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Positivity of sums and integrals for n-convex functions via the Fink identity

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Abstract: We consider the positivity of the sum $\sum_{i=1}^{n}\rho_i F(\xi_i)$, where F is a convex function of higher order, as well as analogous results involving the integral $\int_{a_0}^{b_0}\rho(\xi)F(g(\xi))d\xi$. We use a representation of the function F via the Fink identity and the Green function that leads us to identities from which we obtain conditions for positivity of the above-mentioned sum and integral. We also obtain bounds for the integral remainders which occur in these identities, as well as corresponding mean value results.

Key words: n-convex functions, Fink identity, Green function, Čebyšev functional

1. Introduction

We start this section with some basic definitions and properties regarding n-convex functions, the main object of our study.

Definition 1.1 *The* nth order divided difference *of a function* $F: I \to \mathbb{R}$ *at distinct points* $\xi_i, \xi_{i+1}, \ldots, \xi_{i+n} \in I = [a_0, b_0] \subset \mathbb{R}$ *for some* $i \in \mathbb{N}$ *is defined recursively by:*

$$[\xi_j, F] = F(\xi_j), \quad j \in \{i, \ldots, i+n\},$$
$$[\xi_i, \ldots, \xi_{i+n}, F] = \frac{[\xi_{i+1}, \ldots, \xi_{i+n}, F] - [\xi_i, \ldots, \xi_{i+n-1}, F]}{\xi_{i+n} - \xi_i}.$$

The value $[\xi_i, \ldots, \xi_{i+n}, F]$ is independent of the order of the points $\xi_i, \xi_{i+1}, \ldots, \xi_{i+n}$. We can extend this definition by including the cases in which two or more points coincide by taking respective limits.

Definition 1.2 *A function* $F: I \to \mathbb{R}$ *is called* convex of order n *or* $n-$convex *if for all choices of* $(n+1)$ *distinct points* $\xi_i, \ldots, \xi_{i+n}$ *we have* $[\xi_i, \ldots, \xi_{i+n}, F] \geq 0$.

If the nth order derivative $F^{(n)}$ exists, then F is n-convex if and only if $F^{(n)} \geq 0$. For $1 \leq k \leq n-2$, a function F is n-convex if and only if $F^{(k)}$ exists and is $(n-k)$-convex.

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We will also need the Fink identity given in [1].

Proposition 1.3 *Let* $a_0, b_0 \in \mathbb{R}$, $F : [a_0, b_0] \to \mathbb{R}$, $n \geq 1$ *and* $F^{(n-1)}$ *be absolutely continuous on* $[a_0, b_0]$. *Then*

$$\begin{aligned} F(\xi) = \quad & \frac{n}{b_0 - a_0} \int_{a_0}^{b_0} F(t)\, dt \\ & - \sum_{k=1}^{n-1} \frac{n-k}{k!} \left(\frac{F^{(k-1)}(a_0)(\xi - a_0)^k - F^{(k-1)}(b_0)(\xi - b_0)^k}{b_0 - a_0} \right) \\ & + \frac{1}{(n-1)!\,(b_0 - a_0)} \int_{a_0}^{b_0} (\xi - t)^{n-1} k^{[a_0,b_0]}(t, \xi) F^{(n)}(t)\, dt, \end{aligned} \tag{1.1}$$

where

$$k^{[a_0,b_0]}(t, \xi) = \begin{cases} t - a_0, & a_0 \leq t \leq \xi \leq b_0, \\ t - b_0, & a_0 \leq \xi < t \leq b_0. \end{cases} \tag{1.2}$$

Pečarić in [5] proved the following result (see also [6, p. 262]):

Proposition 1.4 *The inequality*

$$\sum_{i=1}^{m} \rho_i F(\xi_i) \geq 0 \tag{1.3}$$

holds for all convex functions F *if and only if the* $m-$*tuples* $\xi = (\xi_1, \ldots, \xi_m)$, $\mathbf{p} = (\rho_1, \ldots, \rho_m) \in \mathbb{R}^m$ *satisfy*

$$\sum_{i=1}^{m} \rho_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} \rho_i |\xi_i - \xi_k| \geq 0 \text{ for } k \in \{1, \ldots, m\}. \tag{1.4}$$

Since

$$\sum_{i=1}^{m} \rho_i |\xi_i - \xi_k| = 2 \sum_{i=1}^{m} \rho_i (\xi_i - \xi_k)_+ - \sum_{i=1}^{m} \rho_i (\xi_i - \xi_k),$$

where $y_+ = \max(y, 0)$, it is easy to see that condition (1.4) is equivalent to

$$\sum_{i=1}^{m} \rho_i = 0, \quad \sum_{i=1}^{m} \rho_i \xi_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} \rho_i (\xi_i - \xi_k)_+ \geq 0 \text{ for } k \in \{1, \ldots, m-1\}. \tag{1.5}$$

The following result is due to Popoviciu [7, 8] (also see [6, 9]).

Proposition 1.5 *Let* $n \geq 2$. *Inequality* (1.3) *holds for all* n*-convex functions* $F : [a_0, b_0] \to \mathbb{R}$ *if and only if the* $m-$*tuples* $\xi \in [a_0, b_0]^m$, $\mathbf{p} \in \mathbb{R}^m$ *satisfy*

$$\sum_{i=1}^{m} \rho_i \xi_i^k = 0, \quad \text{for all } k \in \{0, 1, \ldots, n-1\}, \tag{1.6}$$

$$\sum_{i=1}^{m} \rho_i (\xi_i - t)_+^{n-1} \geq 0, \quad \text{for every } t \in [a_0, b_0]. \tag{1.7}$$

In fact, Popoviciu proved a stronger result; it is enough to assume that (1.7) holds for every $t \in [\xi_{(1)}, \xi_{(m-n+1)}]$ and then, due to (1.6), it is automatically satisfied for every $t \in [a_0, b_0]$. The integral analogue is given in the next proposition.

Proposition 1.6 *Let $n \geq 2$, $\rho : [\alpha_0, \beta_0] \to \mathbb{R}$ and $g : [\alpha_0, \beta_0] \to [a_0, b_0]$. Then, the inequality*

$$\int_{\alpha_0}^{\beta_0} \rho(\xi) F(g(\xi))\, d\xi \geq 0 \tag{1.8}$$

holds for all n-convex functions $F : [a_0, b_0] \to \mathbb{R}$ if and only if

$$\begin{aligned} &\int_{\alpha_0}^{\beta_0} \rho(\xi) g(\xi)^k\, d\xi = 0, \qquad \text{for all } k \in \{0, 1, \dots, n-1\}, \\ &\int_{\alpha_0}^{\beta_0} \rho(\xi) \left(g(\xi) - t\right)_+^{n-1}\, d\xi \geq 0, \qquad \text{for every } t \in [a_0, b_0]. \end{aligned} \tag{1.9}$$

The paper is organized as follows: in Sections 2 and 3, we derive identities for $\sum_{i=1}^{n} \rho_i F(\xi_i)$ and $\int_{a_0}^{b_0} \rho(\xi) F(g(\xi)) d\xi$ using the Fink identity and the Green function. We also give inequalities for n-convex functions which are based on these identities. Section 4 is devoted to estimations of the integral remainders of the identities by using Čebyšev type inequality and the Hölder inequality. In the last section, we give mean value results for functionals associated to the identities. Here, it is worth mentioning that the contents of this work are part of a monograph [3].

2. Popoviciu-type identities and inequalities via Fink identity

Here, we state our first main result:

Theorem 2.1 *Let $n \in \mathbb{N}$ and $F : [a_0, b_0] \to \mathbb{R}$ be such that $F^{(n-1)}$ is absolutely continuous. Let $\xi_i \in [a_0, b_0]$, $\rho_i \in \mathbb{R}$ $(i \in \{1, \dots, m\})$ be reals such that $\sum_{i=0}^{m} \rho_i = 0$ and let $k^{[a_0, b_0]}$ be the function defined in (1.2). Then we have*

$$\begin{aligned} &\sum_{i=1}^{m} \rho_i F(\xi_i) = \\ &\sum_{k=1}^{n-1} \frac{n-k}{k!\,(b_0 - a_0)} \left(F^{(k-1)}(b_0) \sum_{i=1}^{m} \rho_i (\xi_i - b_0)^k - F^{(k-1)}(a_0) \sum_{i=1}^{m} \rho_i (\xi_i - a_0)^k \right) \\ &+ \frac{1}{(n-1)!\,(b_0 - a_0)} \int_{a_0}^{b_0} F^{(n)}(t) \left(\sum_{i=1}^{m} \rho_i (\xi_i - t)^{n-1} k^{[a_0, b_0]}(t, \xi_i) \right) dt. \end{aligned} \tag{2.1}$$

Proof By using the Fink identity (1.1) for $\xi = \xi_i$, multiplying it with ρ_i and taking the sum over i from 1 to m, we have

$$\sum_{i=1}^{m} \rho_i F(\xi_i) = \frac{n}{b_0 - a_0} \int_{a_0}^{b_0} F(t)\, dt \sum_{i=0}^{m} \rho_i$$
$$+ \sum_{i=1}^{m} \rho_i \sum_{k=1}^{n-1} \frac{n-k}{k!} \frac{F^{(k-1)}(b_0)(\xi_i - b_0)^k - F^{(k-1)}(a_0)(\xi_i - a_0)^k}{b_0 - a_0}$$
$$+ \sum_{i=1}^{m} \rho_i \frac{\int_{a_0}^{b_0} F^{(n)}(t)(\xi_i - t)^{n-1} k^{[a_0,b_0]}(t, \xi_i)\, dt}{(n-1)!(b_0 - a_0)}.$$

After some rearrangement, we get our required result. □

The following theorem is the integral version of Theorem 2.1.

Theorem 2.2 *Let $n \in \mathbb{N}$ and $F : [a_0, b_0] \to \mathbb{R}$ be such that $F^{(n-1)}$ is absolutely continuous on $[a_0, b_0]$ and let $k^{[a_0,b_0]}(t, \xi)$ be the same as defined in (1.2). Let $g : [\alpha_0, \beta_0] \to [a_0, b_0]$ and $\rho : [\alpha_0, \beta_0] \to \mathbb{R}$ be integrable functions such that $\int_{\alpha_0}^{\beta_0} \rho(\xi) d\xi = 0$. Then we have*

$$\int_{\alpha_0}^{\beta_0} \rho(\xi) F(g(\xi))\, d\xi = \sum_{k=1}^{n-1} \frac{n-k}{k!(b_0 - a_0)}$$
$$\times \left(F^{(k-1)}(b_0) \int_{\alpha_0}^{\beta_0} \rho(\xi)(g(\xi) - b_0)^k\, d\xi - F^{(k-1)}(a_0) \int_{\alpha_0}^{\beta_0} \rho(\xi)(g(\xi) - a_0)^k\, d\xi \right)$$
$$+ \frac{1}{(n-1)!(b_0 - a_0)} \int_{a_0}^{b_0} F^{(n)}(t) \left(\int_{\alpha_0}^{\beta_0} \rho(\xi)(g(\xi) - t)^{n-1} k^{[a_0,b_0]}(t, g(\xi))\, d\xi \right) dt.$$

Proof Putting $\xi \to g(\xi)$ in (1.1), multiplying it by $\rho(\xi)$ and integrating with respect to ξ, we get an identity from which, after using the Fubini theorem, we obtain the desired identity. □

Let us now introduce some notations which will be used in the rest of the paper:

$$\Omega_1^{[a_0,b_0]}(m, \xi, \mathbf{p}, t) = \sum_{i=1}^{m} \rho_i (\xi_i - t)^{n-1} k^{[a_0,b_0]}(t, \xi_i), \tag{2.2}$$

$$\Omega_2^{[a_0,b_0]}([\alpha_0, \beta_0], g, \rho, t) = \int_{\alpha_0}^{\beta_0} \rho(\xi)(g(\xi) - t)^{n-1} k^{[a_0,b_0]}(t, g(\xi))\, d\xi. \tag{2.3}$$

$$A_1^{[a_0,b_0]}(m, \xi, \mathbf{p}, F) = \sum_{i=1}^{m} \rho_i F(\xi_i)$$
$$- \sum_{k=1}^{n-1} \frac{n-k}{k!(b_0 - a_0)} \left(F^{(k-1)}(b_0) \sum_{i=1}^{m} \rho_i (\xi_i - b_0)^k - F^{(k-1)}(a_0) \sum_{i=1}^{m} \rho_i (\xi_i - a_0)^k \right) \tag{2.4}$$

$$A_2^{[a_0,b_0]}([\alpha_0,\beta_0],g,\rho,F)=\int_{\alpha_0}^{\beta_0}\rho(\xi)F(g(\xi))\,d\xi-\sum_{k=1}^{n-1}\frac{n-k}{k!\,(b_0-a_0)}$$
$$\times\left(F^{(k-1)}(b_0)\int_{\alpha_0}^{\beta_0}\rho(\xi)(g(\xi)-b_0)^k\,d\xi-F^{(k-1)}(a_0)\int_{\alpha_0}^{\beta_0}\rho(\xi)(g(\xi)-a_0)^k\,d\xi\right) \tag{2.5}$$

The following theorem is our second main result.

Theorem 2.3 *Let all the assumptions of Theorem 2.1 be satisfied and let*

$$\Omega_1^{[a_0,b_0]}(m,\xi,\mathbf{p},t)\geq 0,\quad \textit{for all}\, t\in[a_0,b_0]. \tag{2.6}$$

If F is n-convex, then we have

$$A_1^{[a_0,b_0]}(m,\xi,\mathbf{p},F)\geq 0 \tag{2.7}$$

If the opposite inequality holds in (2.6), then (2.7) holds in the reverse direction.

Proof Since $F^{(n-1)}$ is absolutely continuous on $[a_0,b_0]$, $F^{(n)}$ exists almost everywhere. As F is n-convex, by definition of n-convex functions, we have $F^{(n)}(\xi)\geq 0$ for all $\xi\in[a_0,b_0]$. Now by using $F^{(n)}\geq 0$ and (2.6) in (2.1), we have (2.7). □

Now we state an important consequence.

Theorem 2.4 *Suppose that all the assumptions from Theorem 2.1 hold. Additionally, let $j\in\mathbb{N}$, $2\leq j\leq n$ and let $\xi=(\xi_1,\ldots,\xi_m)\in[a_0,b_0]^m$, $\mathbf{p}=(\rho_1,\ldots,\rho_m)\in\mathbb{R}^m$ satisfy (1.6) and (1.7) with n replaced with j. If F is n-convex and $n-j$ is even, then*

$$\sum_{i=1}^{m}\rho_i F(\xi_i)\geq\sum_{k=j}^{n-1}\frac{n-k}{k!\,(b_0-a_0)}\left(F^{(k-1)}(b_0)\left(\sum_{i=1}^{m}\rho_i(\xi_i-b_0)^k\right)-F^{(k-1)}(a_0)\left(\sum_{i=1}^{m}\rho_i(\xi_i-a_0)^k\right)\right). \tag{2.8}$$

Proof Let $t\in[a_0,b_0]$ be fixed. For $j\leq n-2$, we get

$$\frac{d^j}{d\xi^j}(\xi-t)^{n-1}=(n-1)(n-2)\cdots(n-j)(\xi-t)^{n-j-1}. \tag{2.9}$$

Therefore, (2.9) for $a_0\leq t\leq\xi\leq b_0$ yields

$$(t-a_0)\frac{d^j}{d\xi^j}(\xi-t)^{n-1}\geq 0, \tag{2.10}$$

while for $a_0\leq\xi<t\leq b_0$, we have

$$(-1)^{n-j}(t-b_0)\frac{d^j}{d\xi^j}(\xi-t)^{n-1}\geq 0. \tag{2.11}$$

It is clear that $\xi\mapsto\frac{d^j}{d\xi^j}(\xi-t)^{n-1}k^{[a_0,b_0]}(t,\xi)$ is continuous for $j\leq n-2$. Hence, if $j\leq n-2$ and $n-j$ is even, from (2.10) and (2.11), we can conclude that the function $\xi\mapsto(\xi-t)^{n-1}k^{[a_0,b_0]}(t,\xi)$ is j-convex. Moreover, the conclusion extends to the case $j=n$, *i. e.* the mapping $\xi \mapsto (\xi-t)^{n-1}k^{[a_0,b_0]}(t,\xi)$ is n-convex, since the mapping $\xi \mapsto \frac{d^{n-2}}{d\xi^{n-2}}(\xi-t)^{n-1}k^{[a_0,b_0]}(t,\xi)$ is 2-convex.

Using Proposition 1.5 for j-convex function $\xi \mapsto (\xi-t)^{n-1}k^{[a_0,b_0]}(t,\xi)$ with assumptions (1.6) and (1.7) where n is replaced with j, we get $\sum_{i=1}^{m}\rho_i(\xi_i-t)^{n-1}k^{[a_0,b_0]}(t,\xi) \geq 0$. It means that (2.6) is satisfied and by Theorem 2.3, inequality (2.7) holds. Moreover, due to assumption (1.6), $\sum_{i=1}^{m}\rho_i P(\xi_i)=0$ for every polynomial P of degree $\leq j-1$, so the first $j-2$ terms in the inner sum in (2.4) vanish, i.e. we get inequality (2.8). □

When $j=n$ in (2.8), the notation means that the inner sum is void, i.e. $\sum_{k=n}^{n-1}\cdots=0$. In particular, inequality (2.8) with $j=n$ is inequality (1.3).

Corollary 2.5 *Let all the assumptions of Theorem 2.1 be satisfied and let the function $F:[a_0,b_0]\to\mathbb{R}$ be n-convex for an even n. Let $m-$tuples $\xi=(\xi_1,\ldots,\xi_m)$, $\mathbf{p}=(\rho_1,\ldots,\rho_m)\in\mathbb{R}^m$ satisfy the conditions stated in (1.4). Then inequality (2.7) holds.*

Furthermore, if $F^{(k-1)}(a_0)\leq 0$ and $(-1)^k F^{(k-1)}(b_0)\geq 0$ for $k\in\{2,3,\ldots,n-1\}$, then

$$\sum_{i=1}^{m}\rho_i F(\xi_i)\geq 0. \tag{2.12}$$

Proof Inequality (2.7) holds by Theorem 2.4 applied for $j=2$. Moreover, the functions $\xi\mapsto(\xi-a_0)^k$ and $\xi\mapsto(-1)^k(\xi-b_0)^k$ are convex, so Proposition 1.4 yields

$$\sum_{i=1}^{m}\rho_i(\xi-a_0)^k\geq 0 \tag{2.13}$$

and

$$(-1)^k\sum_{i=1}^{m}\rho_i(\xi-b_0)^k\geq 0. \tag{2.14}$$

Therefore, if $F^{(k-1)}(a_0)\leq 0$ and $(-1)^k F^{(k-1)}(b_0)\geq 0$, then (2.13) and (2.14) together with (2.4) yield inequality (2.12). □

Corollary 2.6 *Suppose all the assumptions from Theorem 2.1 hold and let the function $F:[a_0,b_0]\to\mathbb{R}$ be n-convex. Additionally, let $j\in\mathbb{N}$, $2\leq j\leq n$, let $\xi=(\xi_1,\ldots,\xi_m)\in[a_0,b_0]^m$, $\mathbf{p}=(\rho_1,\ldots,\rho_m)\in\mathbb{R}^m$ satisfy (1.6) and (1.7) with n replaced with j and denote*

$$H(\xi)=\sum_{k=j}^{n-1}\frac{n-k}{k!(b_0-a_0)}\left(F^{(k-1)}(b_0)(\xi-b_0)^k-F^{(k-1)}(a_0)(\xi-a_0)^k\right). \tag{2.15}$$

If H is j-convex on $[a_0,b_0]$ and $n-j$ is even, then

$$\sum_{i=1}^{m}\rho_i F(\xi_i)\geq 0.$$

Proof Applying Proposition 1.5, we conclude that $\sum_{i=1}^{m}\rho_i H(\xi_i)\geq 0$, so the right hand side of inequality (2.8) is nonnegative and we get the desired result. □

Remark 2.7 *For example, since the functions $\xi\mapsto(\xi-a_0)^k$ and $\xi\mapsto(-1)^{k-j}(\xi-b_0)^k$ are j-convex on $[a_0,b_0]$, the function H given by (2.15) is j-convex if $F^{(k-1)}(a_0)\leq 0$ and $(-1)^{k-1-j}F^{(k-1)}(b_0)\geq 0$ for $k\in\{j,\ldots,n-1\}$.*

As we already mentioned, the inequality in Theorem 2.4 and Corollary 2.6 with $j=n$ is the same as inequality (1.3) from Popoviciu's Proposition 1.5. Of course, in the proof of Theorem 2.4, we used Proposition 1.5 to prove that assumption (2.6) holds, so, due to circularity, we did not obtain another proof of Popoviciu's result. However, it is possible, as we will show in the next lemma, to prove directly that conditions (1.6) and (1.7) imply (2.6), i.e. it is possible to prove Theorem 2.4 with $j=n$ independently of Proposition 1.5 and thus provide a new proof of Popoviciu's result.

Lemma 2.8 *Let $n\geq 2$ and let $m-$tuples $\xi\in[a_0,b_0]^m$ and $\mathbf{p}\in\mathbb{R}^m$ satisfy (1.6) and (1.7). Then (2.6) holds.*

Proof Let $t\in[a_0,b_0]$ be fixed. Notice that

$$\Omega_1^{[a_0,b_0]}(m,\xi,\mathbf{p},t)=\sum_{i=1}^{m}\rho_i\varphi_t(\xi_i),$$

where φ_t is the function

$$\varphi_t(\xi)=(\xi-t)^{n-1}k^{[a_0,b_0]}(t,\xi)=(t-b_0)(\xi-t)^{n-1}+(b_0-a_0)(\xi-t)_+^{n-1}.$$

As in the proof of Theorem 2.4 we conclude that (1.6) implies that $\sum_{i=1}^{m}\rho_i P(\xi_i)=0$ for every polynomial P of degree $\leq n-1$. In particular, for $P(\xi)=(\xi-t)^{n-1}$, we have $\sum_{i=1}^{m}\rho_i(\xi_i-t)^{n-1}=0$. Therefore,

$$\sum_{i=1}^{m}\rho_i\varphi_t(\xi_i)=(b_0-a_0)\sum_{i=1}^{m}\rho_i(\xi_i-t)_+^{n-1}\geq 0,$$

where the last inequalities hold due to (1.7). Since the previous inequality holds for every $t\in[a_0,b_0]$, we conclude that (2.6) holds. □

Lemma 2.8 together with Theorem 2.3 gives the "if" part of Popoviciu's Proposition 1.5. On the other hand, the "only if" part is straightforward: since the functions $e_j(\xi)=\xi^j$ are both n-convex and n-concave for $j\in\{0,1,\ldots,n-1\}$, inequality (1.3) yields that $\sum_{i=1}^{m}\rho_i e_k(\xi_i)$ is both ≥ 0 and ≤ 0, so (1.6) holds. Similarly, the function $\xi\mapsto(\xi-t)_+^{n-1}$ is n-convex and applying inequality (1.3) yields (1.7).

In the remainder of the section, we will state integral versions of the previous results, the proofs of which are analogous to the discrete case.

Theorem 2.9 *Let all the assumptions of Theorem 2.2 be satisfied and*

$$\Omega_2^{[a_0,b_0]}([\alpha_0,\beta_0],g,\rho,t)\geq 0,\quad \textit{for all}\, t\in[a_0,b_0]. \tag{2.16}$$

If F is n-convex, then we have

$$A_2^{[a_0,b_0]}([\alpha_0,\beta_0],g,\rho,F) \geq 0. \tag{2.17}$$

If opposite inequality holds in (2.16), then (2.17) holds in the reverse direction.

Proof The idea of the proof is the same as that of Theorem 2.3. □

Remark 2.10 *A result analogous to Corollary 2.5 can be stated for integrals.*

Theorem 2.11 *Suppose all the assumptions from Theorem 2.2 hold. Additionally, let $j \in \mathbb{N}$, $2 \leq j \leq n$ and let $\rho : [\alpha_0,\beta_0] \to \mathbb{R}$ and $g : [\alpha_0,\beta_0] \to [a_0,b_0]$ satisfy (1.9) with n replaced with j. If F is n-convex and $n-j$ is even, then*

$$\int_{\alpha_0}^{\beta_0} \rho(\xi)\, F(g(\xi))\, d\xi$$

$$\geq \frac{1}{b_0-a_0}\left[\sum_{k=j}^{n-1} \frac{n-k}{k!} F^{(k-1)}(b_0) \int_{\alpha_0}^{\beta_0} \rho(\xi)\,(g(\xi)-b_0)^{k+2}\, d\xi \right.$$

$$\left. - \sum_{k=j}^{n-1} \frac{n-k}{k!} F^{(k-1)}(a_0) \int_{\alpha_0}^{\beta_0} \rho(\xi)\,(g(\xi)-a_0)^{k}\, d\xi \right].$$

Corollary 2.12 *Let j, n, f, p, and g be as in Theorem 2.11 and let H be given by (2.15). If H is j-convex, $n-j$ is even, and F is n-convex, then*

$$\int_{\alpha_0}^{\beta_0} \rho(\xi) F(g(\xi))\, d\xi \geq 0.$$

3. Popoviciu-type identities and inequalities via the Fink identity and the Green function

In this section, we will obtain another identity and the corresponding linear inequality by using the Green function and applying again the Fink identity.

The function $G : [a_0,b_0] \times [a_0,b_0] \to \mathbb{R}$ defined by

$$G(s,t) = \begin{cases} \frac{(s-b_0)(t-a_0)}{b_0-a_0} & \text{for } a_0 \leq t \leq s, \\ \frac{(t-b_0)(s-a_0)}{b_0-a_0} & \text{for } s \leq t \leq b_0 \end{cases} \tag{3.1}$$

is the Green function of the boundary value problem

$$z''(\xi) = 0,\ z(a_0) = z(b_0) = 0.$$

The function G is continuous, symmetric, and convex with respect to both variables s and t.

For any function $F : [a_0,b_0] \to \mathbb{R}$, $F \in C^2[a_0,b_0]$, the following integral identity holds

$$F(\xi) = \frac{b_0-\xi}{b_0-a_0} F(a_0) + \frac{\xi-a_0}{b_0-a_0} F(b_0) + \int_{a_0}^{b_0} G(\xi,s) F''(s) ds. \tag{3.2}$$

We now state main results related to the Fink identity and the Green function.

Theorem 3.1 *Let* $n \in \mathbb{N}$, $n \geq 3$, *and* $F : [a_0, b_0] \to \mathbb{R}$ *be such that* $F^{(n-1)}$ *is absolutely continuous. Let* $\xi_i, y_i \in [a_0, b_0]$, $\rho_i \in \mathbb{R}$ *for* $i \in \{1, \ldots, m\}$ *be such that* $\sum_{i=1}^{m} \rho_i = 0$ *and* $\sum_{i=1}^{m} \rho_i \xi_i = 0$ *and let* $k^{[a_0,b_0]}$ *and* G *be as defined in* (1.2) *and* (3.1)*, respectively. Then*

$$\sum_{i=1}^{m} \rho_i F(\xi_i) = \sum_{k=0}^{n-3} \left(\frac{n-k-2}{k!\,(b_0-a_0)} \right) \int_{a_0}^{b_0} \left(\sum_{i=1}^{m} \rho_i G(\xi_i, s) \right) \times \left(F^{(k+1)}(b_0)(s-b_0)^k - F^{(k+1)}(a_0)(s-a_0)^k \right) ds + \frac{1}{(n-3)!\,(b_0-a_0)} \times \int_{a_0}^{b_0} F^{(n)}(t) \left(\int_{a_0}^{b_0} \sum_{i=1}^{m} \rho_i G(\xi_i, s)(s-t)^{n-3} k^{[a_0,b_0]}(t,s)\, ds \right) dt. \tag{3.3}$$

Proof Putting $\xi = \xi_i$ in (3.2), multiplying it with ρ_i, adding all the identities and using the properties $\sum_{i=1}^{m} \rho_i = 0$ and $\sum_{i=1}^{m} \rho_i \xi_i = 0$, we get

$$\sum_{i=1}^{m} \rho_i F(\xi_i) = \int_{a_0}^{b_0} \left(\sum_{i=1}^{m} \rho_i G(\xi_i, s) \right) F''(s)\, ds. \tag{3.4}$$

Applying the Fink identity with $F \to F''$ and $n \to n-2$, it is easy to see that

$$F''(\xi) = \sum_{k=0}^{n-3} \frac{n-k-2}{k!} \frac{F^{(k+1)}(b_0)(\xi-b_0)^k - F^{(k+1)}(a_0)(\xi-a_0)^k}{b_0-a_0} + \frac{1}{(n-3)!\,(b_0-a_0)} \int_{a_0}^{b_0} (\xi-t)^{n-3} k^{[a_0,b_0]}(t,\xi) F^{(n)}(t)\, dt, \tag{3.5}$$

and by using (3.5) in (3.4), we have

$$\sum_{i=1}^{m} \rho_i F(\xi_i) = \int_{a_0}^{b_0} \left(\sum_{i=1}^{m} \rho_i G(\xi_i, s) \right) \times \sum_{k=0}^{n-3} \frac{n-k-2}{k!} \frac{F^{(k+1)}(b_0)(s-b_0)^k - F^{(k+1)}(a_0)(s-a_0)^k}{b_0-a_0} ds + \frac{1}{(n-3)!\,(b_0-a_0)} \int_{a_0}^{b_0} \sum_{i=1}^{m} \rho_i G(\xi_i, s) \left(\int_{a_0}^{b_0} (s-t)^{n-3} k^{[a_0,b_0]}(t,s) F^{(n)}(t)\, dt \right) ds.$$

Now, by interchanging the integral and summation in the second term and by applying Fubini's theorem in the last term, we have (3.3). □

The following theorem is the integral version of Theorem 3.1.

Theorem 3.2 *Let* $n \in \mathbb{N}$, $n \geq 3$, *let* $F : [a_0, b_0] \to \mathbb{R}$ *be such that* $F^{(n-1)}$ *is absolutely continuous on* $[a_0, b_0]$, *let* $\rho : [\alpha_0, \beta_0] \to \mathbb{R}$ *and* $g : [\alpha_0, \beta_0] \to [a_0, b_0]$ *be integrable functions such that* $\int_{\alpha_0}^{\beta_0} \rho(\xi) d\xi = 0$ *and*

$\int_{\alpha_0}^{\beta_0} \rho(\xi)g(\xi)d\xi = 0$ *and let* $k^{[a_0,b_0]}$ *and* G *be as defined in* (1.2) *and* (3.1)*, respectively. Then*

$$\int_{\alpha_0}^{\beta_0} \rho(\xi) F(g(\xi)) d\xi = \sum_{k=0}^{n-3} \frac{n-k-2}{k!(b_0-a_0)} \int_{a_0}^{b_0} \left(\int_{\alpha_0}^{\beta_0} \rho(\xi) G(g(\xi), s) d\xi \right)$$
$$\left(F^{(k+1)}(b_0)(s-b_0)^k - F^{(k+1)}(a_0)(s-a_0)^k \right) ds + \frac{1}{(n-3)!(b_0-a_0)}$$
$$\times \int_{a_0}^{b_0} F^{(n)}(t) \left(\int_{a_0}^{b_0} \left(\int_{\alpha_0}^{\beta_0} \rho(\xi) G(g(\xi), s) d\xi \right) (s-t)^{n-3} k^{[a_0,b_0]}(t,s) ds \right) dt. \tag{3.6}$$

Proof The proof is similar to the proof of the previous theorem, so we omit the details. □

We again introduce some notations here which will be used in the rest of the paper:

$$\Omega_3^{[a_0,b_0]}(m, \xi, \mathbf{p}, t) = \int_{a_0}^{b_0} \sum_{i=1}^{m} \rho_i G(\xi_i, s)(s-t)^{n-3} k^{[a_0,b_0]}(t,s) ds, \tag{3.7}$$

$$\Omega_4^{[a_0,b_0]}([\alpha_0, \beta_0], g, \rho, t) = \int_{a_0}^{b_0} \left(\int_{\alpha_0}^{\beta_0} \rho(\xi) G(g(\xi), s) d\xi \right) (s-t)^{n-3} k^{[a_0,b_0]}(t,s) ds.$$

$$A_3^{[a_0,b_0]}(m, \xi, \mathbf{p}, F) = \sum_{i=1}^{m} \rho_i F(\xi_i) - \sum_{k=0}^{n-3} \left(\frac{n-k-2}{k!(b_0-a_0)} \right) \int_{a_0}^{b_0} \sum_{i=1}^{m} \rho_i G(\xi_i, s)$$
$$\times \left(F^{(k+1)}(b_0)(s-b_0)^k - F^{(k+1)}(a_0)(s-a_0)^k \right) ds \tag{3.8}$$

$$A_4^{[a_0,b_0]}([\alpha_0, \beta_0], g, \rho, F) = \int_{\alpha_0}^{\beta_0} \rho(\xi) F(g(\xi)) d\xi$$
$$- \sum_{k=0}^{n-3} \left(\frac{n-k-2}{k!(b_0-a_0)} \right) \int_{a_0}^{b_0} \left(\int_{\alpha_0}^{\beta_0} \rho(\xi) G(g(\xi), s) d\xi \right)$$
$$\times \left(F^{(k+1)}(b_0)(s-b_0)^k - F^{(k+1)}(a_0)(s-a_0)^k \right) ds.$$

The following theorem is our second main result of this section:

Theorem 3.3 *Let all the assumptions of Theorem* 3.1 *be satisfied and let*

$$\Omega_3^{[a_0,b_0]}(m, \xi, \mathbf{p}, t) \geq 0, \quad \text{for all } t \in [a_0, b_0]. \tag{3.9}$$

If F *is* n*-convex, then we have*

$$A_3^{[a_0,b_0]}(m, \xi, \mathbf{p}, F) \geq 0. \tag{3.10}$$

If opposite inequality holds in (3.9)*, then* (3.10) *holds in the reverse direction.*

Proof Since $F^{(n-1)}$ is absolutely continuous on $[a_0, b_0]$, $F^{(n)}$ exists almost everywhere. Since F is n-convex, we have $F^{(n)}(\xi) \geq 0$ for all $\xi \in [a_0, b_0]$. Now, by using $F^{(n)} \geq 0$ and (3.9) in (3.3), we have (3.10). □

Corollary 3.4 *Let all the assumptions of Theorem 3.1 be satisfied. In addition, let n be even and*

$$\sum_{i=1}^{m} \rho_i (\xi_i - \xi_k)_+ \geq 0 \quad \textit{for} \quad k \in \{1, \ldots, m\}.$$

If the function $F : [a_0, b_0] \to \mathbb{R}$ is n-convex, then inequality (3.10) is satisfied, i.e.

$$\sum_{i=1}^{m} \rho_i F(\xi_i) \geq \sum_{k=0}^{n-3} \frac{n-k-2}{k!\,(b_0 - a_0)} \int_{a_0}^{b_0} \sum_{i=1}^{m} \rho_i G(\xi_i, s)$$
$$\times \left(F^{(k+1)}(b_0)(s - b_0)^k - F^{(k+1)}(a_0)(s - a_0)^k \right) ds. \tag{3.11}$$

Furthermore, if $F^{(k+1)}(a_0) \leq 0$ and $(-1)^k F^{(k+1)}(b_0) \geq 0$ for $k \in \{0, 1, \ldots, n-3\}$, then $\sum_{i=1}^{m} \rho_i F(\xi_i) \geq 0$.

Proof Since ξ and $\mathbf{p}$ are real m-tuples that satisfy assumption (1.5) and the function $\xi \mapsto G(\xi, s)$ is convex, applying inequality (1.3) yields

$$\sum_{i=1}^{m} \rho_i G(\xi_i, s) \geq 0. \tag{3.12}$$

It is easy to see that the assumptions of the corollary for even n imply

$$(s-t)^{n-3} k^{[a_0, b_0]}(t, s) \geq 0$$

for all $s, t \in [a_0, b_0]$. Therefore,

$$\int_{a_0}^{t} \sum_{i=1}^{m} \rho_i G(\xi_i, s)(s-t)^{n-3} k^{[a_0, b_0]}(t, s)\, ds \geq 0 \tag{3.13}$$

and applying Theorem 3.3 when F is n-convex gives inequality (3.11).

Moreover, if $F^{(k+1)}(a_0) \leq 0$ and $(-1)^k F^{(k+1)}(b_0) \geq 0$, then

$$F^{(k+1)}(b_0)(s - b_0)^k - F^{(k+1)}(a_0)(s - a_0)^k \geq 0, \tag{3.14}$$

so from inequalities (3.11), (3.12), and (3.14), we obtain $\sum_{i=1}^{m} \rho_i F(\xi_i) \geq 0$. □

An integral version of our second main result states that:

Theorem 3.5 *Let all the assumptions of Theorem 3.2 be satisfied and let*

$$\Omega_4^{[a_0, b_0]}([\alpha_0, \beta_0], g, \rho, t) \geq 0, \quad \textit{for all } t \in [a_0, b_0]. \tag{3.15}$$

If F is n-convex, then we have

$$A_4^{[a_0, b_0]}([\alpha_0, \beta_0], g, \rho, F) \geq 0. \tag{3.16}$$

If opposite inequality holds in (3.15), then (3.16) holds in the reverse direction.

Proof The idea of the proof is the same as that of the proof of Theorem 2.3. By using $F^{(n)} \geq 0$ and (3.15) in (3.6), we have (3.16). □

Corollary 3.6 *Let all the assumptions of Theorem 3.2 be satisfied. In addition, let n be even and*

$$\int_{\alpha_0}^{\beta_0} \rho(\xi)\,(g(\xi)-t)_+^{n-1}\,d\xi \geq 0, \qquad \textit{for every } t \in [a_0, b_0].$$

If the function $F : [a_0, b_0] \to \mathbb{R}$ is n-convex, then we have

$$\int_{\alpha_0}^{\beta_0} \rho(\xi)\,F(g(\xi))\,d\xi \geq \sum_{k=0}^{n-3} \frac{n-k-2}{k!\,(b_0-a_0)} \int_{a_0}^{b_0} \left(\int_{\alpha_0}^{\beta_0} \rho(\xi)\,G(g(\xi), s)\,d\xi \right) \times \left(F^{(k+1)}(b_0)\,(s-b_0)^k - F^{(k+1)}(a_0)\,(s-a_0)^k \right) ds. \tag{3.17}$$

Furthermore, if $F^{(k+1)}(a_0) \leq 0$ and $(-1)^k F^{(k+1)}(b_0) \geq 0$ for $k \in \{0, 1, \ldots, n-3\}$, then the right-hand side of (3.17) is nonnegative.

Proof The proof is analogous to the proof of Corollary 3.4 but instead of Theorem 3.3, we apply Theorem 3.5. □

4. Bounds for identities and integral remainders

In this section, we give several estimations related to the functionals $A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,F)$, for $k \in \{1,2,3,4\}$. For the sake of brevity, in the present and the next sections, we use notations $A_k(F) := A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,F)$ and $\Omega_k(t) := \Omega_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,t)$ for $k \in \{1,2,3,4\}$. We use the well-known Hölder's inequality and bound for the Čebyšev functional $T(F,h)$. This bound is given in the following proposition in which the pre-Grüss inequality is given [4].

Proposition 4.1 *Let $F, h : [a_0, b_0] \to \mathbb{R}$ be two integrable functions such that Fh is also integrable. If*

$$\gamma \leq h(\xi) \leq \Gamma \quad \textit{for} \quad \xi \in [a_0, b_0],$$

then

$$|T(F,h)| \leq \frac{1}{2}(\Gamma-\gamma)\sqrt{T(F,F)}, \tag{4.1}$$

where

$$T(F,h) = \frac{1}{b_0-a_0}\int_{a_0}^{b_0} F(\xi)h(\xi)d\xi - \left(\frac{1}{b_0-a_0}\int_{a}^{b} F(\xi)d\xi\right)\left(\frac{1}{b_0-a_0}\int_{a_0}^{b_0} h(\xi)d\xi\right). \tag{4.2}$$

Now by using the aforementioned result, we are going to obtain a formula for A_k and estimate the remainder which occurs in this formula.

Theorem 4.2 *Let $n \in \mathbb{N}$ and let $F : [a_0, b_0] \to \mathbb{R}$ be such that $F^{(n-1)}$ is an absolutely continuous function and*

$$\gamma \leq F^{(n)}(\xi) \leq \Gamma \quad \textit{for} \quad \xi \in [a_0, b_0].$$

(i) Let $k \in \{1,2\}$ *and let* $\sum_{i=1}^{m} \rho_i = 0$ *(for* $k = 1$*) or* $\int_{\alpha_0}^{\beta_0} \rho(\xi)d\xi = 0$ *(for* $k = 2$*). Then*

$$A_k(F) = \frac{\left[F^{(n-1)}(b_0) - F^{(n-1)}(a_0)\right]}{(n-1)!(b_0-a_0)^2} \int_a^b \Omega_k(t)dt + R_n^k(F; a_0, b_0), \tag{4.3}$$

where the remainder $R_n^k(F; a_0, b_0)$ *satisfies the estimation*

$$|R_n^k(F; a_0, b_0)| \leq \frac{1}{2(n-1)!}(\Gamma - \gamma)\sqrt{T(\Omega_k, \Omega_k)}. \tag{4.4}$$

(ii) Let $k \in \{3,4\}$ *and* $n \geq 3$*. Let the assumptions stated in Theorem 3.1 for* $\mathbf{p}$ *and* ξ *(for* $k = 3$*) and in Theorem 3.2 for* p *and* g *(for* $k = 4$*) hold. Then (4.3) and (4.4) hold with* $(n-3)!$ *instead of* $(n-1)!$ *in the denominator of* $A_k(F)$ *and in the bound of* R_n^k*.*

Proof Fix $k \in \{1,2\}$. Using the definition of A_k and results from the second section, we have

$$\begin{aligned} A_k(F) &= \frac{1}{(n-1)!(b_0-a_0)} \int_{a_0}^{b_0} F^{(n)}(t)\Omega_k(t)dt \\ &= \frac{1}{(n-1)!(b_0-a_0)^2} \int_{a_0}^{b_0} F^{(n)}(t)dt \int_{a_0}^{b_0} \Omega_k(t)dt + R_n^k(F; a_0, b_0) \\ &= \frac{\left[F^{(n-1)}(b_0) - F^{(n-1)}(a_0)\right]}{(n-1)!(b_0-a_0)^2} \int_{a_0}^{b_0} \Omega_k(t)dt + R_n^k(F; a_0, b_0), \end{aligned}$$

where

$$R_n^k(F; a_0, b_0) = \frac{1}{(n-1)!(b_0-a_0)} \left(\int_{a_0}^{b_0} F^{(n)}(t)\Omega_k(t)dt - \frac{1}{b_0-a_0} \int_{a_0}^{b_0} F^{(n)}(s)ds \int_{a_0}^{b_0} \Omega_k(t)dt \right).$$

If we apply Proposition 4.1 for $F \to \Omega_k$ and $h \to F^{(n)}$, then we obtain

$$|R_n^k(F; a_0, b_0)| = |\frac{1}{(n-1)!} T(\Omega_k, F^{(n)})| \leq \frac{1}{2(n-1)!}(\Gamma - \gamma)\sqrt{T(\Omega_k, \Omega_k)}.$$

The proof for $k \in \{3,4\}$ is done in a similar manner. □

Using the same method as we used in the previous theorem and other type of bounds for the Čebyšev functional, we are able to give another estimation for a remainder. The following theorem gives us some Ostrowski-type inequalities. As usual, the symbol $L_p[a_0, b_0]$ $(1 \leq p < \infty)$ denotes the space of p-power integrable functions on the interval $[a_0, b_0]$ equipped with the norm

$$\|F\|_p = \left(\int_{a_0}^{b_0} |F(t)|^p \, dt \right)^{\frac{1}{p}} < \infty$$

and $L_\infty[a_0, b_0]$ denotes the space of essentially bounded functions on $[a_0, b_0]$ with the norm

$$\|F\|_\infty = \text{ess} \sup_{t \in [a_0, b_0]} |F(t)|.$$

Theorem 4.3 *Let $\digamma^{(n)} \in L_q[a_0, b_0]$ for some $n \in \mathbb{N}$ and let (q, r) be a pair of conjugate exponents, that is, $1 \le q, r \le \infty$, $\frac{1}{q} + \frac{1}{r} = 1$.*

(i) Let $k \in \{1, 2\}$ and let $\sum_{i=1}^{m} \rho_i = 0$ (for $k = 1$) or $\int_{\alpha_0}^{\beta_0} \rho(\xi) d\xi = 0$ (for $k = 2$). Then

$$|A_k(\digamma)| \le \frac{1}{(n-1)!} \|\digamma^{(n)}\|_q \|\Omega_k\|_r. \tag{4.5}$$

The constant on the right hand side of (4.5) is sharp for $1 < q \le \infty$ and the best possible for $q = 1$.

(ii) Let $k \in \{3, 4\}$ and $n \ge 3$. For $k = 3$, we assume that ξ and $\mathbf{p}$ satisfy the assumptions of Theorem 3.1 and for $k = 4$ we assume that p and g satisfy the assumptions of Theorem 3.2. Then inequality (4.5) holds with $(n-3)!$ instead of $(n-1)!$ in the denominator of the bound for A_k.

Proof Fix $k \in \{1, 2\}$. From the definition of A_k and results from the second section together with the Hölder inequality, we get

$$|A_k(\digamma)| = \left| \frac{1}{(n-1)!} \int_{a_0}^{b_0} \digamma^{(n)}(t) \Omega_k(t) dt \right| \le \|\digamma^{(n)}\|_q \|\lambda_k\|_r$$

where we denoted $\frac{1}{(n-1)!} \Omega_k$ by λ_k.

The sharpness of the constant $\left(\int_{a_0}^{b_0} |\lambda_k(t)|^r ds \right)^{1/r}$ can be proved by considering the following function $\digamma$ for which the equality in (4.5) is obtained.

For $1 < q < \infty$ we take $\digamma$ to be such that

$$\digamma^{(n)}(s) = sgn \lambda_k(t) \cdot |\lambda_k(t)|^{1/(q-1)},$$

while for $q = \infty$, we define $\digamma$ such that

$$\digamma^{(n)}(t) = sgn \lambda_k(t).$$

The fact that (4.5) is the best possible for $q = 1$ can be proved as in [2, Theorem 12].

Proof for $k \in \{3, 4\}$ is similar to the previous case. □

5. Mean value results

In this section, we consider mean value theorems involving A_k. Throughout the section, we use the agreement that if $k \in \{1, 2\}$, then $n \in \mathbb{N}$ and if $k \in \{3, 4\}$, then $n \ge 3$. Furthermore, for $k = 1$, we assume that $\sum_{i=1}^{m} \rho_i = 0$, for $k = 2$ we assume that $\int_{\alpha_0}^{\beta_0} \rho(\xi) d\xi = 0$, for $k = 3$ we assume that ξ and $\mathbf{p}$ satisfy the assumptions of Theorem 3.1 and for $k = 4$ we assume that p and g satisfy the assumptions of Theorem 3.2.

Theorem 5.1 *Let $k \in \{1, 2, 3, 4\}$ and let us consider A_k as a functional on $C^n[a_0, b_0]$. If the corresponding conditions from the set $\{(2.6), (2.16), (3.9), (3.15)\}$ related to the fixed k hold, then there exists $\xi_k \in [a_0, b_0]$ such that*

$$A_k(\digamma) = \digamma^{(n)}(\xi_k) A_k(\digamma_0), \tag{5.1}$$

where $\digamma_0(\xi) = \frac{\xi^n}{n!}$.

Proof Let us define the functions

$$F_1(\xi) = MF_0\xi) - F(\xi)$$

and

$$F_2(\xi) = F(\xi) - LF_0(\xi),$$

where L and M are the minimum and maximum of the image of $F^{(n)}$, i.e.

$$F^{(n)}([a_0, b_0]) = [L, M].$$

Then F_1 and F_2 are $n-$convex. Hence, $A_k(F_1) \geq 0$ and $A_k(F_2) \geq 0$ and

$$LA_k(F_0) \leq A_k(F) \leq MA_k(F_0).$$

If $A_k(F_0) = 0$, then the statement obviously holds.

If $A_k(F_0) \neq 0$, then $\frac{A_k(F)}{A_k(f_0)} \in [L, M] = F^{(n)}([a_0, b_0])$, so there exist $\xi_k \in [a_0, b_0]$ such that $\frac{A_k(F)}{A_k(F_0)} = F^{(n)}(\xi_k)$. □

When we apply Theorem 5.1 to the function $\omega = A_k(h)F - A_k(F)h$, we get the following result.

Theorem 5.2 *Let $k \in \{1, 2, 3, 4\}$ and let us consider A_k as a functional on $C^n[a_0, b_0]$. If the corresponding conditions from the set $\{(2.6), (2.16), (3.9), (3.15)\}$ related to the fixed k hold, then there exists $\xi_k \in [a_0, b_0]$ such that*

$$\frac{A_k(F)}{A_k(h)} = \frac{F^{(n)}(\xi_k)}{h^{(n)}(\xi_k)}$$

assuming that both of the denominators are non-zero.

Remark 5.3 *If the inverse of $\frac{F^{(n)}}{h^{(n)}}$ exists, then from the above mean value theorems, we can give generalized means*

$$\xi_k = \left(\frac{F^{(n)}}{h^{(n)}}\right)^{-1} \left(\frac{A_k(F)}{A_k(h)}\right). \tag{5.2}$$

Remark 5.4 *Using the same method as in [2], we can construct new families of exponentially convex functions and Cauchy type means. Also, using the idea described in [2] we can obtain results for the $n-$ convex functions at a point.*

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