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Additive derivative and multiplicative coderivative operators on MV-algebras

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Abstract: In this paper we introduce derivative MV-algebras (or MV-algebras with additive derivative operators). We indicate that the derivative MV-algebras are generalizations of closure MV-algebras. Then we investigate the connection between additive derivative operators on MV-algebras and the derivative operators on the greatest Boolean subalgebras of MV-algebras. Finally, we study some properties of the derivative MV-algebras.

Key words: Derivative MV-algebra, additive derivative operator, closure MV-algebra, topological Boolean algebra

1. Introduction

The notion of an MV-algebra was introduced by Chang in [1] and [2] for providing algebraic proof of the completeness theorem of the Łukasiewicz infinite valued propositional logic. The closure MV-algebra, a natural generalization of topological Boolean algebra, was introduced by Rachůnek and Švrček in [8]. On p. 182 of [6] McKinsey and Tarski state "Like the topological operation of closure, other topological operations can be treated in an algebraic way. This may be especially interesting in regards to those operations which are not definable in terms of closure. An especially important notion is that of the derivative of a point set A which will be denoted by $D(A)$". Thus, the derivative algebra was introduced by McKinsey and Tarski in [6]. Then derivative algebra was redefined by Esakia in [4]. On p. 157 the authors say "It must be pointed out that we weaken the definition of derivative algebra in [6] slightly; namely, we postulate the condition (3*) $\mathbf{dd}a \leq a \vee \mathbf{d}a$ instead of (3) $\mathbf{dd}a \leq \mathbf{d}a$. We justify this weakening by noting that there are topological spaces, in which condition (3) is not valid".

In this paper, we introduce and study additive derivative and multiplicative coderivative operators on MV-algebras. Then we study the relation between the derivative MV-algebras and closure MV-algebras. At the same time, we investigate the connection between additive derivative operators on complete MV-algebras and the derivative operators on the greatest Boolean subalgebras of MV-algebras. Finally, we introduce the d-ideal and study some properties of d-ideals.

2. Preliminaries

Definition 2.1 ([3, 8]) An algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ of signature $< 2, 1, 0 >$ is called an *MV-algebra* if it satisfies the following identities:

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

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(MV2) $x \oplus y = y \oplus x$,

(MV3) $x \oplus 0 = x$,

(MV4) $\neg\neg x = x$,

(MV5) $x \oplus \neg 0 = \neg 0$,

(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$.

For arbitrary $x, y \in A$ we put:

$x \odot y := \neg(\neg x \oplus \neg y)$, $x \vee y := \neg(\neg x \oplus y) \oplus y$, $x \wedge y := \neg(\neg x \vee \neg y)$, $1 := \neg 0$. Then $(A, \odot, 1)$ is an commutative monoid, $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice, and $(A, \oplus, 0, \vee, \wedge)$ and $(A, \odot, 1, \vee, \wedge)$ are lattice ordered monoids. Hence, the binary operations " $\oplus$ " and " $\odot$ " are mutually dual as well as lattice operations " $\vee$ " and " $\wedge$ ".

The following identities are immediate consequences of (MV4) and (MV6):

(MV7) $\neg 1 = 0$,

(MV8) $x \oplus \neg x = 1$.

Lemma 2.1 ([3]) *Let $\mathcal{A} = (A, \oplus, \neg, 0)$ be an MV-algebra and $x, y \in A$. Then the following conditions are equivalent:*

(i) $\neg x \oplus y = 1$,

(ii) $x \odot \neg y = 0$,

(iii) $y = x \oplus (y \odot \neg x)$,

(iv) there is an element $z \in A$ such that $x \oplus z = y$.

Let $\mathcal{A}$ be an MV-algebra. For any two elements x and y of A, $x \leq y$ iff x and y satisfy the above equivalent conditions (i)–(iv). It follows that $\leq$ is a partial order, called the natural order of $\mathcal{A}$.

Lemma 2.2 ([3]) *In every MV-algebra $\mathcal{A}$ the natural order $\leq$ has the following properties:*

(i) $x \leq y$ *iff* $\neg y \leq \neg x$,

(ii) if $x \leq y$, then $x \oplus z \leq y \oplus z$ and $x \odot z \leq y \odot z$ for each $z \in A$,

(iii) $x \odot y \leq z$ *iff* $x \leq \neg y \oplus z$.

Note that in every MV-algebra $\mathcal{A}$ we have $x \odot y \leq x \wedge y$ and $x \vee y \leq x \oplus y$ for all $x, y \in A$.

Definition 2.2 ([3]) An element x of a bounded lattice L with 0 and 1 is said to be *complemented* iff there is an element $y \in L$ (the complement of x) such that $x \vee y = 1$ and $x \wedge y = 0$.

When L is distributive each $z \in L$ has at most one complement, denoted $\neg z$. Let $B(L)$ be the set of all complemented elements of the distributive lattice L. Note that 0 and 1 are elements of $B(L)$. As a matter of fact, $B(L)$ is a sublattice of L, which is also Boolean algebra. For any MV-algebra $\mathcal{A}$ we shall write $B(\mathcal{A})$. Elements of $B(\mathcal{A})$ are called the *Boolean elements* of $\mathcal{A}$ (for details, see [3]).

Theorem 2.1 ([3]) *For every element x in an MV-algebra $\mathcal{A}$ the following conditions are equivalent:*

(i) $x \in B(\mathcal{A})$,

(ii) $x \vee \neg x = 1$,

(iii) $x \wedge \neg x = 0$,

(iv) $x \oplus x = x$,

(v) $x \odot x = x$,

(vi) $x \oplus y = x \vee y$, *for all* $y \in A$,

(vii) $x \odot y = x \wedge y$, *for all* $y \in A$.

Corollary 2.1 ([3]) *$B(\mathcal{A})$ is a subalgebra of the MV-algebra $\mathcal{A}$. A subalgebra $\mathcal{B}$ of $\mathcal{A}$ is a Boolean algebra iff $\mathcal{B} \subseteq B(\mathcal{A})$.*

Corollary 2.2 ([3]) *An MV-algebra $\mathcal{A}$ is Boolean algebra if and only if the operation $\oplus$ is idempotent, i.e. the equation $x \oplus x = x$ is satisfied by $\mathcal{A}$.*

Lemma 2.3 ([8]) *If $\mathcal{A}$ is an MV-algebra and $a \in B(\mathcal{A})$, then*

$$a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$$

for each $x, y \in A$.

3. Derivative MV-algebras

The following definition of an additive derivative operator on an MV-algebra generalizes that of a derivative operator on Boolean algebra (see [4]).

Definition 3.1 Let $\mathcal{A} = (A, \oplus, \neg, 0)$ be an MV-algebra. An *additive derivative operator* on $\mathcal{A}$ is a mapping $d : A \longrightarrow A$ with the following properties for each $x, y \in A$:

(d1) $d(x \oplus y) = d(x) \oplus d(y)$,

(d2) $dd(x) \leq x \oplus d(x)$,

(d3) $d(0) = 0$.

When d has the stronger property

(d4) $dd(x) \leq d(x)$,

it is a *strong additive derivative operator* on $\mathcal{A}$.

If d is an additive derivative operator on $\mathcal{A}$, then $\mathcal{A} = (A, \oplus, \neg, 0, d)$ is called a *derivative MV-algebra.*

We note that an additive derivative operator on MV-algebra $\mathcal{A}$ reflects algebraic properties of a derivative operator d on topological space X. Recall that $d(S)$ is the set of all accumulation points of a subset S of a topological space X. A point $x \in X$ is called an accumulation point of S if every open neighborhood of x contains a point in S different from x.

Theorem 3.1 *The axioms of an additive derivative operator on MV-algebra $\mathcal{A}$ are independent.*

Proof We need to construct a model for each axiom in which the axiom is false while the others are true.

(d1) Consider $L_3 = \{0, \frac{1}{2}, 1\}$, and for all $x, y \in L_3$, let $x \oplus y = min\{1, x + y\}$ and $\neg x = 1 - x$. It is easy to see that $_3 = (L_3, \oplus, \neg, 0)$ is an MV-algebra. Define $d : L_3 \longrightarrow L_3$ as

$$d(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{2}, & otherwise. \end{cases}$$

d satisfies (d2) and (d3) axioms. We can show that (d1) is not satisfied. Indeed, we take $x = 1$ and $y = 1$. Then:

$$d(x \oplus y) = d(x) \oplus d(y),$$

$$d(1 \oplus 1) = d(1) \oplus d(1),$$

$$d(1) = d(1) \oplus d(1),$$

$$\frac{1}{2} \neq \frac{1}{2} \oplus \frac{1}{2} = 1.$$

(d2) Let $A = \{0, a, b, c, e, 1\}$, where $0 < a, b < c < 1$ and $0 < b < e < 1$. Define $\oplus$ and $\neg$, as follows:

$\oplus$	0	a	b	c	e	1
0	0	a	b	c	e	1
a	a	a	c	c	1	1
b	b	c	e	1	e	1
c	c	c	1	1	1	1
e	e	1	e	1	e	1
1	1	1	1	1	1	1

x	0	a	b	c	e	1
$\neg x$	1	e	c	b	a	0

Then the structure $\mathcal{A} = (\{0, a, b, c, e, 1\}, \oplus, \neg, 0)$ is an MV-algebra (see [5]). We define the operator $d : A \longrightarrow A$ as follows:

x	0	a	b	c	e	1
$d(x)$	0	e	a	1	a	1

Then it is clear that d satisfies (d1) and (d3) axioms. We take $x = b$. Then we have $d(d(b)) = e$ and $b \oplus d(b) = b \oplus a = c$. Moreover, we must have $dd(b) \leq b \oplus d(b)$, but e and c are not comparable. Thus, axiom (d2) is not satisfied.

(d3) Let $\mathcal{A} = (A, \oplus, \neg, 0)$ be an MV-algebra. We define the operator $g : A \longrightarrow A$ as $d(x) = 1$ for all $x \in A$. Then it is obvious that d satisfies (d1) and (d2) axioms, but axiom (d3) is not satisfied. □

We shall denote by $\mathcal{D}(\mathcal{A})$ the set of all the additive derivative operators on $\mathcal{A}$, and by $\mathcal{S}(\mathcal{A})$ the set of all the strong additive derivative operators on $\mathcal{A}$. It is clear that $\mathcal{S}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$.

Example 3.1 *Consider* $\mathcal{A} = (\{0, a, b, 1\}, \oplus, \neg, 0)$ *with the operators as follows:*

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

x	0	a	b	1
$\neg x$	1	b	a	0

This structure is an MV-algebra. We define operator d *as follows:*

x	0	a	b	1
$d(x)$	0	b	a	1

It is easy to check that d *is an additive derivative operator on MV-algebra* $\mathcal{A}$ *and* $d \in \mathcal{D}(\mathcal{A})$. *We take* $x = a$. *Then we have* $dd(a) = d(d(a)) = d(b) = a$ *and* $d(a) = b$. *Since* a *and* b *are not comparable, axiom (d4) is not satisfied. Thus,* $d \notin \mathcal{S}(\mathcal{A})$.

Let $\mathcal{A} = (A, \oplus, \neg, 0)$ be an MV-algebra. We set $0_A : A \longrightarrow A$ operator by $0_A(x) = 0$ for all $x \in A$. Then it is clear that $\mathcal{A} = (A, \oplus, \neg, 0, 0_A)$ is a derivative MV-algebra.

Now we define $d_A : A \longrightarrow A$ operator as follows:

$$d_A = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

for all $x \in A$. Then it is obvious that $\mathcal{A} = (A, \oplus, \neg, 0, d_A)$ is a derivative MV-algebra.

Finally, we describe $I_A : A \longrightarrow A$ operator as identity function I_A. It is clear that $\mathcal{A} = (A, \oplus, \neg, 0, I_A)$ is a derivative MV-algebra.

Every MV-algebra has the additive derivative 0_A, d_A, and I_A operators. These operators are trivial additive derivative operators. Of course, these three operators belong to $\mathcal{S}(\mathcal{A})$.

Example 3.2 *Consider* $L_3 = \{0, \frac{1}{2}, 1\}$, *and for all* $x, y \in L_3$, *let* $x \oplus y = min\{1, x + y\}$ *and* $\neg x = 1 - x$. $_3 = (L_3, \oplus, \neg, 0)$ *is an MV-algebra. It has only trivial additive derivative operators.*

Proposition 3.1 *Let* $d : A \longrightarrow A$ *be an additive derivative operator on MV-algebra* $\mathcal{A}$. *Then* d *is a monotone mapping.*

Proof Let $x, y \in A$ such that $x \leq y$. Then:
$x \vee y = y$,
$d(x \vee y) = d(y)$,
$d(\neg(x \oplus \neg y) \oplus x) = d(y)$ by definition of $\vee$ and (MV6),
$d(\neg(\neg\neg x \oplus \neg y) \oplus x) = d(y)$ by (MV4),
$d((\neg x \odot y) \oplus x) = d(y)$ by definition of $\odot$,
$d((\neg x \odot y)) \oplus d(x) = d(y)$ by (d1),
and so $d(x) \leq d(y)$ by Lemma 2.1 (iv). □

Let $f : A \to A$ be a mapping on an MV-algebra $\mathcal{A}$. We define a mapping

$$f^{\neg} : A \to A$$

such that

$$f^{\neg}(x) = \neg f(\neg x)$$

for any $x \in A$ (see [7]).

Proposition 3.2 *Let $d : A \longrightarrow A$ be an additive derivative operator on MV-algebra $\mathcal{A}$. Then for any $x, y \in A$ we have:*

(i) $d^{\neg}(x \odot y) = d^{\neg}(x) \odot d^{\neg}(y)$,

(ii) $x \odot d^{\neg}(x) \leq d^{\neg}d^{\neg}(x)$,

(iii) $d^{\neg}(1) = 1$.
When d has the stronger property,

(iv) $d^{\neg}(x) \leq d^{\neg}d^{\neg}(x)$.

Proof **(i)** $d^{\neg}(x \odot y) = \neg d(\neg(x \odot y))$ by definition of $d^{\neg}$
$= \neg d(\neg\neg(\neg x \oplus \neg y))$ by definition of $\odot$
$= \neg d(\neg x \oplus \neg y)$ by (MV4)
$= \neg(d(\neg x) \oplus d(\neg y))$ by (d1)
$= \neg(\neg\neg d(\neg x) \oplus \neg\neg d(\neg y))$ by (MV4)
$= \neg d(\neg x) \odot \neg d(\neg y)$ by definition of $\odot$
$= d^{\neg}(x) \odot d^{\neg}(y)$ by definition of $d^{\neg}$.
(ii) $x \odot d^{\neg}(x) = x \odot \neg d(\neg x)$ by definition of $d^{\neg}$
$= \neg(\neg x \oplus d(\neg x))$ by definition of $\odot$
$\leq \neg dd(\neg x)$ by (d2) and Lemma 2.2 (i)
$= \neg d\neg(\neg d(\neg x))$ by (MV4)
$= d^{\neg}d^{\neg}(x)$ by definition of $d^{\neg}$.
(iii) $d^{\neg}(1) = \neg d(\neg 1) = \neg d(0) = \neg 0 = 1$.
(iv) If d has the stronger property, then
$d^{\neg}(x) = \neg d(\neg x)$ by definition of $d^{\neg}$
$\leq \neg dd(\neg x)$ by (d4) and Lemma 2.2 (i)
$= \neg d\neg(\neg d(\neg x))$ by (MV4)
$= d^{\neg}d^{\neg}(x)$ by definition of $d^{\neg}$. □

Definition 3.2 Let $\mathcal{A} = (A, \oplus, \neg, 0)$ be an MV-algebra. A *multiplicative coderivative operator* on $\mathcal{A}$ is a mapping $t : A \longrightarrow A$ with the following properties for each $x, y \in A$:

(t1) $t(x \odot y) = t(x) \odot t(y)$,

(t2) $x \odot t(x) \leq tt(x)$,

(t3) $t(1) = 1$.

When t has the stronger property,

(t4) $t(x) \leq tt(x)$,

we call it a *strong multiplicative coderivative operator* on $\mathcal{A}$. $t(x)$ will be called *the coderivative* of x.

Proposition 3.3 *Let $t : A \longrightarrow A$ be a multiplicative coderivative operator on MV-algebra $\mathcal{A}$. Then t is a monotone mapping.*

Proof. Let $x, y \in A$ such that $x \leq y$. We get
$x \wedge y = x$,
$t(x \wedge y) = t(x)$,
$t(\neg(\neg x \vee \neg y) = t(x)$ by duality,
$t(\neg[\neg(\neg(\neg x) \oplus \neg y) \oplus \neg y]) = t(x)$ by definition of $\vee$,
$t(\neg[\neg(x \oplus \neg y) \oplus \neg y]) = t(x)$ by (MV4),
$t((x \oplus \neg y) \odot y) = t(y)$ by definition of $\odot$,
$t(x \oplus \neg y) \odot t(y) = t(y)$ by (t1),
and hence $t(x) \leq t(y)$. □

Theorem 3.2 *Let $d : A \longrightarrow A$ be an additive derivative operator on MV-algebra $\mathcal{A}$. Then the mapping $d^{\neg} : A \longrightarrow A$ is a multiplicative coderivative operator on $\mathcal{A}$. Moreover, if d is an strong additive derivative operator on $\mathcal{A}$, then $d^{\neg}$ is a strong multiplicative coderivative operator on $\mathcal{A}$.*

Proof. It follows from Proposition 3.2. □

Proposition 3.4 *Let $t : A \longrightarrow A$ be a multiplicative coderivative operator on MV-algebra $\mathcal{A}$. Then for any $x, y \in A$ we have:*

(i) $t^{\neg}(x \oplus y) = t^{\neg}(x) \oplus t^{\neg}(y)$,

(ii) $t^{\neg}t^{\neg}(x) \leq x \oplus t^{\neg}(x)$,

(iii) $t^{\neg}(0) = 0$.

When t has the stronger property,

(iv) $t^{\neg}t^{\neg}(x) \leq t^{\neg}(x)$.

Proof. (i)
$$\begin{aligned} t^{\neg}(x \oplus y) &= \neg t(\neg(x \oplus y)) && \text{by definition of } t^{\neg} \\ &= \neg t(\neg(\neg\neg x \oplus \neg\neg y)) && \text{by (MV4)} \\ &= \neg t(\neg x \odot \neg y) && \text{by definition of } \odot \\ &= \neg(t(\neg x) \odot t(\neg y)) && \text{by (t1)} \\ &= \neg(\neg(\neg t(\neg x) \oplus \neg t(\neg y)) && \text{by definition of } \odot \\ &= \neg t(\neg x) \oplus \neg t(\neg y) && \text{by (MV4)} \end{aligned}$$

$= t^{\neg}(x) \oplus t^{\neg}(y)$ by definition of $t^{\neg}$.

(ii) $t^{\neg}t^{\neg}(x) = t^{\neg}(\neg t(\neg x))$ by definition of $t^{\neg}$

$= \neg t(\neg(\neg t(\neg x)))$ by definition of $t^{\neg}$

$= \neg tt(\neg x))$ by (MV4)

$\leq \neg(\neg x \odot t(\neg x))$ by (t2) and Lemma 2.2 (i)

$= \neg(\neg(\neg\neg x \oplus \neg t(\neg x)))$ by definition of $\odot$

$= x \oplus \neg t(\neg x)$ by (MV4)

$= x \oplus t^{\neg}(x)$ by definition of $t^{\neg}$.

(iii) $t^{\neg}(0) = \neg t(\neg 0) = \neg t(1) = \neg 1 = 0$.

(iv) If t has the stronger property, then

$t^{\neg}t^{\neg}(x) = t^{\neg}(\neg t(\neg x))$ by definition of $t^{\neg}$

$= \neg t(\neg(\neg t(\neg x)))$ by definition of $t^{\neg}$

$= \neg tt(\neg x)$ by (MV4)

$\leq \neg t(\neg x)$ by (t4) and Lemma 2.2 (i)

$= t^{\neg}(x)$ by definition of $t^{\neg}$. □

Theorem 3.3 *Let $t : A \longrightarrow A$ be a multiplicative coderivative operator on MV-algebra $\mathcal{A}$. Then the mapping $t^{\neg} : A \longrightarrow A$ is an additive derivative operator on $\mathcal{A}$. Moreover, if t is a strong multiplicative coderivative operator on $\mathcal{A}$, then $t^{\neg}$ is a strong additive derivative operator on $\mathcal{A}$.*

Proof. It follows from Proposition 3.2. □

Definition 3.3 ([8]) Let $\mathcal{A} = (A, \oplus, \neg, 0)$ be an MV-algebra and $Cl : A \longrightarrow A$ a mapping. Then Cl is called an *additive closure operator* on $\mathcal{A}$ if for each $x, y \in A$:

(c1) $Cl(x \oplus y) = Cl(x) \oplus Cl(y)$,

(c2) $x \leq Cl(x)$,

(c3) $Cl(Cl(x)) = Cl(x)$,

(c4) $Cl(0) = 0$.

If Cl is an additive operator on $\mathcal{A}$, then $\mathcal{A} = (A, \oplus, \neg, 0, Cl)$ is called a *closure MV-algebra*.

Remark 3.1 *Let $\mathcal{A} = (A, \oplus, \neg, 0, Cl)$ be a closure MV-algebra. Clearly, if Cl is a closure operator, then Cl satisfies conditions (d1)–(d4), but the converse is not true generally.*

Example 3.3 *Let $\mathcal{A} = (\{0, a, b, 1\}, \oplus, \neg, 0)$ be an MV-algebra in Example 3.1. We define the operator d as follows:*

x	0	a	b	1
$d(x)$	0	0	1	1

Then the structure $\mathcal{A} = (\{0, a, b, 1\}, \oplus, \neg, 0, d)$ is a derivative MV-algebra. We can easily check that (c2) does not hold. In fact, if $x = a$, then we have $d(a) = 0$ and $a \nleq d(a)$.

Now, we generalize the well-known definitions on derivative algebras.

Definition 3.4 Let $\mathcal{A} = (A, \oplus, \neg, 0, d)$ be a derivative MV-algebra. The element x of $\mathcal{A}$ is called *dense-in-itself* if $x \leq d(x)$.

Definition 3.5 Let $\mathcal{A} = (A, \oplus, \neg, 0, d)$ be a derivative MV-algebra. $\mathcal{A}$ is called *dense-in-itself* if $d(1) = 1$.

Definition 3.6 A derivative MV-algebra is called a *scattered space* if it has no dense-in-itself nonzero elements.

Example 3.4 *Let $\mathcal{A} = (A, \oplus, \neg, 0)$ be an MV-algebra. Then it is clear that $\mathcal{A} = (A, \oplus, \neg, 0, 0_A)$ derivative MV-algebra is a scattered space.*

The following theorem shows that the d operator is more expressive than the Cl operator. We known that the class of dense-in-itself spaces is not Cl-definable.

Theorem 3.4 *Let $\mathcal{A} = (A, \oplus, \neg, 0, d)$ be a derivative MV-algebra. $\mathcal{A}$ is dense-in-itself iff $d^{\neg}(x) \leq d(x)$ for all $x \in \mathcal{A}$.*

Proof. Let $\mathcal{A}$ be dense-in-itself. Then, for any $x \in \mathcal{A}$,

$x \oplus \neg x = 1$ by (MV8),

$d(x \oplus \neg x) = d(1)$,

$d(x) \oplus d(\neg x) = d(1)$ by (d1),

$d(x) \oplus d(\neg x) = 1$ by hypothesis,

$d(x) \oplus \neg(\neg d(\neg x)) = 1$ by (MV4),

$d(x) \oplus \neg(d^{\neg}(x)) = 1$ by definition of $d^{\neg}$,

$d^{\neg}(x) \leq d(x)$ by Lemma 2.1 (i).

Conversely, let $d^{\neg}(x) \leq d(x)$ for all $x \in \mathcal{A}$. Then,

$\neg(d^{\neg}(x)) \oplus d(x) = 1$ by Lemma 2.1 (i),

$\neg(\neg d(\neg x)) \oplus d(x) = 1$ by definition of $d^{\neg}$,

$d(\neg x) \oplus d(x) = 1$ by (MV4),

$d(\neg x \oplus x) = 1$ by (d2),

$d(1) = 1$ by (MV8). □

The notion of subspace of a topological space generalizes to derivative MV-algebra. Given a derivative MV-algebra $\mathcal{A}$ and its the Boolean elements a, set $A_a = \{x \in A : x \leq a\}$. Then $\mathcal{A}_a$ has a "natural structure" of derivative MV-algebra induced by the structure $\mathcal{A}$. In $\mathcal{A}_a$, the operation $\oplus$ is unchanged. We put

$x \oplus_a y = x \oplus y$, $0_a = 0$, $\neg_a x = a \odot \neg x$, $d_a(x) = a \odot d(x)$ for any $x, y \in A_a$.

Theorem 3.5 *Let $\mathcal{A}$ be a derivative MV-algebra and a be a Boolean element in A. Then $\mathcal{A}_a = (A_a, \oplus_a, \neg_a, 0_a, d_a)$ is a derivative MV-algebra.*

Proof. Clearly, $((A, \oplus_a, \neg_a, 0_a)$ is an MV-algebra (for detail, see [8]). We will show that d_a is a derivative operator on $\mathcal{A}_a$. Let $x, y \in A_a$.

(d1) $d_a(x) \oplus_a d_a(y) = (a \odot d(x)) \oplus (a \odot d(y))$ by the definition of operators
$= a \odot (d(x) \oplus d(y))$ by Lemma 2.3
$= a \odot (d(x \oplus y))$ by (d1)
$= d_a(x \oplus y)$ by definition of d_a.

(d2) $a \odot d(x) \leq d(x)$.
$d(a \odot d(x)) \leq d(d(x))$ by Proposition 3.1.
$a \odot d(a \odot d(x)) \leq a \odot dd(x) \leq a \odot (x \oplus d(x))$ by Lemma 2.2 and (d2).
$a \odot (x \oplus d(x)) = (a \odot x) \oplus (a \odot d(x)) = x \oplus d_a(x)$ by Lemma 2.3.

$$d_a d_a(x) = d_a(a \odot d(x)) = a \odot d(a \odot d(x)) \leq a \odot dd(x) \leq a \odot (x \oplus d(x)) = x \oplus d_a(x).$$

(d3) $d_a(0_a) = d_a(0) = a \odot 0 = 0$.

(d4) if d has the stronger property, then

$$d_a d_a(x) = d_a(a \odot d(x)) = a \odot d(a \odot d(x)) \leq a \odot dd(x) \leq a \odot d(x) = d_a(x).$$

□

At the same time, we have for any $x, y \in A_a$
$x \odot_a y = x \odot y$
and

$$d_a^{\neg}(x) = \neg_a(d_a(\neg_a(x)) = \neg_a(d_a(\neg x \odot a)) = \neg_a[a \odot d(\neg x \odot a)] = \neg_a[a \odot d(\neg(x \oplus \neg a))]$$
$$= a \odot \neg[a \odot d(\neg(x \oplus \neg a))] = a \odot [\neg a \oplus \neg d(\neg(x \oplus \neg a))] = a \odot [\neg a \oplus d^{\neg}(x \oplus \neg a)]$$
$$= a \odot d^{\neg}(x \oplus \neg a).$$

Definition 3.7 A subalgebra $\mathcal{B}$ of a derivative MV-algebra $\mathcal{A}$ is called a *derivative subalgebra* if $d(x) \in \mathcal{B}$ for every $x \in \mathcal{B}$.

Note that a subalgebra $\mathcal{B}$ is a derivative subalgebra iff $d^{\neg}(x) \in \mathcal{B}$ for every $x \in \mathcal{B}$.

Theorem 3.6 *The Boolean algebra $B(\mathcal{A})$ of a derivative MV-algebra $\mathcal{A}$ is a derivative subalgebra of $\mathcal{A}$.*

Proof. Let $x \in B(\mathcal{A})$. Then:
$x \oplus x = x$ by Theorem 2.1 (iv),
$d(x \oplus x) = d(x)$,
$d(x) \oplus d(x) = d(x)$ by (d1),
so, $d(x) \in B(\mathcal{A})$. □

Let $\mathcal{A} = (A, \oplus, \neg, 0, d)$ be a derivative MV-algebra. We define the Cl operator as follows:
$Cl_d := x \oplus d(x)$ for all $x \in B(\mathcal{A})$.

Theorem 3.7 *The derivative subalgebra $B(\mathcal{A})$ of a derivative MV-algebra $\mathcal{A}$ is a closure MV-algebra with Cl_d an additive closure operator.*

Proof. We will show that Cl_d is a closure operator on $B(\mathcal{A})$. Let $x, y \in B(\mathcal{A})$.

(c1) $Cl_d(x \oplus y) = (x \oplus y) \oplus d(x \oplus y)$ by definition of Cl_d
$= (x \oplus y) \oplus [d(x) \oplus d(y)]$ by (d1)
$= (x \oplus d(x)) \oplus [y \oplus d(y)]$ by (MV1),(MV2)
$= Cl_d(x) \oplus Cl_d(y)$ by definition of Cl_d.

(c2) $x \leq x \oplus d(x) = Cl_d(x)$.

(c3) $Cl_d Cl_d(x) = Cl_d(x \oplus d(x))$
$= Cl_d(x) \oplus Cl_d(d(x))$ by (c1)
$= (x \oplus d(x)) \oplus (d(x) \oplus dd(x))$ by definition of Cl_d
$= x \oplus [d(x) \oplus d(x)] \oplus dd(x)$ by (MV1)
$= (x \oplus d(x)) \oplus dd(x)$ by Theorem 3.6
$\leq (x \oplus d(x)) \oplus (x \oplus d(x))$ by (d2)
$= (x \oplus d(x)) = Cl_d(x)$ by Theorem 3.6.

On the other hand,

$$Cl_d(x) = x \oplus d(x) \leq (x \oplus d(x)) \oplus (d(x) \oplus dd(x)) = Cl_d Cl_d(x).$$

(c4) $Cl_d(0) = 0 \oplus d(0) = 0$ by (d3),(MV3). □

3.1. Extending a derivative operation

Let $\mathcal{A}$ be a complete derivative MV-algebra, i.e. a derivative MV-algebra in which every subset has a lowest upper bound and therefore also a greatest lower bound with respect to induced order. Can every derivative operator on $B(\mathcal{A})$ be extended to an additive derivative operator on $\mathcal{A}$? The answer to that question is the following theorem.

Theorem 3.8 *Let $\mathcal{A}$ be a complete derivative MV-algebra and g be a derivative operator on the Boolean algebra $B(\mathcal{A})$. Then there may be not an additive derivative operator d on $\mathcal{A}$ such that its restriction on $B(\mathcal{A})$ is equal to g.*

Proof. Let $A = \{0, a, b, c, e, 1\}$, where $0 < a, b < c < 1$ and $0 < b < e < 1$. Define $\oplus$ and $\neg$ as follows:

$\oplus$	0	a	b	c	e	1
0	0	a	b	c	e	1
a	a	a	c	c	1	1
b	b	c	e	1	e	1
c	c	c	1	1	1	1
e	e	1	e	1	e	1
1	1	1	1	1	1	1

x	0	a	b	c	e	1
$\neg x$	1	e	c	b	a	0

Then the structure $\mathcal{A} = (\{0, a, b, c, e, 1\}, \oplus, \neg, 0)$ is a complete MV-algebra. The Boolean algebra of the Boolean elements of the MV-algebra is
$B(\mathcal{A}) = \{0, a, e, 1\}$. Define $g : B(\mathcal{A}) \longrightarrow B(\mathcal{A})$ as

$$g(0) = 0,\ g(a) = e,\ g(e) = a,\ g(1) = 1.$$

Then g is a derivative operator on the Boolean algebra $B(\mathcal{A})$.

Suppose that $d : A \longrightarrow A$ is an additive derivative operator on the MV-algebra $\mathcal{A}$ and that d extends g. Since we have $b \leq e$ we must have $d(b) \leq d(e) = a$. Since we have $b \oplus b = e$, we must have $a = d(e) = d(b \oplus b) = d(b) \oplus d(b)$. Thus, $d(b) \neq 0$, such that $d(b) = a$ and $dd(b) = d(a) = e$. We must have $dd(b) \leq b \oplus d(b) = b \oplus a = c$. This means that we must have $e \leq c$, which is not the case. □

Remark 3.2 *Let $A = \{0, a, b, c, e, 1\}$, where $0 < a, b < c < 1$ and $0 < b < e < 1$. Define $\oplus$ and $\neg$ as in the above proof. We define the operator d as follows:*

x	0	a	b	c	e	1
$d(x)$	0	a	0	a	o	a

Then the structure $\mathcal{A} = (\{0, a, b, c, e, 1\}, \oplus, \neg, 0)$ is a derivative complete MV-algebra.

When the MV-algebra $\mathcal{A}$ is complete, so is its Boolean subalgebra $B(\mathcal{A})$. Moreover, given $x \in A$, set

$$u(x) = \bigwedge\{a : x \leq a \in B(\mathcal{A})\}.$$

This is the element in $B(\mathcal{A})$ *closest to* x *from above.*
Of course, $x \in B(\mathcal{A}) \iff u(x) = x$.

Remark 3.3 *Observe the following. For each x and y in A, we have*

(u1) $u(x) \oplus u(y) = u(x \oplus y)$.

Indeed, since $x \leq u(x)$ and $y \leq u(y)$, we get $x \oplus y \leq u(x) \oplus u(y)$. The element $u(x) \oplus u(y)$ belongs to $B(\mathcal{A})$. Thus, on one hand, we have $u(x \oplus y) \leq u(x) \oplus u(y)$ by the definition of u. We also have $x \leq x \oplus y$ and $y \leq x \oplus y$, so $u(x) \leq u(x \oplus y)$ and $u(y) \leq u(x \oplus y)$. We thus also have $u(x) \oplus u(y) \leq u(x \oplus y)$.

Theorem 3.9 *Given a complete MV-algebra $\mathcal{A} = (A, \oplus, \neg, 0)$ and a strong derivative operator $g : B(\mathcal{A}) \longrightarrow B(\mathcal{A})$ on the Boolean algebra $B(\mathcal{A})$, set $d(x) = g(u(x))$ for each $x \in A$. The mapping $d : A \longrightarrow A$ thus defined is a strong additive derivative operator on A, which extends g to A, i.e. $d(x) = g(x)$ for each $x \in B(\mathcal{A})$.*

Proof. Condition (d3) holds because $d(0) = g(u(0)) = g(0) = 0$.

Using identity (u1), we get

$d(x \oplus y) = g(u(x \oplus y)) = g(u(x) \oplus u(y)) = g(u(x)) \oplus g(u(y)) = d(x) \oplus d(y)$. Condition (d1) holds.

Finally, notice that $g(u(x))$ belong to $B(\mathcal{A})$, so that, in fact, the mapping d sends A into $B(\mathcal{A}) \subset A$, i.e. we have $d(x) \in B(\mathcal{A})$. Thus, $d(d(x)) = g(u(g(u(x)))) = g(g(u(x))) \leq g(u(x)) = d(x)$. Condition (d4) holds for d.
□

Now the above result can be formulated as follows. When $\mathcal{A}$ is a complete MV-algebra, each $g \in \mathcal{S}(\mathcal{B}(\mathcal{A}))$ has an extension $d \in \mathcal{S}(\mathcal{A})$ with $d(x) = g(u(x))$. Notice that, if $\delta \in \mathcal{S}(\mathcal{A})$ is any extension of g, we have $\delta(x) \leq d(x)$, because $\delta(x) \leq \delta(u(x)) = g(u(x)) = d(x)$.

Let us recall that an ideal of an MV-algebra $\mathcal{A}$ (see [3,8]) is a subset I of A satisfying the following conditions:

(I1) $0 \in I$;

(I2) if $x \in I$, $y \in A$ and $y \leq x$ then $y \in I$;

(I3) if $x \in I$ and $y \in I$ then $x \oplus y \in I$.

Definition 3.8 Let $\mathcal{A}$ be a derivative MV-algebra and I be an *ideal* of $\mathcal{A}$. Then I is called a *d-ideal* if $d(x) \in I$ for every $x \in I$.

Definition 3.9 Let $\mathcal{A}$ be a derivative MV-algebra and I be an ideal of $\mathcal{A}$. Then I is called a $d^{\neg}$*-ideal* if $d^{\neg}(x) \in I$ for every $x \in I$.

Lemma 3.1 *Let $\mathcal{A} = (A, \oplus, \neg, 0, d)$ be a derivative MV-algebra. Then the subset $I = \{x \in A : d(x) = 0\}$ of A is a d-ideal.*

Proof. **(I1)** $0 \in I$ is clear by (d1).

(I2) Let $x \in I$, $y \in A$ and $y \leq x$. Then we have $d(y) \leq d(x) = 0$, so $d(y) = 0$ and $y \in I$.

(I3) Let $x, y \in I$. Then we have $d(y) = 0$ and $d(x) = 0$. Since $d(x \oplus y) = d(x) \oplus d(y)$ by $(d1)$, we must have $d(x \oplus y) = 0$, so $x \oplus y \in I$.

Finally, let $x \in I$. Then we get $d(x) = 0$. Hence, $dd(x) = d(d(x)) = d(0) = 0$. It means that $d(x) \in I$. □

Lemma 3.2 *Let $\mathcal{A} = (A, \oplus, \neg, 0, d)$ be a derivative MV-algebra and $\mathcal{A}$ be dense-in-itself. If I is a d-ideal of $\mathcal{A}$, then I is a $d^{\neg}$-ideal of $\mathcal{A}$.*

Proof. Suppose that I is a d-ideal of $\mathcal{A}$. Let $x \in I$. Then $d(x) \in I$. Since $\mathcal{A}$ is dense-in-itself, $d^{\neg}(x) \leq d(x)$ by Theorem 3.4. Thus, we must have $d^{\neg}(x) \in I$ by (I2). □

Lemma 3.3 ([3]) *Let I be an ideal of an MV-algebra $\mathcal{A}$. Then the binary relation $\equiv_I$ on $\mathcal{A}$ defined by $x \equiv_I y$ iff $(x \odot \neg y) \oplus (\neg x \odot y) \in I$ is a congruence relation.*

Conversely, if $\equiv$ is a congruence on $\mathcal{A}$, then $\{x \in A : x \equiv 0\}$ is an ideal, and $x \equiv y$ iff $(x \odot \neg y) \oplus (\neg x \odot y) \equiv 0$. Therefore, the correspondence $I \longmapsto \equiv_I$ is a bijection from the set of ideals of $\mathcal{A}$ onto the set of congruences on $\mathcal{A}$.

Given $x \in A$, the equivalence class of x with respect to $\equiv$ will be denoted by $\overline{x}$ and the quotient set $A/\equiv_I$ by A/I. Since $\equiv_I$ is a congruence, define on the set A/I the operators

$$\overline{x} \oplus \overline{y} =_{def} \overline{x \oplus y} \tag{1}$$

and

$$\neg\overline{x} =_{def} \overline{\neg x}. \tag{2}$$

The system $(A/I, \oplus, \neg, \overline{0})$ becomes an MV-algebra, called the quotient algebra of $\mathcal{A}$ by the ideal (for details, see [3]).

If $\mathcal{A}$ is a derivative MV-algebra and I is an d-ideal of $\mathcal{A}$, set

$$d(\overline{x}) =_{def} \overline{d(x)}. \tag{3}$$

Theorem 3.10 *If $\mathcal{A}$ is a derivative MV-algebra and I is a d-ideal of $\mathcal{A}$, then the quotient MV-algebra $\mathcal{A}/I$ endowed with d is a derivative MV-algebra.*

Proof. We know by Lemma 3.3 that $\equiv_I$ satisfies the following conditions:

$\equiv_I$ is an equivalence relation.

If $x \equiv_I y$, then $\neg x \equiv_I \neg y$.

If $x \equiv_I z$ and $y \equiv_I t$, then $x \oplus y \equiv_I z \oplus t$.

Let $x, y \in A$ and $x \equiv_I y$. Then $(x \odot \neg y) \oplus (\neg x \odot y) \in I$. Clearly, we have $x \odot \neg y \in I$ and $\neg x \odot y \in I$, and therefore $d(x \odot \neg y) \in I$, $d(\neg x \odot y) \in I$. At the same time,

$$\begin{aligned} d(y) \oplus d(x \odot \neg y) &= d(y \oplus (x \odot \neg y)) \text{ by (d1)}, \\ &= d(x \vee y) \text{ by definition of } \vee, \\ &\geq d(x) \text{ by Proposition 3.1}. \end{aligned}$$

Hence, we get $d(x \odot \neg y) \geq d(x) \odot \neg d(y)$ by Lemma 2.2 (iii) and $d(x) \odot \neg d(y) \in I$.
Similarly, $\neg d(x) \odot d(y) \in I$ is clear. We must have $(d(x) \odot \neg d(y)) \oplus (\neg d(x) \odot d(y)) \in I$ by (I3). That means $d(x) \equiv_I d(y)$. Therefore, the relation $\equiv_I$ on A is a congruence relation.

Finally, we only show that $d : A/I \longrightarrow A/I$ satisfies conditions (d1)–(d3):

(d1) $$\begin{aligned} d(\overline{x} \oplus \overline{y}) = d(\overline{x \oplus y}) &= \overline{d(x \oplus y)} \text{ by (3)}, \\ &= \overline{d(x) \oplus d(y)} \text{ by (d1)}, \\ &= \overline{d(x)} \oplus \overline{d(y)} \text{ by (1)}, \\ &= d(\overline{x}) \oplus d(\overline{y}) \text{ by (3)}. \end{aligned}$$

(d2) $$\begin{aligned} dd(\overline{x}) = d(d(\overline{x})) = d(\overline{d(x)}) &= \overline{dd(x)} \text{ by (3)}, \\ &\leq \overline{x \oplus d(x)} \text{ by (d2)}, \\ &\leq \overline{x} \oplus \overline{d(x)} \text{ by (1)}, \\ &= \overline{x} \oplus d(\overline{x}) \text{ by (3)}. \end{aligned}$$

(d3) $$\begin{aligned} d(\overline{0}) &= \overline{d(0)} \text{ by (3)}, \\ &= \overline{0} \text{ by (d3)}, \end{aligned}$$

(d4) if d has the stronger property, then

$$\begin{aligned} dd(\overline{x}) &= \overline{dd(x)} \text{ by (3)}, \\ &\leq \overline{d(x)} \text{ by (d4)}, \\ &= d(\overline{x}) \text{ by (3)}. \end{aligned}$$

□

Corollary 3.1 *There is one to one correspondence between the d-ideals and congruences of derivative MV-algebras.*

Definition 3.10 Let $\mathcal{A}$ and $\mathcal{B}$ derivative MV-algebras and $h : \mathcal{A} \longrightarrow \mathcal{B}$ be a homomorphism of MV-algebras. Then h is called a *derivative homomorphism (or d-homomorphism)* if

$$h(d(x)) = d(h(x))$$

for each $x \in A$.
Note that a homomorphism h of MV-algebras is a d-homomorphism iff

$$h(d^{\neg}(x)) = d^{\neg}(h(x))$$

for each $x \in A$

The *kernel* of a homomorphism $h : \mathcal{A} \longrightarrow \mathcal{B}$ is the set

$$Ker(h) =_{def} h^{-1}(0) = \{x \in A | h(x) = 0\}.$$

Proposition 3.5 *Let $\mathcal{A}$, $\mathcal{B}$ derivative MV-algebras and $h : \mathcal{A} \longrightarrow \mathcal{B}$ be a d-homomorphism. For each d-ideal J of $\mathcal{B}$, the set $h^{-1}(J) = \{x \in A | h(x) \in J\}$ is a d-ideal of $\mathcal{A}$. Thus, in particular, $Ker(h)$ is a d-ideal.*

Proof. We know that $h^{-1}(J)$ is an ideal of $\mathcal{A}$ for any ideal $J \in \mathcal{B}$. Then let $x \in h^{-1}(J)$. $x \in h^{-1}(J)$ implies $h(x) \in J$. Since J is a d-ideal, $d(h(x)) \in J$. We must have $d(h(x)) = h(d(x)) \in J$. By the definition, $d(x) \in h^{-1}(J)$.

Finally, let $x \in Ker(h)$. Then $h(x) = 0$. Thus, we get $h(d(x)) = d(h(x)) = d(0) = 0$. Hence, $d(x) \in Ker(h)$. □

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