^* + \Sigma_3^* + \Sigma_4^* > 3M/4$, we find that $\Sigma_1^* < 1 - 3M/4$. By Lemma 3.1 and using the inequality $\sigma(1 - 3M/4)\sigma(x_5^*) < \sigma(\Sigma_1^*)\sigma(x_5^*) = f_5(\mathbf{x}^*) \leq \alpha$, we compute that

$$x_5^* > \frac{\sigma(1 - 3M/4)}{\alpha + \sigma(1 - 3M/4)} = \frac{3M}{(4 - 3M)\alpha + 3M} \approx 0.0405.$$

We have $x_1^* + \Sigma_2^* + \Sigma_3^* + \Sigma_4^* < 1$. By (3.11), we get $\Sigma_2^* + \Sigma_3^* + \Sigma_4^* < 1 - L \approx 0.6487$. By the inequality $\sigma(1 - L)\sigma(x_7^*) < \sigma(\Sigma_2^* + \Sigma_3^* + \Sigma_4^*)\sigma(x_7^*) = f_7(\mathbf{x}^*) \leq \alpha$, we derive that

$$x_7^* > \frac{\sigma(1 - L)}{\alpha + \sigma(1 - L)} = \frac{L}{(1 - L)\alpha + L} \approx 0.0213.$$

We claim that $\Sigma_2^* < \frac{1}{4}$ because otherwise we calculate that

$$x_1^* + x_5^* + x_7^* + \Sigma_2^* + \Sigma_3^* + \Sigma_4^* > L + \frac{3M}{(4 - 3M)\alpha + 3M} + \frac{L}{(1 - L)\alpha + L} + \frac{M}{2} + \frac{1}{4} \approx 1.0047 > 1, \tag{3.12}$$

a contradiction. Similarly, we find a contradiction in each case if we assume $\Sigma_3^* \geq \frac{1}{4}$ or $\Sigma_4^* \geq \frac{1}{4}$. Therefore, we have $\Sigma_r^* < \frac{1}{4}$ for every $r \in \{2, 3, 4\}$. Then, for every $l \in \{9, 13, 15, 19, 23, 27\}$, we obtain

$$x_l^* > \frac{\sigma(1/4)}{\alpha + \sigma(1/4)} \approx 0.1076$$

by the inequalities $\sigma(M/4)\sigma(x_l^*) \leq \sigma(\Sigma_r^*)\sigma(x_l^*) \leq \alpha$. Finally, we get the contradiction

$$x_1^* + x_5^* + x_7^* + x_9^* + x_{13}^* + x_{15}^* + x_{19}^* + x_{23}^* + x_{27}^* \approx 1.0591 > 1.$$

This shows that (C) is not the case either, which completes the proof. □

Lemma 3.5 *Let $\boldsymbol{x}^* \in \Delta^{27}$ so that $\alpha_* = F(\boldsymbol{x}^*)$. Then $\boldsymbol{x}^* \in C_{f_l}$, defined in (3.3), for $l \in \{5, 9, 13, 15, 19, 23, 27\}$.*

Proof The proof of Lemma 3.4 is symmetric in the sense that it can be repeated for every index $l \in \{5, 9, 13, 15, 19, 23, 27\}$. In particular, if $l = 5$, we interchange x_1^* with x_5^* and let $\Sigma(\mathbf{x}) = \Sigma_J^1(\mathbf{x}) - x_5$. Then we reiterate the computations carried out in the proof above by keeping the same organizations in (3.6), (3.7), (3.10), and (3.12).

For some $l \in \{9, 13, 15, 19, 23, 27\}$, we replace x_1^* with x_l^*, let $\Sigma(\mathbf{x}) = \Sigma_J^1(\mathbf{x}) - x_l$ for some $i \in \{1, 2, 3, 4\}$, and reorganize the inequalities in (3.6), (3.7), (3.10), and (3.12) by choosing relevant sums from Σ_1^*, Σ_2^*, Σ_3^*, and Σ_4^*. Then we carry out analogous calculations given in the proof of Lemma 3.4 for the chosen index l. □

We shall also need the observation below about g, defined in (3.2), in the computation of α_*. Its proof is elementary. Therefore, we shall omit it. We have:

Lemma 3.6 *For* $(x,y)\in C_g$, *the inequality* $g(x,y)<\alpha=24.8692...$ *holds if and only if* $0.1670...<y<\frac{1}{2}$ *and* $0<x<(-3+8y-4y^2)/(-4+4y)$ *or*

$$0.0134...=\frac{1+3\alpha-\sqrt{1-10\alpha+9\alpha^2}}{8\alpha}<y<\frac{1}{1-\alpha}+\sqrt{\frac{\alpha}{(\alpha-1)^2}}=0.1670...$$
$$\text{and}\quad \frac{1-2y+(1-\alpha)y^2}{1+(\alpha-1)y}<x<\frac{-3+8y-4y^2}{-4+4y}.$$

As mentioned earlier, the displacement functions $\{f_l\}$ for $l\in J=\{1,5,9,13,15,19,23,27\}$ play a more important role in the computation of α_*. These functions take larger values on $C=\bigcap_{l\in J}C_{f_l}$ than the values of the rest of the displacement functions in $\mathcal{F}$ at the points that are significant to calculate the infimum of the maximum of F. In other words, we have the following:

Lemma 3.7 *Let* $\widetilde{F}(\boldsymbol{x})=\max_{\boldsymbol{x}\in C}\{f_l(\boldsymbol{x}):l\in J\}$ *for* $C=\bigcap_{l\in J}C_{f_l}$. *Then,* $\widetilde{F}(\boldsymbol{x})\geq\alpha_*$.

Proof Assume on the contrary that $\widetilde{F}(\mathbf{z})<\alpha_*$ for some $\mathbf{z}\in C$. Then, by Lemma 3.1 for every $l\in J$, we have $f_l(\mathbf{z})<\alpha_*\leq\alpha=24.8692...$. Let $\mathbf{z}=(z_1,z_2,\ldots,z_{28})$.

Assume that $z_l>3/(3+\alpha)$ for every $l\in\{1,5\}$. Also assume that $z_l\leq 3/(3+\alpha)$ for every $l\in\{9,15,23\}$. By the inequalities $f_l(\mathbf{z})=\sigma(\Sigma_J^i(\mathbf{z}))\sigma(z_l)<\alpha$ for every $l\in\{9,15,23\}$, for every $i\in\{2,3,4\}$, we get

$$\Sigma_J^i(\mathbf{z})>\frac{\sigma\left(\frac{3}{3+\alpha}\right)}{\alpha+\sigma\left(\frac{3}{3+\alpha}\right)}=\frac{1}{4}. \tag{3.13}$$

Since $\Sigma_J^1(\mathbf{z})+\Sigma_J^2(\mathbf{z})+\Sigma_J^3(\mathbf{z})+\Sigma_J^4(\mathbf{z})=1$, we have $\Sigma_J^1(\mathbf{z})<\frac{1}{4}$. This implies that

$$\Sigma_J^1(\mathbf{z})-z_1<\frac{1}{4}-\frac{3}{3+\alpha}=0.1423.... \tag{3.14}$$

Because $\mathbf{z}\in C\subset C_{f_1}$, by Lemma 3.6 for $g=f_1$, $x=\Sigma_J^1-z_1$, and $y=z_1$, we find $z_1>0.4237...>\Sigma_J^1(\mathbf{z})$, a contradiction. Thus, $z_l>3/(3+\alpha)$ for some $l\in\{9,15,23\}$.

Assume without loss of generality that $z_9>3/(3+\alpha)$ and $z_l\leq 3/(3+\alpha)$ for every $l\in\{15,23\}$. Then we have $\Sigma_J^i(\mathbf{z})>\frac{1}{4}$ for every $i\in\{3,4\}$ by the inequalities $f_l(\mathbf{z})=\sigma(\Sigma_J^i(\mathbf{z}))\sigma(z_l)<\alpha$ for $l\in\{15,23\}$. This implies that $\Sigma_J^1(\mathbf{z})+\Sigma_J^2(\mathbf{z})<1/2$. If $\Sigma_J^1(\mathbf{z})<\frac{1}{4}$, then by the argument in the previous paragraph, we obtain a contradiction. If $\Sigma_J^2(\mathbf{z})<\frac{1}{4}$, we have $\Sigma_J^2(\mathbf{z})-z_9<0.1423....$ Using Lemma 3.6 for $g=f_9$, $x=\Sigma_J^2(\mathbf{z})-z_9$, and $y=z_9$, we find the contradiction $z_9>\Sigma_J^2(\mathbf{z})$. This implies that $z_l>3/(3+\alpha)$ for at least two distinct $l\in\{9,15,23\}$.

Assume again without loss of generality that $z_l>3/(3+\alpha)$ for every $l\in\{9,15\}$ and $z_{23}\leq 3/(3+\alpha)$. Then $\Sigma_J^4(\mathbf{z})>\frac{1}{4}$ by the inequality $f_{23}(\mathbf{z})=\sigma(\Sigma_J^4(\mathbf{z}))\sigma(z_{23})<\alpha$. This implies that $\Sigma_J^1(\mathbf{z})+\Sigma_J^2(\mathbf{z})+\Sigma_J^3(\mathbf{z})<\frac{3}{4}$, which in turn gives that $\Sigma_J^i(\mathbf{z})<\frac{1}{4}$ for some $i\in\{1,2,3\}$. Since $z_l>3/(3+\alpha)$ for every $l\in\{1,5,9,15\}$, depending on i, using z_1 and $g=f_1$, or z_9 and $g=f_9$, or z_{15} and $g=f_{15}$ in (3.14) and Lemma 3.6, we obtain a contradiction in each case by repeating the arguments given above. We must have $z_l>3/(3+\alpha)$ for every $l\in\{9,15,23\}$.

We already know that $\Sigma_J^1(\mathbf{z}) + \Sigma_J^2(\mathbf{z}) + \Sigma_J^3(\mathbf{z}) + \Sigma_J^4(\mathbf{z}) = 1$ as $\mathbf{z} \in C \subset \Delta^{27}$. Then we get $\Sigma_J^i(\mathbf{z}) \leq \frac{1}{4}$ for some $i \in \{1, 2, 3, 4\}$. Given i, by choosing appropriate z_l from the list $\{z_1, z_9, z_{15}, z_{23}\}$, we repeat the relevant argument carried out above and derive a contradiction using Lemma 3.6. As a result, we conclude that $z_l \leq 3/(3 + \alpha)$ for some $l \in \{1, 5\}$.

Notice that the computations used to show that $z_l \leq 3/(3+\alpha)$ for some $l \in \{1, 5\}$ are symmetric in the sense that they can be deployed to prove $z_l \leq 3/(3 + \alpha)$ for some l in any given pair $\{9, 13\}$, $\{15, 19\}$, and $\{23, 27\}$. This implies that there exist entries z_m, z_n, z_r, and z_s for $m \in \{1, 5\}$ $n \in \{9, 13\}$, $r \in \{15, 19\}$, and $s \in \{23, 27\}$ such that $z_l \leq 3/(3 + \alpha)$ for every $l \in \{m, n, r, s\}$. By the inequalities $f_l(\mathbf{z}) = \sigma(\Sigma_J^i(\mathbf{z}))\sigma(z_l) < \alpha$ for $l \in \{m, n, r, s\}$, we find that $\Sigma_J^i(\mathbf{z}) > \frac{1}{4}$ for every $i \in \{1, 2, 3, 4\}$, a contradiction. Hence, the conclusion of the lemma follows. □

Before we proceed to prove Proposition 3.11, we review three facts from convex analysis. These facts were also used in [19, Theorem 3.2, Theorem 3.3, and Proposition 3.3]. For their proofs interested readers may refer to this source and the references therein.

Theorem 3.8 *If $\{C_i\}$ for $i \in I$ is a collection of finitely many nonempty convex sets in $\mathbb{R}^d$ with $C = \cap_{i\in I} C_i \neq \emptyset$, then C is also convex.*

Theorem 3.9 *If $\{f_i\}$ for $i \in I$ is a finite set of strictly convex functions defined on a convex set $C \subset \mathbb{R}^d$, then $\max_{\boldsymbol{x}\in C}\{f_i(\boldsymbol{x}) : i \in I\}$ is also a strictly convex function on C.*

Proposition 3.10 *Let F be a convex function on an open convex set $C \subset \mathbb{R}^d$. If $\boldsymbol{x}^*$ is a local minimum of F, then it is a global minimum of F, and the set $\{\boldsymbol{y}^* \in C : F(\boldsymbol{y}^*) = F(\boldsymbol{x}^*)\}$ is a convex set. Furthermore, if F is strictly convex and $\boldsymbol{x}^*$ is a global minimum, then the set $\{\boldsymbol{y}^* \in C : F(\boldsymbol{y}^*) = F(\boldsymbol{x}^*)\}$ consists of $\boldsymbol{x}^*$ alone.*

With these facts, we can prove the following statement, which gives the first part of Property B:

Proposition 3.11 *Let $\mathcal{F} = \{f_i\}$ for $i \in I = \{1, 2, \ldots, 28\}$ be the set of displacement functions listed in Proposition 2.3 and F be as in (3.1). If $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are two points in Δ^{27} so that $\alpha_* = F(\boldsymbol{x}^*) = F(\boldsymbol{y}^*)$, then $\boldsymbol{x}^* = \boldsymbol{y}^*$.*

Proof We know by Lemma 3.3 that each f_l for $l \in J$ is a strictly convex function over the open convex set C_{f_l}. Therefore, $\widetilde{F}(\mathbf{x})$ defined in Lemma 3.6 is also strictly convex on $C = \cap_{l\in J} C_{f_l}$, which is itself an open convex set by Theorem 3.8 and Theorem 3.9. By Lemma 3.4 and Lemma 3.5, we have $\mathbf{x}^*$, $\mathbf{y}^* \in C$. Since $\widetilde{F}(\mathbf{x}) \geq \alpha_*$ for every $\mathbf{x} \in C$ and $\widetilde{F}(\mathbf{x}^*) = \alpha_*$ by Lemma 3.2 and Lemma 3.7, the value α_* is the global minimum of $\widetilde{F}$. As a result, we find that $\mathbf{x}^* = \mathbf{y}^*$ by Proposition 3.10. □

The uniqueness of $\mathbf{x}^*$ established by Proposition 3.11 simplifies the task of determining the relations among the coordinates of $\mathbf{x}^*$ considerably. In fact, we have the following statement:

Lemma 3.12 *If $\boldsymbol{x}^* = (x_1^*, x_2^*, \ldots, x_{28}^*) \in \Delta^{27}$ so that $F(\boldsymbol{x}^*) = \alpha_*$, then $x_i^* = x_j^*$ for all indices $i, j \in \{1, 5, 9, 13, 15, 19, 23, 27\}$. Also, for every $i, j \in \{2, 6, 8, 12, 16, 20, 22, 26\}$, $i, j \in \{3, 4, 10, 11, 17, 18, 24, 25\}$, and $i, j \in \{7, 14, 21, 28\}$, the equality $x_i^* = x_j^*$ holds.*

Proof Consider the permutations τ_1, τ_2, and τ_3 in the symmetric group S_{28} defined below:

$$\tau_1 = (1\ \ 5)(2\ \ 6)(3\ \ 4)(8\ \ 16)(9\ \ 15)(10\ \ 17)(11\ \ 18)(12\ \ 20)(13\ \ 19)(14\ \ 21)(22\ \ 26)(23\ \ 27)(24\ \ 25),$$

$$\tau_2 = (1\ \ 23)(2\ \ 22)(3\ \ 24)(4\ \ 25)(5\ \ 27)(6\ \ 26)(7\ \ 28)(8\ \ 12)(9\ \ 13)(10\ \ 11)(15\ \ 19)(16\ \ 20)(17\ \ 18),$$

$$\tau_3 = (1\ \ 13)(2\ \ 12)(3\ \ 11)(4\ \ 10)(5\ \ 9)(6\ \ 8)(7\ \ 14)(15\ \ 27)(16\ \ 26)(17\ \ 25)(18\ \ 24)(19\ \ 23)(20\ \ 22)(21\ \ 28).$$

Let $T_l : \Delta^{27} \to \Delta^{27}$ be the transformation defined by $x_i \mapsto x_{\tau_l(i)}$ for $l = 1, 2, 3$. Note that $T_l(\Delta^{27}) = \Delta^{27}$ for every l. Let $H_l : \Delta^{27} \to \mathbb{R}$ be the map so that $H_l(\mathbf{x}) = \max\{(f_i \circ T_l)(\mathbf{x}) : i = 1, 2, \ldots, 28\}$. Then we have $f_i(T_l(\mathbf{x})) = f_{\tau_l(i)}(\mathbf{x})$ for every $\mathbf{x} \in \Delta^{27}$ for every $i = 1, 2, \ldots, 28$ for every $l = 1, 2, 3$. This implies that $F(\mathbf{x}) = H_l(\mathbf{x})$ for every $\mathbf{x}$ and for every l. Since $\mathbf{x}^*$ is unique by Proposition 3.11, we obtain $T_l^{-1}(\mathbf{x}^*) = \mathbf{x}^*$ for $l = 1, 2, 3$. Then the lemma follows. □

Lemma 3.12 implies that $f_i(\mathbf{x}^*) = f_j(\mathbf{x}^*)$ for every $i, j \in \{1, 5, 9, 13, 15, 19, 23, 27\}$. Also, for every $i, j \in \{2, 6, 8, 12, 16, 20, 22, 26\}$, $i, j \in \{3, 4, 10, 11, 17, 18, 24, 25\}$, and $i, j \in \{7, 14, 21, 28\}$ we have $f_i(\mathbf{x}^*) = f_j(\mathbf{x}^*)$. Therefore, there are four values to consider at $\mathbf{x}^*$ to compute α_*: $f_1(\mathbf{x}^*)$, $f_2(\mathbf{x}^*)$, $f_3(\mathbf{x}^*)$, and $f_7(\mathbf{x}^*)$, which are given as

$$\frac{1 - 2(x_1^* + x_2^* + x_3^*) - x_7^*}{2(x_1^* + x_2^* + x_3^*) + x_7^*} \cdot \frac{1 - x_1^*}{x_1^*} = \alpha_*, \tag{3.15}$$

$$\frac{1 - 7(x_1^* + x_2^* + x_3^*) - 4x_7^*}{7(x_1^* + x_2^* + x_3^*) + 4x_7^*} \cdot \frac{1 - x_2^*}{x_2^*} \leq \alpha_*, \tag{3.16}$$

$$\frac{1 - 7(x_1^* + x_2^* + x_3^*) - 4x_7^*}{7(x_1^* + x_2^* + x_3^*) + 4x_7^*} \cdot \frac{1 - x_3^*}{x_3^*} \leq \alpha_*, \tag{3.17}$$

$$\frac{1 - 6(x_1^* + x_2^* + x_3^*) - 3x_7^*}{6(x_1^* + x_2^* + x_3^*) + 3x_7^*} \cdot \frac{1 - x_7^*}{x_7^*} \leq \alpha_*. \tag{3.18}$$

We shall show next that $f_2(\mathbf{x}^*) = f_3(\mathbf{x}^*) = f_7(\mathbf{x}^*) = \alpha_*$. For this, we will need the statement below:

Lemma 3.13 *For $1 \leq k \leq n-1$, let $f_1, \ldots, f_k$ be smooth functions on an open neighborhood U of the $(n-1)-$simplex Δ^{n-1} in $\mathbb{R}^n$. If at some $\boldsymbol{x} \in \Delta^{n-1}$ the collection $\{\nabla f_1(\boldsymbol{x}), \nabla f_2(\boldsymbol{x}), \ldots, \nabla f_k(\boldsymbol{x}), \langle 1, \ldots, 1\rangle\}$ of vectors in $\mathbb{R}^n$ is linearly independent, then there exists a vector $\vec{u} \in T_{\boldsymbol{x}}\Delta^{n-1}$ such that each f_i for $i = 1, \ldots, k$ decreases in the direction of $\vec{u}$ at $\boldsymbol{x}$.*

Interested readers may refer to [18, Lemma 4.10] for its proof. We have the following statement:

Proposition 3.14 *Let $\mathcal{F} = \{f_i\}$ for $i \in I = \{1, 2, \ldots, 28\}$ be the set of displacement functions listed in Proposition 2.3 and F be as in (3.1). If $\boldsymbol{x}^*$ is the point such that $F(\boldsymbol{x}^*) = \alpha_*$, then $\boldsymbol{x}^*$ is in the set* $\Delta_{27} = \{\boldsymbol{x} \in \Delta^{27} : f_i(\boldsymbol{x}) = f_j(\boldsymbol{x}) \text{ for every } i, j \in I\}$.

Proof By Lemma 3.12, it is enough to show that $f_2(\mathbf{x}^*) = f_3(\mathbf{x}^*) = f_7(\mathbf{x}^*) = \alpha_*$. Remember that C_i^j denotes the partial derivative of f_i with respect to x_j at $\mathbf{x}^*$. We calculate the constants below:

$$C_1^1 = -\frac{\sigma(x_1^*)}{\left(\Sigma_J^1(\mathbf{x}^*)\right)^2} - \frac{\sigma\left(\Sigma_J^1(\mathbf{x}^*)\right)}{(x_1^*)^2}, \quad C_1^2 = -\frac{\sigma(x_1^*)}{\left(\Sigma_J^1(\mathbf{x}^*)\right)^2}, \quad C_5^1 = -\frac{\sigma(x_5^*)}{\left(\Sigma_J^1(\mathbf{x}^*)\right)^2}, \quad C_{15}^{16} = -\frac{\sigma(x_{15}^*)}{\left(\Sigma_J^3(\mathbf{x}^*)\right)^2},$$

$$C_5^5 = -\frac{\sigma(x_5^*)}{\left(\Sigma_J^1(\mathbf{x}^*)\right)^2} - \frac{\sigma\left(\Sigma_J^1(\mathbf{x}^*)\right)}{(x_5^*)^2}, \quad C_9^8 = -\frac{\sigma(x_9^*)}{\left(\Sigma_J^2(\mathbf{x}^*)\right)^2}, \quad C_{13}^8 = -\frac{\sigma(x_{13}^*)}{\left(\Sigma_J^2(\mathbf{x}^*)\right)^2}, \quad C_{19}^{15} = -\frac{\sigma(x_{19}^*)}{\left(\Sigma_J^3(\mathbf{x}^*)\right)^2},$$

$$C_9^9 = -\frac{\sigma(x_9^*)}{\left(\Sigma_J^2(\mathbf{x}^*)\right)^2} - \frac{\sigma\left(\Sigma_J^2(\mathbf{x}^*)\right)}{(x_9^*)^2}, \quad C_{13}^{13} = -\frac{\sigma(x_{13}^*)}{\left(\Sigma_J^2(\mathbf{x}^*)\right)^2} - \frac{\sigma\left(\Sigma_J^2(\mathbf{x}^*)\right)}{(x_{13}^*)^2}, \quad C_{15}^{15} = -\frac{\sigma(x_{15}^*)}{\left(\Sigma_J^3(\mathbf{x}^*)\right)^2} - \frac{\sigma\left(\Sigma_J^3(\mathbf{x}^*)\right)}{(x_{15}^*)^2},$$
$$C_{19}^{19} = -\frac{\sigma(x_{19}^*)}{\left(\Sigma_J^3(\mathbf{x}^*)\right)^2} - \frac{\sigma\left(\Sigma_J^3(\mathbf{x}^*)\right)}{(x_{19}^*)^2}, \quad C_{23}^{23} = -\frac{\sigma(x_{23}^*)}{\left(\Sigma_J^4(\mathbf{x}^*)\right)^2} - \frac{\sigma\left(\Sigma_J^4(\mathbf{x}^*)\right)}{(x_{23}^*)^2}, \quad C_{27}^{27} = -\frac{\sigma(x_{27}^*)}{\left(\Sigma_J^4(\mathbf{x}^*)\right)^2} - \frac{\sigma\left(\Sigma_J^4(\mathbf{x}^*)\right)}{(x_{27}^*)^2},$$

$$C_{23}^{22} = -\frac{\sigma(x_{23}^*)}{\left(\Sigma_J^4(\mathbf{x}^*)\right)^2}, \quad C_{27}^{22} = -\frac{\sigma(x_{27}^*)}{\left(\Sigma_J^4(\mathbf{x}^*)\right)^2}.$$

Since we have $\Sigma_J^1(\mathbf{x}^*) = \Sigma_J^2(\mathbf{x}^*) = \Sigma_J^3(\mathbf{x}^*) = \Sigma_J^4(\mathbf{x}^*)$, we derive that $C_1^1 = C_5^5 = C_9^9 = C_{13}^{13} = C_{15}^{15} = C_{19}^{19} = C_{23}^{23} = C_{27}^{27}$ and $C_1^2 = C_5^1 = C_9^8 = C_{13}^8 = C_{15}^{16} = C_{19}^{15} = C_{23}^{22} = C_{27}^{22}$ by Lemma 3.12. Again by the same lemma, we have $\Sigma_I^4(\mathbf{x}^*) = \Sigma_I^6(\mathbf{x}^*) = \Sigma_I^7(\mathbf{x}^*) = \Sigma_I^1(\mathbf{x}^*) = \Sigma_I^8(\mathbf{x}^*) = \Sigma_I^2(\mathbf{x}^*) = \Sigma_I^3(\mathbf{x}^*) = \Sigma_I^5(\mathbf{x}^*)$. For the constants given in Lemma 3.2, this implies that $C_2^2 = C_6^6 = C_8^8 = C_{12}^{12} = C_{16}^{16} = C_{20}^{20} = C_{22}^{22} = C_{26}^{26}$, $C_2^1 = C_6^1 = C_8^1 = C_{12}^4 = C_{16}^1 = C_{20}^1 = C_{21}^1 = C_{26}^1$, $C_3^3 = C_4^4 = C_{10}^{10} = C_{11}^{11} = C_{17}^{17} = C_{18}^{18} = C_{24}^{24} = C_{25}^{25}$, and $C_3^4 = C_4^1 = C_{10}^1 = C_{11}^1 = C_{17}^1 = C_{18}^1 = C_{24}^1 = C_{25}^1$. Note that we get $\Sigma_1^J(\mathbf{x}^*) = \Sigma_2^J(\mathbf{x}^*) = \Sigma_3^J(\mathbf{x}^*) = \Sigma_4^J(\mathbf{x}^*)$ by Lemma 3.12. As a result, we also see that $C_7^7 = C_{14}^{14} = C_{21}^{21} = C_{28}^{28}$ and $C_7^8 = C_{14}^1 = C_{21}^1 = C_{28}^1$.

Consider the 28×28 matrix below whose rows are $\nabla f_1(\mathbf{x}^*)$, $\nabla f_2(\mathbf{x}^*)$, ..., $\nabla f_{28}(\mathbf{x}^*)$:

$$\begin{bmatrix}
C_1^1 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_2^1 & C_2^2 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & 0 & 0 & 0 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
0 & 0 & C_3^3 & C_3^4 \\
C_3^4 & C_3^4 & C_3^4 & C_3^3 & 0 & 0 & C_3^4 \\
C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^1 & C_1^2 & C_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^2 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & 0 & 0 & 0 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
0 & 0 & 0 & 0 & 0 & 0 & C_7^7 & C_7^8 \\
C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^2 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & 0 & 0 & 0 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & C_1^2 & C_1^1 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & 0 & 0 & C_3^3 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 \\
C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^3 & 0 & 0 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 \\
0 & 0 & 0 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^2 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^1 & C_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & 0 & 0 & 0 & 0 & 0 & 0 & C_7^7 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_1^1 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^2 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & 0 & 0 & 0 & C_2^1 \\
C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & 0 & 0 & C_3^3 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 \\
C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^3 & 0 & 0 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 & C_3^4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^1 & C_1^2 & C_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_2^1 & C_2^1 & C_2^1 & 0 & 0 & 0 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^2 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & 0 & 0 & 0 & 0 & 0 & 0 & C_7^7 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 & C_7^8 \\
C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & 0 & 0 & 0 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^2 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_1^2 & C_1^1 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^2 \\
C_3^4 & 0 & 0 & C_3^3 & C_3^4 & C_3^4 & C_3^4 & C_3^4 \\
C_3^4 & C_3^3 & 0 & 0 & C_3^4 \\
C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & 0 & 0 & 0 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^2 & C_2^1 & C_2^1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^1 & C_1^2 \\
C_7^8 & 0 & 0 & 0 & 0 & 0 & 0 & C_7^7
\end{bmatrix}.$$

Assume that $f_2(\mathbf{x}^*) < \alpha_*$. Consider the vector $\vec{v}_1 \in T_{\mathbf{x}^*}\Delta^{27}$ with the following coordinates:

$$(\vec{v}_1)_i = \begin{cases} 1 & \text{if } i = 1,3,4,5,7,9,10,11,13,14,15,17,18,19,21,23,24,25,27,28, \\ -2 & \text{if } i = 6,12,20,26, \\ -3 & \text{if } i = 2,8,16,22. \end{cases}$$

For any given indices $l \in J = \{1,5,9,13,15,19,23,27\}$, $i \in K = \{3,10,17,24\}$, $j \in L = \{4,11,18,25\}$, and

$k \in N = \{7, 14, 21, 28\}$, we calculate that

$$\nabla f_l(\mathbf{x}^*) \cdot \vec{v}_1 = C_1^1 - C_1^2 = -\frac{\sigma\left(\Sigma_J^1(\mathbf{x}^*)\right)}{(x_1^*)^2} < 0, \quad \nabla f_i(\mathbf{x}^*) \cdot \vec{v}_1 = C_3^3 + C_3^4 = -\frac{\sigma\left(\Sigma_I^1(\mathbf{x}^*)\right)}{(x_3^*)^2} - \frac{(x_3^*)^2}{\sigma\left(\Sigma_I^1(\mathbf{x}^*)\right)} < 0,$$
$$\nabla f_j(\mathbf{x}^*) \cdot \vec{v}_1 = C_3^3 = -\frac{\sigma\left(\Sigma_I^1(\mathbf{x}^*)\right)}{(x_3^*)^2} < 0, \qquad \nabla f_k(\mathbf{x}^*) \cdot \vec{v}_1 = C_7^7 = -\frac{\sigma\left(\Sigma_J^1(\mathbf{x}^*)\right)}{(x_7^*)^2} < 0.$$

This implies that values of f_l for $l \in J \cup K \cup L \cup N$ decrease along a line segment in the direction of $\vec{v}_1$. For a short distance along $\vec{v}_1$, values of f_l for $l \in \{2, 6, 8, 12, 16, 20, 22, 26\}$ are smaller than α_*. There exists a point $\mathbf{z} \in \Delta^{27}$ such that $f_l(\mathbf{z}) < \alpha_*$ for every $l \in I = \{1, 2, \ldots, 28\}$. This is a contradiction. Hence, we find that $f_2(\mathbf{x}^*) = \alpha_*$. Assume that $f_3(\mathbf{x}^*) < \alpha_*$. We introduce the vector $\vec{v}_2 \in T_{\mathbf{x}^*}\Delta^{27}$ with the coordinates

$$(\vec{v}_2)_i = \begin{cases} 1 & \text{if } i = 1, 2, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 19, 20, 21, 22, 23, 26, 27, 28, \\ -2 & \text{if } i = 4, 11, 18, 25, \\ -3 & \text{if } i = 3, 10, 17, 24. \end{cases}$$

For $l \in J$, $i \in K' = \{2, 6, 16, 20\}$, $j \in L' = \{8, 12, 22, 26\}$, and $k \in N$, we calculate

$$\nabla f_l(\mathbf{x}^*) \cdot \vec{v}_2 = C_1^1 - C_1^2 = -\frac{\sigma\left(\Sigma_J^1(\mathbf{x}^*)\right)}{(x_1^*)^2} < 0, \quad \nabla f_k(\mathbf{x}^*) \cdot \vec{v}_1 = C_7^7 = -\frac{\sigma\left(\Sigma_J^1(\mathbf{x}^*)\right)}{(x_7^*)^2} < 0,$$
$$\nabla f_i(\mathbf{x}^*) \cdot \vec{v}_2 = C_2^2 - C_2^1 = -\frac{\sigma\left(\Sigma_I^4(\mathbf{x}^*)\right)}{(x_2^*)^2} < 0,$$

which show that values of f_l for $l \in J \cup K' \cup L' \cup N$ decrease along a line segment in the direction of $\vec{v}_2$. Values of f_l for $l \in \{2, 6, 8, 12, 16, 20, 22, 26\}$ are smaller than α_* for a short distance along $\vec{v}_2$. As a result, there exists a point $\mathbf{w} \in \Delta^{27}$ such that $f_l(\mathbf{w}) < \alpha_*$ for every $l \in I = \{1, 2, \ldots, 28\}$, a contradiction. We derive that $f_3(\mathbf{x}^*) = \alpha_*$. Since we have $f_2(\mathbf{x}^*) = f_3(\mathbf{x}^*) = \alpha_*$, we obtain $x_2^* = x_3^*$. Then we see that $C_2^1 = C_3^4$. Also, we find that $C_3^3 = C_2^2 - C_2^1$. Now assume that $f_7(\mathbf{x}^*) < \alpha_*$. Then we construct the 25×28 matrix A below:

$$\left[\begin{array}{cccccccccccccccccccccccccccc}
C_1^1 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_2^1 & C_2^2 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & 0 & 0 & 0 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
0 & 0 & C_2^2 - C_2^1 & C_2^1 \\
C_2^1 & C_2^1 & C_2^1 & C_2^2 - C_2^1 & 0 & 0 & C_2^1 \\
C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^1 & C_1^2 & C_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^2 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & 0 & 0 & 0 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^2 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & 0 & 0 & 0 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & C_1^2 & C_1^1 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & 0 & 0 & C_2^2 - C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^2 - C_2^1 & 0 & 0 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
0 & 0 & 0 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^2 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^1 & C_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_1^1 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^2 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & 0 & 0 & 0 & C_2^1 \\
C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & 0 & 0 & C_2^2 - C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^2 - C_2^1 & 0 & 0 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^1 & C_1^2 & C_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_2^1 & C_2^1 & C_2^1 & 0 & 0 & 0 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^2 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & 0 & 0 & 0 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^2 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_1^2 & C_1^1 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^2 \\
C_2^1 & 0 & 0 & C_2^2 - C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 \\
C_2^1 & C_2^2 - C_2^1 & 0 & 0 & C_2^1 \\
C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & 0 & 0 & 0 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^1 & C_2^2 & C_2^1 & C_2^1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^2 & C_1^1 & C_1^2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right].$$

Let R_l denote the lth row of A for $l \in \{1, 2, \ldots, 25\}$. Applying from left to right and row by row, we perform on A the row reduction operations listed in Table 11 simultaneously.

Table 11. Row reduction operations on A.

$-C_2^1R_{25}+R_{23}\to R_{23}$	$-C_2^1R_{25}+R_{22}\to R_{22}$	$-C_2^1R_{25}+R_{21}\to R_{21}$	$-C_2^1R_{25}+R_{19}\to R_{19}$	$-C_2^1R_{25}+R_{18}\to R_{18}$
$-C_2^1R_{25}+R_{16}\to R_{16}$	$-C_2^1R_{25}+R_{15}\to R_{15}$	$-C_2^1R_{25}+R_{14}\to R_{14}$	$-C_2^1R_{25}+R_{10}\to R_{10}$	$-C_2^1R_{25}+R_9\to R_9$
$-C_2^1R_{25}+R_7\to R_7$	$-C_2^1R_{25}+R_6\to R_6$	$-C_2^1R_{25}+R_4\to R_4$	$-C_2^1R_{25}+R_2\to R_2$	$-C_1^1R_{25}+R_1\to R_1$
$-C_1^2R_{25}+R_5\to R_5$	$R_{18}+R_{11}\to R_{11}$	$R_{19}+R_{11}\to R_{11}$	$R_{18}+R_3\to R_3$	$R_{19}+R_3\to R_3$
$-R_{11}+R_3\to R_3$	$-2R_{18}+R_4\to R_4$	$-R_{19}+R_7\to R_7$	$R_{18}+R_4\to R_4$	$\dfrac{1}{C_2^2-C_2^1}R_3\to R_3$
$\dfrac{1}{C_2^2-C_2^1}R_4\to R_4$	$-R_{19}+R_9\to R_9$	$R_{19}+R_7\to R_7$	$-R_{21}+R_7\to R_7$	$\dfrac{1}{C_2^2-C_2^1}R_7\to R_7$
$R_{12}+R_1\to R_1$	$R_{13}+R_1\to R_1$	$R_{20}+R_1\to R_1$	$\dfrac{1}{C_1^2-C_1^1}R_1\to R_1$	$R_8+R_5\to R_5$
$R_{13}+R_5\to R_5$	$R_{20}+R_5\to R_5$	$\dfrac{1}{C_1^1-C_1^2}R_5\to R_5$	$-R_{10}+R_2\to R_2$	$\dfrac{1}{C_2^2-C_2^1}R_2\to R_2$
$-R_{16}+R_6\to R_6$	$\dfrac{1}{C_2^2-C_2^1}R_6\to R_6$	$-R_{12}+R_8\to R_8$	$\dfrac{1}{C_1^1-C_1^2}R_8\to R_8$	$\dfrac{1}{C_2^2-C_2^1}R_9\to R_9$
$-R_{17}+R_{13}\to R_{13}$	$\dfrac{1}{C_1^1-C_1^2}R_{13}\to R_{13}$	$-R_{24}+R_{20}\to R_{20}$	$\dfrac{1}{C_1^1-C_1^2}R_{20}\to R_{20}$	$C_2^1R_4+R_{18}\to R_{18}$
$C_2^1R_5+R_{18}\to R_{18}$	$C_2^1R_6+R_{18}\to R_{18}$	$C_2^1R_7+R_{19}\to R_{19}$	$C_2^1R_8+R_{19}\to R_{19}$	$C_2^1R_9+R_{19}\to R_{19}$
$-C_2^1R_8+R_{18}\to R_{18}$	$-C_1^2R_7+R_{12}\to R_{12}$	$-C_1^2R_8+R_{12}\to R_{12}$	$-C_1^2R_9+R_{12}\to R_{12}$	$-R_2+R_1\to R_1$
$-R_3+R_1\to R_1$	$-R_4+R_1\to R_1$	$-R_5+R_1\to R_1$	$-R_6+R_1\to R_1$	$-R_7+R_1\to R_1$
$-R_8+R_1\to R_1$	$-R_9+R_1\to R_1$	$-C_2^1R_1+R_{11}\to R_{11}$	$-C_2^1R_8+R_{11}\to R_{11}$	$-R_{11}+R_{10}\to R_{10}$
$2R_{18}+R_{10}\to R_{10}$	$-R_{19}+R_{10}\to R_{10}$	$R_{19}+R_{11}\to R_{11}$	$R_{19}+R_{18}\to R_{18}$	$C_2^1R_{13}+R_{15}\to R_{15}$
$-2C_2^1R_{13}+R_{11}\to R_{11}$	$-C_1^2R_{13}+R_{17}\to R_{17}$	$-C_2^1R_{13}+R_{18}\to R_{18}$	$C_2^1R_{13}+R_{23}\to R_{23}$	$-R_{23}+R_{15}\to R_{15}$
$-R_{22}+R_{14}\to R_{14}$	$-R_{18}+R_{11}\to R_{11}$	$\dfrac{1}{C_2^2-C_2^1}R_{14}\to R_{14}$	$\dfrac{1}{C_2^2-C_2^1}R_{15}\to R_{15}$	$-C_1^2R_{14}+R_{17}\to R_{17}$
$-C_1^2R_{15}+R_{17}\to R_{17}$	$C_2^1R_{14}+R_{23}\to R_{23}$	$C_2^1R_{15}+R_{23}\to R_{23}$	$-C_2^1R_{20}+R_{18}\to R_{18}$	$C_2^1R_{20}+R_{21}\to R_{21}$
$-C_1^2R_{20}+R_{24}\to R_{24}$	$R_{18}\leftrightarrow R_{19}$	$R_{17}\leftrightarrow R_{18}$	$R_{16}\leftrightarrow R_{17}$	$R_{15}\leftrightarrow R_{16}$
$R_{14}\leftrightarrow R_{15}$	$R_{13}\leftrightarrow R_{14}$	$R_{20}\leftrightarrow R_{21}$	$R_{22}\leftrightarrow R_{23}$	$R_{21}\leftrightarrow R_{22}$
	$R_{23}\leftrightarrow R_{24}$	$R_{22}\leftrightarrow R_{23}$	$R_{20}\leftrightarrow R_{21}$	

Then we see that A is row equivalent to the matrix $\widetilde{A}$ below:

$$
\left[\begin{array}{cccccccccc|cccc|ccc|cccccccc|ccc}
0&0&0&0&0&0&1&0&-1&0&2&2&1&1&-1&1&1&2&1&2&1&2&-1&2&1&1&1&1\\
0&1&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&C_2^2-C_2^1&C_2^1-C_2^2&C_2^1&0&0&0&0&-C_2^1&0&C_2^2-2C_2^1&0&4C_2^1-2C_2^2&0&2C_2^1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-C_2^1&C_2^2-2C_2^1&0&0&0&0&0&0&C_2^1&0&0&C_2^2-2C_2^1&C_2^1&-C_2^1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&C_1^2&C_1^2&C_1^1+C_1^2&C_1^2&0&0&0&0&0&0&0&C_1^2&0&C_1^2&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&-C_2^1&0&0&0&0&0&0&0&0&C_2^2-2C_2^1&0&-C_2^1&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&-1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&-1&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&C_2^2-2C_2^1&-C_2^1&-C_2^1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&C_1^2&C_1^1+C_1^2&C_1^2&C_1^2&0&0&0&C_1^2&C_1^2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-C_2^1&C_2^1&C_2^2-2C_2^1&0&C_2^2-2C_2^1&0&-C_2^1&0&0&C_2^1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-C_2^1&0&0&0&0&0&-C_2^1&C_2^2-2C_2^1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-C_2^1&0&C_2^2-2C_2^1&0&0&-C_2^1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&C_1^2&0&C_1^2&C_1^2&C_1^2&C_1^1+C_1^2&C_1^2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&-1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&C_2^2-2C_2^1&-C_2^1&-C_2^1&0\\
\hline
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{array}\right].
$$

Note that in the presentation $\widetilde{A}$ is partitioned. Let $\widetilde{A}_{2,2}$ and $\widetilde{A}_{4,4}$ denote the $(2,2)$ and $(4,4)$ partitions, respectively, of $\widetilde{A}$ counting from left to right and top to bottom. The matrix $\widetilde{A}$ has full rank if and only if $det(\widetilde{A}_{2,2})\neq 0$ and $det(\widetilde{A}_{4,4})\neq 0$. We have

$$det(\widetilde{A}_{2,2})=C_1^2C_2^1\left(C_2^1-C_2^2\right)\left(3C_2^1-C_2^2\right),\quad det(\widetilde{A}_{4,4})=\left(C_1^2\right)^2C_2^1\left(C_2^1-C_2^2\right)^2\left(2C_2^1-C_2^2\right)\left(3C_2^1-C_2^2\right).$$

We know that $C_1^2 \neq 0$, $C_2^1 \neq 0$ and $C_2^1 - C_2^2 \neq 0$, so $\widetilde{A}$ has full rank if and only if $3C_2^1 - C_2^2 \neq 0$ and $2C_2^1 - C_2^2 \neq 0$, where $\Sigma_I^4(\mathbf{x}^*) = x_{11}^* + x_{12}^* + x_{13}^* = x_1^* + 2x_2^*$,

$$3C_2^1 - C_2^2 = \frac{\sigma(\Sigma_I^4(\mathbf{x}^*))}{(x_2^*)^2} - \frac{2\sigma(x_2^*)}{(\Sigma_I^4(\mathbf{x}^*))^2} = \frac{\Sigma_I^4(\mathbf{x}^*)(1-\Sigma_I^4(\mathbf{x}^*)) - 2x_2^*(1-x_2^*)}{(x_2^*)^2(\Sigma_I^4(\mathbf{x}^*))^2},$$

$$2C_2^1 - C_2^2 = \frac{\sigma(\Sigma_I^4(\mathbf{x}^*))}{(x_2^*)^2} - \frac{\sigma(x_2^*)}{(\Sigma_I^4(\mathbf{x}^*))^2} = \frac{\Sigma_I^4(\mathbf{x}^*)(1-\Sigma_I^4(\mathbf{x}^*)) - x_2^*(1-x_2^*)}{(x_2^*)^2(\Sigma_I^4(\mathbf{x}^*))^2}.$$

Assume on the contrary that $3C_2^1 - C_2^2 = 0$. We simplify the previous equality and get

$$(x_1^* + 2x_2^*)(1 - x_1^* - 2x_2^*) - 2x_2^*(1 - x_2^*) = 0 \quad \text{or} \quad x_2^* = -x_1^* + \sqrt{\frac{x_1^* + (x_1^*)^2}{2}} \tag{3.19}$$

as $x_2^* > 0$. Since $\mathbf{x}^* \in \Delta^{27}$, we have $8(x_1^* + 2x_2^*) + 4x_7^* = 1$. This implies $0 < x_1^* < \Sigma_I^4(\mathbf{x}^*) = x_1^* + 2x_2^* < \frac{1}{8}$. By (3.19), we have $x_2^* < x_1^*$ if and only if $x_1^* > \frac{1}{7}$. Using the equality $f_2(\mathbf{x}^*) = f_3(\mathbf{x}^*)$ and the formulas of $f_1(\mathbf{x}^*)$, $f_2(\mathbf{x}^*)$, and $f_3(\mathbf{x}^*)$ in (3.15), (3.16), and (3.17), we find that $\sigma(x_2^*) = 3\sigma(\Sigma_I^4(\mathbf{x}^*))\sigma(x_1^*)$, where $\sigma(\Sigma_I^4(\mathbf{x}^*)) > 1$. Thus, we deduce that $x_2^* < x_1^*$. This is a contradiction.

Next, assume that $2C_2^1 - C_2^2 = 0$. Then we get $(x_1^* + 2x_2^*)(1 - x_1^* - 2x_2^*) - x_2^*(1 - x_2^*) = 0$. This gives

$$x_2^* = \frac{1 - x_1^*}{3} \quad \text{or} \quad x_2^* = -x_1^*.$$

Since $x_2^* > 0$, we obtain $x_1^* + 3x_2^* = 1$ or $7x_1^* + 13x_2^* + 4x_7^* = 0$, a contradiction. This shows that A has full rank.

By Lemma 3.13, there exists a direction $\vec{v}_3 \in T_{\mathbf{x}^*}\Delta^{27}$ such that values of f_l for $l \in I - \{7, 14, 21, 28\}$ decrease along a line segment in the direction of $\vec{v}_3$. Values of f_l for $l \in \{7, 14, 21, 28\}$ are smaller than α_* for a short distance along $\vec{v}_3$. As a result, there exists a point $\mathbf{w} \in \Delta^{27}$ such that $f_l(\mathbf{w}) < \alpha_*$ for every $l \in I = \{1, 2, \ldots, 28\}$, a contradiction. Therefore, we obtain that $f_7(\mathbf{x}^*) = \alpha_*$. This concludes the proof. □

Propositions 3.11 and 3.14 establish the properties of F given in Property B in the introduction. Once these properties are verified, the computation of α_*, and consequently the infimums of the maximum of the displacement functions in $\mathcal{F}$ and $\mathcal{G}$ on Δ^{27}, is straightforward. In other words, we have the statements below:

Theorem 3.15 *Let $F : \Delta^{27} \to \mathbb{R}$ be defined by $\boldsymbol{x} \to \max\{f(\boldsymbol{x}) : f \in \mathcal{F}\}$, where $\mathcal{F}$ is the set of functions listed in 2.3. Then $\inf_{\boldsymbol{x} \in \Delta^{27}} F(\boldsymbol{x}) = \alpha_* = 24.8692...$, the unique real root of the polynomial $21x^4 - 496x^3 - 654x^2 + 24x + 81$ greater than 9.*

Proof Since $\mathbf{x}^* \in \Delta^{27}$, we have $8x_1^* + 8x_2^* + 8x_3^* + 4x_7^* = 1$ by Lemma 3.12. We plug $x_1^* + x_2^* + x_3^* = \frac{1}{8} - x_7^*/2$ into $f_7(\mathbf{x}^*) = \alpha_*$ in (3.18). Then we find $x_7^* = 1/(1 + 3\alpha_*)$. Using x_7^*, we obtain from $f_1(\mathbf{x}^*) = \alpha_*$ in (3.15) that $x_1^* = 3/(3 + \alpha_*)$. Because we have $f_2(\mathbf{x}^*) = f_3(\mathbf{x}^*)$ by Proposition 3.14, using the formulas in (3.16) and (3.17), we find

$$x_2^* = x_3^* = \frac{3(\alpha_* - 1)}{21\alpha_*^2 + 14\alpha_* - 3}.$$

When we plug all these values into the equation $2x_1^* + 2x_2^* + 2x_3^* + x_7^* = \frac{1}{4}$, we see that α_* satisfies the equation $21x^4 - 496x^3 - 654x^2 + 24x + 81 = 0$, which has the roots

$$\alpha_1 = -1.1835..., \quad \alpha_2 = -0.3968..., \quad \alpha_3 = 0.3302..., \quad \alpha_4 = 24.8692....$$

The conclusion of the theorem follows from Lemma 3.1. □

Theorem 3.16 *Let $G : \Delta^{27} \to \mathbb{R}$ be defined by $\boldsymbol{x} \to \max\{f(\boldsymbol{x}) : f \in \mathcal{G}\}$, where $\mathcal{G}$ is the set of functions listed in 2.3. Then $\inf_{\boldsymbol{x}\in\Delta^{27}} G(\boldsymbol{x}) = 24.8692...$.*

Proof Since $\mathcal{F} \subset \mathcal{G}$, we have $G(\mathbf{x}) \geq F(\mathbf{x})$ for every $\mathbf{x} \in \Delta^{27}$. Note that we obtain the coordinates of $\mathbf{x}^*$ as

$$x_1^* = 0.1076..., \quad x_2^* = x_3^* = 0.0053..., \quad x_7^* = 0.0132...$$

by Theorem 3.15. Then, for the indices $l \in \{3, 4, 10, 11, 17, 18, 24, 25\}$, we find that $g_l(\mathbf{x}^*) = 2.4822...$. For the indices $l \in \{1, 5, 9, 13, 15, 17, 19, 23, 27\}$ we have $g_l(\mathbf{x}^*) = 1.1131...$. Similarly, we compute that $h_l(\mathbf{x}^*) = u_l(\mathbf{x}^*) = 0.4028...$ for $l \in \{7, 14, 21, 28\}$ and $h_l(\mathbf{x}^*) = 0.1111...$ for $l \in \{1, 5, 9, 13, 15, 19, 23, 27\}$. Because $G(\mathbf{x}^*) = F(\mathbf{x}^*)$, we are done. □

4. Proof of the main theorem

To prove the main theorem of this paper, we shall require two preliminary statements. The first one is the following:

Lemma 4.1 *Let ξ and η be two noncommuting loxodromic isometries of $\mathbb{H}^3$. If z_2 is the midpoint of the shortest geodesic segment connecting the axes of ξ and $\eta^{-1}\xi\eta$, then $d_\xi z_2 < d_{\eta\xi\eta^{-1}} z_2$.*

Proof Let us denote the λ-displacement cylinder for a loxodromic isometry γ by $Z_\lambda(\gamma)$. Let $\lambda = d_\xi z_2$. The point $z_2 \in Z_\lambda(\xi)$ is the only point in the set $Z_\lambda(\xi) \cap Z_\lambda(\eta^{-1}\xi\eta)$. Because $\eta \cdot z_2 \neq z_2$ and $\eta \cdot z_2$ is the only element in $Z_\lambda(\eta\xi\eta^{-1}) \cap Z_\lambda(\xi)$, the point z_2 cannot be in $Z_\lambda(\eta\xi\eta^{-1})$. Hence, the conclusion follows. □

The second statement below is proved using arguments analogous to the ones introduced in [4, Theorem 9.1], [18, Theorem 5.1], and [19, Theorem 4.1]. Therefore, we shall not provide a detailed proof.

Theorem 4.2 *Let ξ and η be two noncommuting isometries of $\mathbb{H}^3$. If $\Gamma = \langle \xi, \eta \rangle$ is a purely loxodromic free Kleinian group so that $\Gamma_* = \{1\} \cup \Gamma_1 \cup \{\xi\eta\xi^{-1}, \xi^{-1}\eta\xi, \eta\xi\eta^{-1}, \eta^{-1}\xi\eta, \xi\eta^{-1}\xi^{-1}, \xi^{-1}\eta^{-1}\xi, \eta\xi^{-1}\eta^{-1}, \eta^{-1}\xi^{-1}\eta\}$, where $\Gamma_1 = \{\xi, \eta, \eta^{-1}, \xi^{-1}\}$, then we have $\max_{\gamma\in\Gamma_*}\{d_\gamma z\} \geq 1.6068...$ for any $z \in \mathbb{H}^3$.*

Proof Assume that $\Gamma = \langle \xi, \eta \rangle$ is geometrically infinite. The conclusion of the theorem follows from Proposition 2.3, Theorem 3.16, and the following inequality:

$$\max_{\gamma\in\Gamma_*}\{d_\gamma z\} \geq \frac{1}{2}\log G(\mathbf{m}) \geq \frac{1}{2}\log\left(\inf_{\mathbf{x}\in\Delta^{27}} G(\mathbf{x})\right) = \frac{1}{2}\log 24.8692... = 1.6068...,$$

where $\mathbf{m} = (\nu_{\xi\eta^{-1}\xi^{-1}}(S_\infty), \ldots, \nu_{\xi^{-2}}(S_\infty)) \in \Delta^{27}$.

Assume that $\Gamma = \langle \xi, \eta \rangle$ is geometrically finite. Because $\Gamma = \langle \xi, \eta \rangle$ is torsion-free, each isometry $\gamma \in \Gamma_*$ has infinite order. This implies that $\gamma \cdot z \neq z$ for every $z \in \mathbb{H}^3$. Since $\mathrm{dist}(z, \gamma_1\gamma_2 \cdot z) = \mathrm{dist}(\gamma_1^{-1} \cdot z, \gamma_2 \cdot z)$ and $\mathrm{dist}(z, \gamma_1 \cdot z) = \mathrm{dist}(z, \gamma_1^{-1} \cdot z)$ for all $\gamma_1, \gamma_2 \in \Gamma = \langle \xi, \eta \rangle$, we have

$$\begin{aligned}
\mathrm{dist}(z, \xi\eta\xi^{-1} \cdot z) &= \mathrm{dist}(\xi^{-1} \cdot z, \eta\xi^{-1} \cdot z) = \mathrm{dist}(\xi^{-1} \cdot z, \eta^{-1}\xi^{-1} \cdot z) = \mathrm{dist}(z, \xi\eta^{-1}\xi^{-1} \cdot z),\\
\mathrm{dist}(z, \xi^{-1}\eta\xi \cdot z) &= \mathrm{dist}(\xi \cdot z, \eta\xi \cdot z) = \mathrm{dist}(\xi \cdot z, \eta^{-1}\xi \cdot z) = \mathrm{dist}(z, \xi^{-1}\eta^{-1}\xi \cdot z),\\
\mathrm{dist}(z, \eta\xi\eta^{-1} \cdot z) &= \mathrm{dist}(\eta^{-1} \cdot z, \xi\eta^{-1} \cdot z) = \mathrm{dist}(\eta^{-1} \cdot z, \xi^{-1}\eta^{-1} \cdot z) = \mathrm{dist}(z, \eta\xi^{-1}\eta^{-1} \cdot z),\\
\mathrm{dist}(z, \eta^{-1}\xi\eta \cdot z) &= \mathrm{dist}(\eta \cdot z, \xi\eta \cdot z) = \mathrm{dist}(\eta \cdot z, \xi^{-1}\eta \cdot z) = \mathrm{dist}(z, \eta^{-1}\xi^{-1}\eta \cdot z).
\end{aligned}$$

Therefore, all of the hyperbolic displacements under the isometries in Γ_* are realized by the geodesic line segments joining the points $\{z\} \cup \{\gamma \cdot z : \gamma \in \Phi\}$, where $\Phi = \{\xi, \eta^{-1}, \eta, \xi^{-1}\} \cup \{\xi\eta^{-1}, \xi\eta, \eta\xi, \eta\xi^{-1}\}$. We enumerate the elements of Φ for some index set $I' \subset \mathbb{N}$ such that $P_0 = z$ and $P_i = \gamma_i \cdot z$ for $i \in I'$ and $\gamma_i \in \Phi$. Let $\Delta_{ij} = \triangle P_i P_0 P_j$ represent the geodesic triangle with vertices P_i, P_0, and P_j for $i, j \in I'$ and $i \neq j$.

Let $\mathfrak{X}$ denote the character variety $PSL(2,\mathbb{C}) \times PSL(2,\mathbb{C}) \simeq \mathrm{Isom}^+(\mathbb{H}^3) \times \mathrm{Isom}^+(\mathbb{H}^3)$ and $\mathfrak{GF}$ be the set $\{(\gamma, \beta) \in \mathfrak{X} : \langle \gamma, \beta \rangle$ is free, geometrically finite, and without any parabolic$\}$. For a fixed $z \in \mathbb{H}^3$, let us define the real-valued function $f_z : \mathfrak{X} \to \mathbb{R}$ with the formula

$$f_z(\xi, \eta) = \max_{\psi \in \Gamma_*} \{\mathrm{dist}(z,\ \psi \cdot z)\}.$$

The function f_z is continuous and proper. Therefore, it takes a minimum value at some point (ξ_0, η_0) in $\overline{\mathfrak{GF}}$. The value $f_z(\xi_0, \eta_0)$ is the unique longest side length of one geodesic triangle Δ_{ij} for some $i, j \in I'$. Let us denote this geodesic triangle with Δ and their vertices by $\widetilde{P}_i$, P_0, and $\widetilde{P}_j$. There are two cases to consider: (1) Δ is acute or (2) Δ is not acute.

Assume that (2) is the case. Then there is a one-step process analogous to the ones described in the proofs of [18, Theorem 5.1] and [19, Theorem 4.1]. This one-step process is illustrated in Figure 1, proving

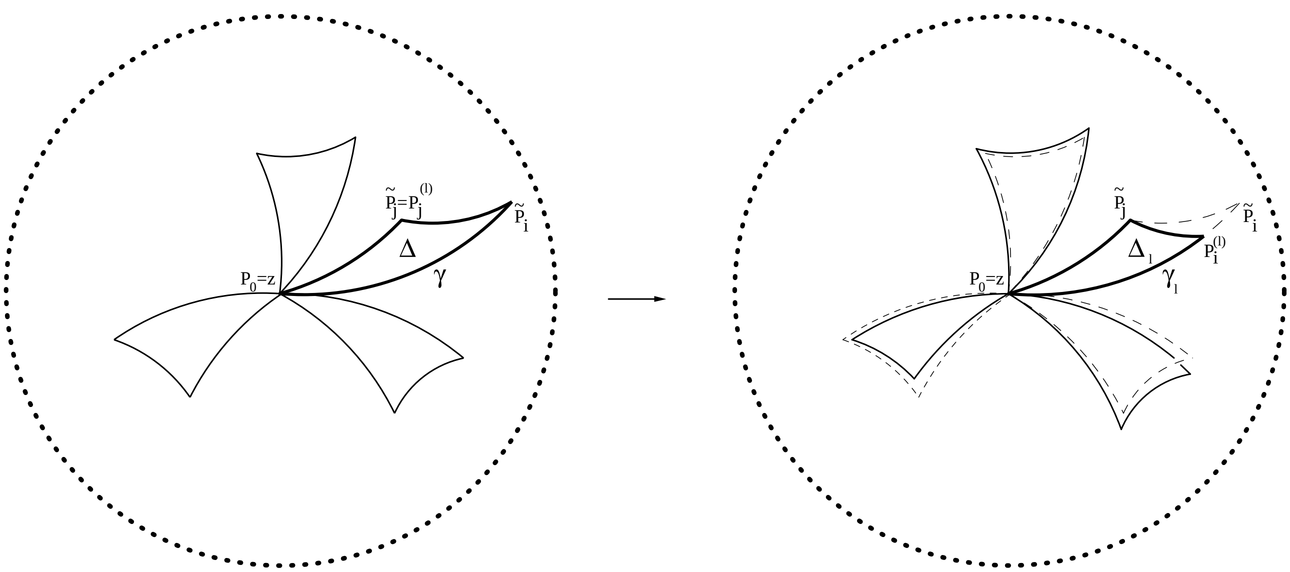

Figure 1. Case (2): Δ is not acute.

that $(\xi_0, \eta_0) \in \overline{\mathfrak{GF}} - \mathfrak{GF}$. If (1) is the case, then there is a two-step process analogous to the ones described in

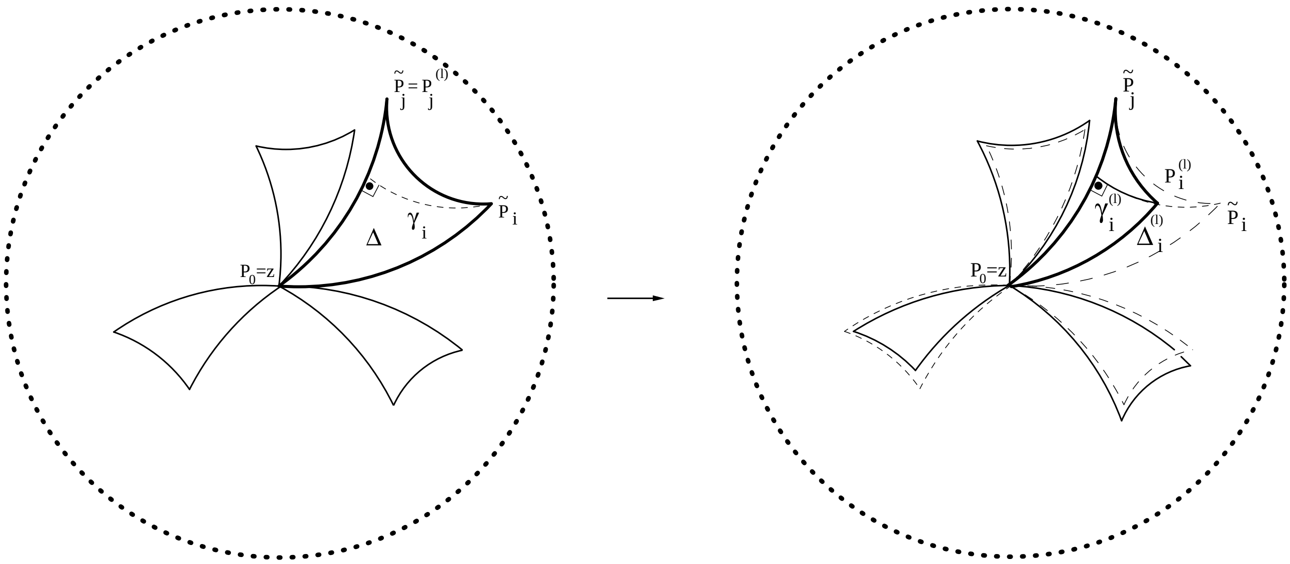

Figure 2. Case (1): Δ is acute.

the proofs of [18, Theorem 5.1] and [19, Theorem 4.1]. This two-step process is illustrated in Figures 2 and 3, proving again that $(\xi_0, \eta_0) \in \overline{\mathfrak{GF}} - \mathfrak{GF}$.

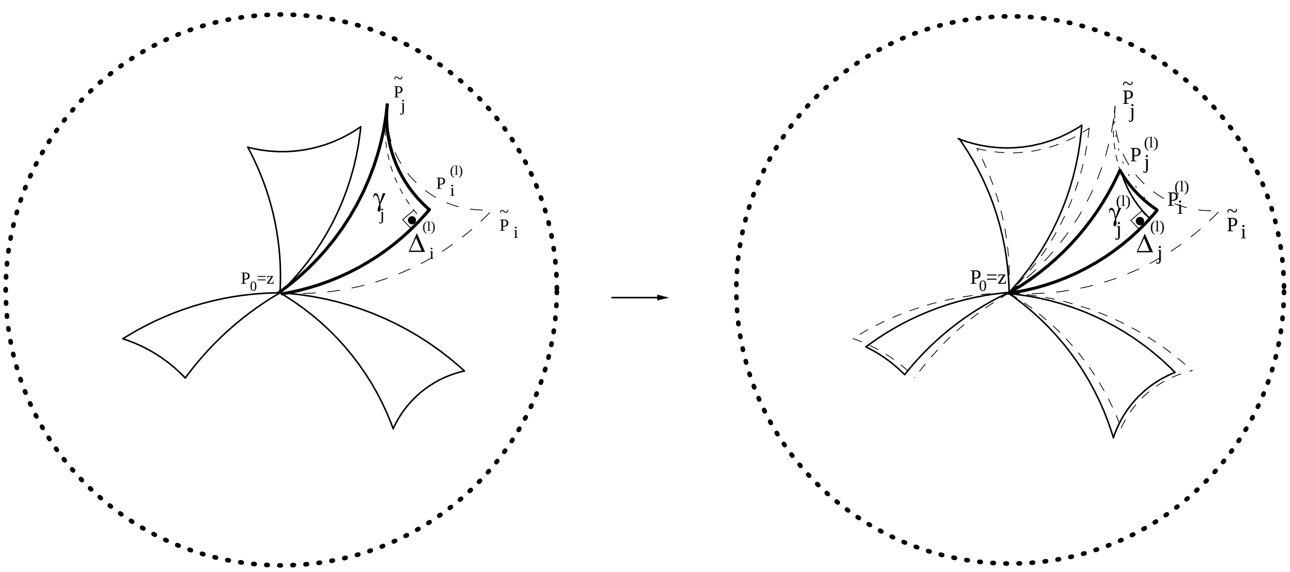

Figure 3. Case (1): Δ is acute.

Since the geometrically finite case reduces to the geometrically infinite case by the facts that the set of (ξ, η) such that $\langle \xi, \eta \rangle$ is free, geometrically infinite, and without any parabolic is dense in $\overline{\mathfrak{GF}} - \mathfrak{GF}$ and every $(\xi, \eta) \in \mathfrak{X}$ with $\langle \xi, \eta \rangle$ that is free and without any parabolic is in $\overline{\mathfrak{GF}}$, the conclusion of the theorem follows when $\Gamma = \langle \xi, \eta \rangle$ is geometrically finite as well. For the details of this crucial final step in the proof, readers may refer to [4, Propositions 8.2 and 9.3], [3, Main Theorem], and [2]. □

Using Lemma 4.1 and Theorem 4.2, we can prove the following statement, the main result of this paper.

Theorem 4.3 *Let ξ and η be two noncommuting isometries of $\mathbb{H}^3$. Suppose that $\Gamma = \langle \xi, \eta \rangle$ is a purely loxodromic free Kleinian group. If $d_\gamma z_2 < 1.6068...$ for every $\gamma \in \Phi_2 = \{\eta, \xi^{-1}\eta\xi, \xi\eta\xi^{-1}\}$ and $d_{\eta\xi\eta^{-1}} z_2 \leq d_{\eta\xi\eta^{-1}} z_1$ for the midpoints z_1 and z_2 of the shortest geodesic segments joining the axis of ξ to the axes of $\eta\xi\eta^{-1}$ and $\eta^{-1}\xi\eta$, respectively, then we have $|\mathrm{trace}^2(\xi) - 4| + |\mathrm{trace}(\xi\eta\xi^{-1}\eta^{-1}) - 2| \geq 1.5937....$*

Proof We shall mostly follow the computations given in the proof of Theorem 5.4.5 in [1, Section 5.4]. Readers who are interested in further details should refer to this source.

Considering conjugate elements, for $u = |u|e^{i\theta}$ and $ad - bc = 1$, we can assume that

$$\xi = \begin{pmatrix} u & 0 \\ 0 & 1/u \end{pmatrix} \quad \text{and} \quad \eta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $\mathcal{A}$ and T_ξ denote the axis and translation length of ξ, respectively. Above θ denotes the angle of rotation of ξ about its axis. Then we have

$$|\mathrm{trace}^2(\xi) - 4| + |\mathrm{trace}(\xi\eta\xi^{-1}\eta^{-1}) - 2| = |u - 1/u|^2(1 + |bc|),$$

where $\sinh^2(\frac{1}{2}T_\xi) + \sin^2\theta = \frac{1}{4}|u - 1/u|^2$; see [1, Equations (5.4.8) and (5.4.10)]. First, we shall determine a lower bound for the term $1 + |bc|$.

By construction $\mathcal{A}$ is the geodesic with end-points 0 and ∞ and $\mathcal{B} = \eta\mathcal{A}$ is the geodesic with end-points $\eta 0$ and $\eta\infty$. Since $\Gamma = \langle \xi, \eta \rangle$ is nonelementary, $\mathcal{A}$ and $\mathcal{B}$ do not have a common end-point. This implies that $bc \neq 0$. Thus, the equation

$$bc = \frac{(1 - w)^2}{4w} \tag{4.1}$$

obtained by the cross-ratios $[1, -1, w, -w] = [0, \infty, b/d, a/c]$ has two solutions. Let $w = \exp 2(x_0 + iy_0)$ be one of the solutions. We may assume that $|w| \geq 1$.

Plugging $w = \exp 2(x_0 + iy_0)$ in (4.1) we obtain $bc = \sinh^2(x_0 + iy_0)$. Then we derive

$$\begin{aligned} 4|bc|^2 &= |\cosh 2(x_0 + iy_0) - 1|^2 \\ &= (\cosh 2x_0 - \cos 2y_0)^2 \\ &\geq (\cosh 2x_0 - 1)^2 = (\cosh^2 x_0 + \sinh^2 x_0 - 1)^2 \geq (\cosh^2 x_0 - 1)^2, \end{aligned}$$

which gives that $2|bc| \geq \cosh^2 x_0 - 1 = \sinh^2 x_0$. This implies the following inequality:

$$1 + |bc| \geq \tfrac{1}{2}\sinh^2 x_0 + 1 = \tfrac{1}{2}\cosh^2 x_0 + \tfrac{1}{2} \geq \tfrac{1}{2}\cosh^2 x_0. \tag{4.2}$$

Let $d_z\mathcal{A}$ denote the shortest distance between z and $\mathcal{A}$. Since ξ and $\eta\xi\eta^{-1}$ have the same trace squared, the same translation length, and consequently the same value of $\sin^2\theta$, for every $z \in \mathbb{H}^3$, we obtain

$$\sinh^2 \tfrac{1}{2}d_\xi z = \sinh^2(\tfrac{1}{2}T_\xi)\cosh^2 d_z\mathcal{A} + \sin^2\theta \sinh^2 d_z\mathcal{A} \leq \big(\sinh^2(\tfrac{1}{2}T_\xi) + \sin^2\theta\big)\cosh^2 d_z\mathcal{A}, \tag{4.3}$$

$$\sinh^2 \tfrac{1}{2}d_{\eta\xi\eta^{-1}} z = \sinh^2(\tfrac{1}{2}T_\xi)\cosh^2 d_z\mathcal{B} + \sin^2\theta \sinh^2 d_z\mathcal{B} \leq \big(\sinh^2(\tfrac{1}{2}T_\xi) + \sin^2\theta\big)\cosh^2 d_z\mathcal{B}. \tag{4.4}$$

Then, by using the inequalities in (4.3) and (4.4) and the fact that $\sinh^2 x$ and $\cosh^2 x$ are increasing for $x > 0$, for every $z \in \mathbb{H}^3$, we derive that

$$\sinh^2 \tfrac{1}{2} \max\{d_\xi z, d_{\eta\xi\eta^{-1}} z\} \leq \tfrac{1}{4} \left|u - 1/u\right|^2 \cosh^2 \max\{d_z \mathcal{A}, d_z \mathcal{B}\}. \tag{4.5}$$

At this point, we consider the Möbius transformation ψ taking 0, ∞, $\beta 0$, $\beta\infty$ to 1, -1, w, $-w$. Then we have

$$d_{\mathcal{A}}\mathcal{B} = d_{\psi\mathcal{A}}\psi\mathcal{B} = \log|w| = 2x_0,$$

where $d_{\mathcal{A}}\mathcal{B}$ denotes the shortest distance between $\mathcal{A}$ and $\mathcal{B}$. Since we have $d_{z_1}\mathcal{A} = d_{z_1}\mathcal{B} = x_0$ and $d_\xi z_1 = d_{\eta\xi\eta^{-1}} z_1$, by the inequalities in (4.2) and (4.5), we derive that

$$\sinh^2 \tfrac{1}{2} d_\xi z_1 \leq \tfrac{1}{4}\left|u - 1/u\right|^2 \cosh^2 d_{z_0}\mathcal{A} \leq \tfrac{1}{2}\left|u - 1/u\right|^2 (1 + |bc|). \tag{4.6}$$

Now, assume on the contrary that $|\text{trace}^2(\xi) - 4| + |\text{trace}(\xi\eta\xi^{-1}\eta^{-1}) - 2| < 1.5937\ldots$. Because we have $d_{\eta\xi\eta^{-1}} z_2 \leq d_{\eta\xi\eta^{-1}} z_1 = d_\xi z_1$ and $d_\gamma z_2 < 1.6068...$ for every $\gamma \in \{\eta, \xi\eta\xi^{-1}, \xi^{-1}\eta\xi\}$ by the hypothesis, we get $d_\gamma z_2 < 1.6068...$ for every $\gamma \in \Gamma$ by the inequality in (4.6) and Lemma 4.1. This contradicts with Theorem 4.2. □

Notice that all of the computations given in this paper to prove Theorems 4.2 and 4.3 can be repeated also for a finitely generated purely loxodromic free Kleinian group $\Gamma = \langle \xi_1, \xi_2, \ldots, \xi_n \rangle$ satisfying a hypothesis similar to the one in Theorem 4.3. An analog of the decomposition $\Gamma_{\mathcal{D}}$ defined in (1.3) is required. For a fixed $n > 2$, let

$$\Psi^n = \{\xi_i^2, \xi_i^{-2} : i = 1, \ldots, n\} \cup \{\xi_i \xi_j \xi_k^{-1} : i \neq j,\ j \neq k,\ i, j, k = 1, \ldots, n\}$$

and $\Gamma_1^n = \Psi_r^n = \Xi \cup \Xi^{-1}$, where $\Xi = \{\xi_i : i = 1, \ldots, n\}$ and $\Xi^{-1} = \{\xi_i^{-1} : i = 1, \ldots, n\}$. When the group $\Gamma = \langle \xi_1, \xi_2, \ldots, \xi_n \rangle$ is geometrically infinite, the following is the relevant decomposition:

$$\Gamma = \{1\} \cup \Psi_r^n \cup \bigcup_{\psi \in \Psi^n} J_\psi. \tag{4.7}$$

Let us name this decomposition $\Gamma_{\mathcal{D}_n}$. The rest follows again from the Culler–Shalen machinery introduced in [4] and the solution method for the optimization problems described in this text and [18, 19]. Consider the subset of isometries

$$\Gamma_*^n = \Gamma_1^n \cup \{\xi_i \xi_j \xi_i^{-1} : i \neq j,\quad i, j = 1, 2, \ldots, n\} \tag{4.8}$$

of $\Psi_r^n \cup \Psi^n$. We first prove an analog of Theorem 2.2 for $\Gamma_{\mathcal{D}_n}$. We list all of the group-theoretical relations as in Lemma 2.1 for the isometries in Γ_*^n. By Lemma 1.6 and the group-theoretical relations, we state analog of Proposition 2.3 to list all of the displacement functions $\mathcal{G}^n = \{f_l\}$ for the indices $l = 1, 2, \ldots, 2n(8n^2 - 10n + 3)$ for the isometries in Γ_*^n.

These displacement functions satisfy generalized versions of Properties A and B for the decomposition $\Gamma_{\mathcal{D}_n^*}$. In other words, we can prove statements similar to Propositions 3.11 and 3.14. With a suitable enumeration of the isometries in Γ_*^n as in (1.2), an analog of Proposition 3.11 for $\Gamma_{\mathcal{D}_n^*}$ implies that it is enough to compare the values

$$\frac{1-2(n-1)(x_1^*+(n-1)x_2^*+(n-1)x_3^*)-x_{2(n-1)(2n-1)+1}^*}{2(n-1)(x_1^*+(n-1)x_2^*+(n-1)x_3^*)+x_{2(n-1)(2n-1)+1}^*}\cdot\frac{1-x_1^*}{x_1^*}=\alpha_*,$$
$$\frac{1-(4n^2-4n-1)(x_1^*+(n-1)x_2^*+(n-1)x_3^*)-2nx_{2(n-1)(2n-1)+1}^*}{(4n^2-4n-1)(x_1^*+(n-1)x_2^*+(n-1)x_3^*)+2nx_{2(n-1)(2n-1)+1}^*}\cdot\frac{1-x_2^*}{x_2^*}\leq\alpha_*,$$
$$\frac{1-(4n^2-4n-1)(x_1^*+(n-1)x_2^*+(n-1)x_3^*)-2nx_{2(n-1)(2n-1)+1}^*}{(4n^2-4n-1)(x_1^*+(n-1)x_2^*+(n-1)x_3^*)+2nx_{2(n-1)(2n-1)+1}^*}\cdot\frac{1-x_3^*}{x_3^*}\leq\alpha_*,$$
$$\frac{1-(2n-1)(2(n-1)(x_1^*+(n-1)x_2^*+(n-1)x_3^*)+x_{2(n-1)(2n-1)+1}^*)}{(2n-1)(2(n-1)(x_1^*+(n-1)x_2^*+(n-1)x_3^*)+x_{2(n-1)(2n-1)+1}^*)}\cdot\frac{1-x_{2(n-1)(2n-1)+1}^*}{x_{2(n-1)(2n-1)+1}^*}\leq\alpha_*,$$

of four functions, where α_* is the infimum of the maximum of the displacement functions in $\mathcal{G}^n$ on the simplex $\Delta^{(2n-1)^3}$. Using an analog of Proposition 3.14 for $\Gamma_{\mathcal{D}_n}$ and the computations given in Theorems 3.15 and 3.16, we can prove the following generalization of Theorem 4.2:

Conjecture 4.4 *Let $\Xi=\{\xi_1,\xi_2,\dots,\xi_n\}$ be a set of noncommuting isometries of $\mathbb{H}^3$ for $n>2$ and $\Xi^{-1}=\{\xi_1^{-1},\xi_2^{-1},\dots,\xi_n^{-1}\}$. Suppose that $\Gamma=\langle\xi_1,\xi_2,\dots,\xi_n\rangle$ is a purely loxodromic free Kleinian group. Let $\Gamma_1^n=\Xi\cup\Xi^{-1}$ and Γ_*^n be as in (4.8). Then we have*

$$\max_{\gamma\in\Gamma_*^n} d_\gamma z\geq\tfrac{1}{2}\log\alpha_n$$

for every $z\in\mathbb{H}^3$. Above α_n is the only real root of the polynomial $p_n(x)$ greater than $(2n-1)^2$, where

$$\begin{aligned}p_n(x) \;=\;& (8n^3-12n^2+2n+1)\;x^4+(-64n^6+192n^5-192n^4+64n^3+4n^2+2n-4)x^3+\\ &(-96n^5+224n^4-168n^3+52n^2-18n+6)\;x^2+\\ &(32n^5-112n^4+128n^3-68n^2+22n-4)\;x+16n^4-32n^3+24n^2-8n+1.\end{aligned}$$

The proof of Conjecture 4.4 goes along the same lines as the proof of Theorem 4.2 when $\Gamma=\langle\xi_1,\xi_2\dots,\xi_n\rangle$ is geometrically finite. This conjecture and arguments analogous to the ones presented in the proof of Theorem 4.2 imply the following generalization of Theorem 4.3:

Conjecture 4.5 *Let $\Gamma=\langle\xi_1,\xi_2,\dots,\xi_n\rangle$ and α_n be as described in Conjecture 4.4. Assume that there exists an isometry ξ_i for $i\neq1$ so that $d_{\xi_i\xi_1\xi_i^{-1}}z_2\leq d_{\xi_i\xi_1\xi_i^{-1}}z_1$ and $d_\gamma z_2<\frac{1}{2}\log\alpha_n$ for every isometry $\gamma\in\Phi_n=\Gamma^n-\{\xi_1,\xi_1^{-1},\xi_i^{-1}\xi_1\xi_i,\xi_i^{-1}\xi_1^{-1}\xi_i,\xi_i\xi_1\xi_i^{-1},\xi_i\xi_1^{-1}\xi_i^{-1}\}$, where z_1 and z_2 are the midpoints of the shortest geodesic segments connecting the axis of ξ_1 to the axes of $\xi_i\xi_1\xi_i^{-1}$ and $\xi_i^{-1}\xi_1\xi_i$, respectively. Then we have*

$$|\mathrm{trace}^2(\xi_1)-4|+|\mathrm{trace}(\xi_1\xi_i\xi_1^{-1}\xi_i^{-1})-2|\geq2\sinh^2\left(\tfrac{1}{4}\log\alpha_n\right).$$

The details of the outlines of the proofs of Conjectures 4.4 and 4.5 given above will be left to future studies.

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