Research Article

Covariant differential calculus on $\mathcal{SP}_h^{2|1}$

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Abstract: The h-deformed symplectic superspaces via a contraction of the q-deformed symplectic superspaces are introduced and a covariant differential calculus on the quantum symplectic superspace $\mathcal{SP}_h^{2|1}$ is presented.

Key words: Quantum symplectic superspace, quantum supergroup, super-Hopf algebra, quantum differential calculus

1. Introduction

The theory of compact matrix quantum groups introduced by Woronowicz [15] has very rich mathematical and mathematical physics structures. These groups were considered as a noncommutative extension of the Lie groups and thus provided a very powerful tool for investigating noncommutative geometry [6]. A space in which the quantum group acts by linear transformations and its coordinates belong to a noncommutative associative algebra [12] is called a quantum space. The pioneer work of Woronowicz [16] presented concrete examples of noncommutative differential geometry, adding a consistent differential calculus on the noncommutative spaces of quantum groups. Wess and Zumino [14] developed a covariant differential calculus on quantum spaces by interpreting the dual space of quantum space as their differentials. The natural extension of their scheme to superspace [13] was introduced in [11].

The quantum groups admit two distinct deformations. One of them is the well-known q-deformation and the other is the so-called Jordanian (h-)deformation [8]. The quantum group $\mathrm{GL}_q(n)$ has been obtained by deforming the coordinates of a plane to be noncommutative objects [13]. In [1], the authors showed that the h-deformed group can be obtained from the q-deformed Lie group through a singular limit $q \to 1$ of a linear transformation. This method is known as the contraction procedure [9]. In this paper, we investigate the noncommutative geometry of the quantum symplectic $(2 + 1)$-superspace, denoted by $\mathcal{SP}_h^{2|1}$.

Throughout the paper, we will fix a base field $\mathbb{K}$ as the set of real numbers $\mathbb{R}$ or the set of complex numbers $\mathbb{C}$. We will denote by Greek letters the odd generators and by Latin letters the even generators.

2. Preliminaries: super structures

In this section, some basic information is presented from classical theory in order to ensure that the present article presents integrity within it and the readers easily adapt to the subject.

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2.1. Superalgebras

The theory of superalgebras starts with definition of the concept of supervector space, which is a vector space that decomposes as a direct sum. In some cases, it would be more appropriate to refer to supervector spaces (or superalgebras) as $\mathbb{Z}_2$-graded vector spaces (or $\mathbb{Z}_2$-graded superalgebras).

Definition 2.1 *A* $\mathbb{Z}_2$*-graded (or* supervector*) space* V *is a vector space over a field* $\mathbb{K}$ *together with two subspaces* V_0 *and* V_1 *of* V *such that* $V = V_0 \oplus V_1$.

The subspace V_0 is called the *even* part of V, and its elements even. The subspace V_1 is called the *odd* part of V, and its elements odd. The degree (or grade) of an even or odd element $v \in V$ is denoted by $\tau(v)$ and is equal to 0 and 1, respectively. The even and odd elements of V are collectively said to be homogeneous.

Definition 2.2 *Let* $f : V \longrightarrow W$ *be a linear map of supervector spaces. Then* f *is called a* supervector space homomorphism *if it satisfies*

$$\tau(f(v)) = \tau(f) + \tau(v) \pmod 2$$

for all $v \in V$.

Definition 2.3 *An algebra* $\mathcal{A}$ *over* $\mathbb{K}$ *is called a* superalgebra *(or* $\mathbb{Z}_2$*-graded algebra) if it is a supervector space over* $\mathbb{K}$*, with a bilinear map* $\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ *such that* $\mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j}$ *for* $i, j = 0, 1$.

The superalgebra $\mathcal{A}$ is called *supercommutative* if

$$ab = (-1)^{\tau(a)\tau(b)}ba$$

for homogeneous elements a and b of $\mathcal{A}$. Even elements commute with all elements, but odd elements anticommute with one another and the square of an odd element is zero.

Definition 2.4 *Let* $\mathcal{A}$ *be a supercommutative algebra and the map* $f : \mathcal{A} \longrightarrow \mathcal{A}$ *be a supervector space homomorphism. If it satisfies the super-Leibniz rule*

$$f(ab) = f(a)b + (-1)^{\tau(a)\tau(f)}af(b)$$

for all $a, b \in \mathcal{A}$*, then* f *is called a* superderivation.

Definition 2.5 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be superalgebras and* $f : \mathcal{A} \longrightarrow \mathcal{B}$ *be a map of definite degree. If it is a supervector space homomorphism and*

$$f(ab) = (-1)^{\tau(a)\tau(f)}f(a)f(b)$$

for all $a, b \in \mathcal{A}$*, then* f *is called a* superalgebra homomorphism.

Note that a superderivation is a vector space homomorphism, but not a superalgebra homomorphism.

2.2. Modules of superalgebras

Since a general algebra not need have invertible elements, modules do not always have bases. In the super case, there is an extra requirement of compatible degree.

Definition 2.6 *Let $\mathcal{A}$ be a supercommutative algebra and M be a supervector space. If there exists a mapping*

$$\mathcal{A} \times M \longrightarrow M, \quad (a,m) \mapsto am$$

such that

$$\tau(am) = \tau(a) + \tau(m) \quad \text{and} \quad a(bm) = (ab)m$$

for all $a,b \in \mathcal{A}$ and all $m \in M$, then M is called a left super $\mathcal{A}$-module.

A left super $\mathcal{A}$-module M over a supercommutative algebra $\mathcal{A}$ is called free of rank $k|n$ if there exists a homogeneous basis $\{e_1, \ldots, e_k, e_{k+1}, \ldots, e_{k+n}\}$ for M, where $e_1, \ldots, e_k$ are elements of M_0 and $e_{k+1}, \ldots, e_{k+n}$ are elements of M_1. Every element $m \in M$ can uniquely be expressed as

$$m = \sum_i^{k+n} a^i e_i, \quad a^i \in \mathcal{A}.$$

For a supercommutative algebra $\mathcal{A}$, there exist differences between left and right $\mathcal{A}$-modules. A left module can also be given the structure of a right module by defining the map

$$V \times \mathcal{A} \longrightarrow V, \quad (v,a) \mapsto (-1)^{\tau(v)\tau(a)} av.$$

The set of superderivations of $\mathcal{A}$ is an important example of a super $\mathcal{A}$-module.

2.3. Supermatrices

If $\mathcal{A}$ is a superalgebra, supermatrices with entries in $\mathcal{A}$ define even homomorphisms of free super $\mathcal{A}$-modules in terms of particular bases.

Definition 2.7 *An $(m|n) \times (r|s)$ supermatrix over a superalgebra $\mathcal{A}$ is an $(m+n) \times (r+s)$ matrix T whose entries are elements of $\mathcal{A}$ and which has the block form*

$$T = \begin{pmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{pmatrix},$$

where T_{00} is an $m \times r$-matrix, T_{01} an $m \times s$-matrix, T_{10} an $n \times r$-matrix, and T_{11} an $n \times s$-matrix.

The addition and multiplication of supermatrices are defined in a manner similar to that for conventional matrices, with matching of superorder being required. The resulting matrices are supermatrices.

The set $\mathcal{M}(m|n, \mathcal{A})$ of supermatrices over $\mathcal{A}$ is closed under multiplication, and the subset $\mathrm{GL}(m|n, \mathcal{A})$ of invertible matrices is a group with identity element of the matrix with the unit element of $\mathcal{A}$ along the leading diagonal and zeros elsewhere. When $\mathcal{A}$ is of suitable form, $\mathrm{GL}(m|n, \mathcal{A})$ becomes a Lie supergroup.

2.4. Super-Hopf algebras

The $\mathbb{Z}_2$-graded tensor product of two superalgebras $\mathcal{A}$ and $\mathcal{B}$ may be regarded as a superalgebra $\mathcal{A} \otimes \mathcal{B}$ with a product rule determined by the following:

Definition 2.8 *If $\mathcal{A}$ and $\mathcal{B}$ are two superalgebras, then the product rule in the superalgebra $\mathcal{A} \otimes \mathcal{B}$ is defined by*

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\tau(b_1)\tau(a_2)} a_1 a_2 \otimes b_1 b_2,$$

where a_is and b_is are homogeneous elements in the superalgebras $\mathcal{A}$ and $\mathcal{B}$, respectively.

Definition 2.9 *A super-Hopf algebra is a supervector space $\mathcal{A}$ over $\mathbb{K}$ with two superalgebra homomorphisms Δ (the coproduct), ϵ (the counit), and a superalgebra antihomomorphism S (the coinverse) such that*

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta,$$

$$m \circ (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = m \circ (\mathrm{id} \otimes \epsilon) \circ \Delta,$$

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta,$$

together with $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$, $\epsilon(\mathbf{1}) = 1$, $S(\mathbf{1}) = \mathbf{1}$, and for any $a, b \in \mathcal{A}$, where $m : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ is the product map, $\mathrm{id} : \mathcal{A} \longrightarrow \mathcal{A}$ is the identity map and $\eta : \mathbb{K} \longrightarrow \mathcal{A}$.

3. Quantum symplectic superspaces $\mathcal{SP}_h^{2|1}$ and $\mathcal{SP}_h^{1|2}$

In this section, we introduce an h-deformation of the superspaces $\mathcal{SP}^{2|1}$ and $\mathcal{SP}^{1|2}$ from the corresponding q-deformed parts via a contraction. Here we will denote q-deformed objects by primed quantities and denote transformed coordinates by unprimed quantities. In the first subsection, we will give brief information about (2+1)- and (1+2)-superspaces.

3.1. The superspaces $\mathcal{SP}^{2|1}$ and $\mathcal{SP}^{1|2}$

The elements of the symplectic (2+1)-superspace are supervectors generated by two even components and one odd component. We define the symplectic superspace $\mathcal{SP}^{2|1}$ of column vectors with a decomposition $\mathcal{SP}^{2|1} = U_0 \oplus U_1$. A vector is an element of U_0 (resp. U_1) and is of degree 0 (resp. 1) if it has the form

$$\begin{pmatrix} x \\ 0 \\ y \end{pmatrix}, \quad (\text{resp.} \quad \begin{pmatrix} 0 \\ \theta \\ 0 \end{pmatrix}).$$

The elements of the symplectic (1+2)-superspace are supervectors generated by one even and two odd components. We define the symplectic superspace $SP^{1|2}$ of column vectors with a decomposition $\mathcal{SP}^{1|2} = V_0 \oplus V_1$ where an element of V_0 (resp. V_1) has the form

$$\begin{pmatrix} 0 \\ z \\ 0 \end{pmatrix}, \quad (\text{resp.} \quad \begin{pmatrix} \xi \\ 0 \\ \eta \end{pmatrix}).$$

3.2. The algebra of functions on the quantum superspace $\mathcal{SP}_q^{2|1}$

The superalgebra $\mathcal{O}(\mathcal{SP}_q^{2|1})$ is the q-deformed algebra of functions on the quantum superspace $\mathcal{SP}_q^{2|1}$ generated by x', θ', and y' with the quadratic relations [3, 4]

$$x'\theta' = q\theta'x', \quad \theta'y' = qy'\theta', \quad x'y' = q^2y'x', \quad \theta'^2 = q^{1/2}(q-1)y'x', \tag{3.1}$$

where $q \in \mathbb{C} - \{0\}$. Obviously, in the limit $q \to 1$ this algebra becomes the superalgebra of functions on the superspace $\mathcal{SP}^{2|1}$.

Example 3.1 *There exist some representations that satisfy (3.1).*

Proof   For instance, the representation $\rho : \mathcal{O}(\mathcal{SP}_q^{2|1}) \to \mathrm{M}(3, \mathbb{C})$, acting on the generators x', θ', and y', defined by matrices*

$$\rho(x') = \begin{pmatrix} q & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(\theta') = \begin{pmatrix} 0 & q-1 & 0 \\ 0 & 0 & 0 \\ q^{1/2} & 0 & 0 \end{pmatrix}, \quad \rho(y') = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{3.2}$$

satisfies the relations (3.1). □

Remark 3.2 *The last two relations in (3.1) can also be written as a single relation. In this case, we say that $\mathcal{O}(\mathcal{SP}_q^{2|1})$ is the superalgebra with generators $x'_\pm$ and θ' satisfying the relations [5]*

$$x'_\pm\theta' = q^{\pm 1}\theta'x'_\pm, \quad x'_+x'_- = x'_-x'_+ + q^{-1/2}(q+1)\theta'^2. \tag{3.3}$$

Of course, the representation ρ in (3.2) preserves the relations (3.3) where $\rho(x'_+) = \rho(x')$ and $\rho(x'_-) = \rho(y')$.

Using the relations (3.3), it is easily shown that the element [5]

$$r_q = q^{1/2}x'_-x'_+ + \theta'^2 - q^{-1/2}x'_+x'_- \tag{3.4}$$

is a central element of the algebra $\mathcal{O}(\mathcal{SP}_q^{2|1})$.

3.3. The algebra of functions on the quantum superspace $\mathcal{SP}_q^{1|2}$

The quantum symplectic superalgebra $\mathcal{O}(\mathcal{SP}_q^{1|2})$ is defined as follows: The superalgebra $\mathcal{O}(\mathcal{SP}_q^{1|2})$ is the q-deformed algebra of functions on the quantum superspace $\mathcal{SP}_q^{1|2}$ generated by ξ', z', and η' with the quadratic relations [3]

$$\xi'z' = q^{-1}z'\xi', \quad z'\eta' = q^{-1}\eta'z', \quad \eta'\xi' = -q^2\xi'\eta' + q^{1/2}(1-q)z'^2, \quad \xi'^2 = 0 = \eta'^2, \tag{3.5}$$

where $q \in \mathbb{C} - \{0\}$.

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*arXiv.org e-Print archive (2018). Hopf algebra structure on symplectic superspace $SP_q^{2|1}$ [online]. Website http://arxiv.org/abs/1811.06381 [14 November 2018].

Example 3.3 *There exist some representations that satisfy (3.5).*

Proof For instance, the representation $\rho : \mathcal{O}(\mathcal{SP}_q^{1|2}) \to \mathrm{M}(3, \mathbb{C})$, acting on the generators ξ', z', and η', defined by matrices [4]

$$\rho(\xi') = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q^{1/2} & 0 & 0 \end{pmatrix}, \quad \rho(z') = \begin{pmatrix} q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(\eta') = \begin{pmatrix} 0 & 0 & 1-q \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.6}$$

satisfies the relations (3.5). $\quad\square$

3.4. The superalgebras $\mathcal{O}(\mathcal{SP}_h^{2|1})$ and $\mathcal{O}(\mathcal{SP}_h^{1|2})$ via a contraction

We introduce new coordinates x, θ, and y with the change of basis in the coordinates of the superspace $\mathcal{SP}_q^{2|1}$ as follows:

$$X' = \begin{pmatrix} x' \\ \theta' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{h}{2(q-1)} & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ \theta \\ y \end{pmatrix} = g\,X. \tag{3.7}$$

Here h is a new deformation parameter and it will replace q in the limit $q \to 1$. The factor $\frac{1}{2}$ appearing in the matrix g has been taken in order to prevent the factor 2 from appearing in many places in the future. There is no other feature.

If we now use the relations (3.1), then we have the relations

$$x\theta = q\theta x, \quad xy = q^2 yx + (1+q)\,\tfrac{h}{2}\,x^2, \quad \theta y = qy\theta + (1+q)\,\tfrac{h}{2}\,\theta x, \quad \theta^2 = q^{1/2}(q-1)yx + q^{1/2}\tfrac{h}{2}\,x^2. \tag{3.8}$$

Example 3.4 *There exist some representations that satisfy (3.8).*

Proof The representation ρ, acting on the generators x, θ, and y, defined by matrices

$$\rho(x) = \begin{pmatrix} q & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(\theta) = \begin{pmatrix} 0 & q-1 & 0 \\ 0 & 0 & 0 \\ q^{1/2} & 0 & 0 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} -qh' & 0 & 0 \\ 0 & -q^2 h' & 0 \\ 0 & 1 & -h' \end{pmatrix}, \tag{3.9}$$

satisfies the relations (3.8), where $h' = \frac{h}{2(q-1)}$. $\quad\square$

In the limit $q \to 1$ of the relations (3.8) we get the following quadratic relations. The resulting relations define superalgebra $\mathcal{O}(\mathcal{SP}_h^{2|1})$ of functions on the symplectic h-superspace $\mathcal{SP}_h^{2|1}$:

Definition 3.5 *Let $\mathbb{K}\{x, \theta, y\}$ be a free associative algebra generated by x, θ, y and I_h is a two-sided ideal generated by $x\theta - \theta x$, $xy - yx - hx^2$, $\theta y - y\theta - hx\theta$, $\theta^2 - \frac{h}{2}x^2$. The quantum superspace $\mathcal{SP}_h^{2|1}$ with the function algebra*

$$\mathcal{O}(\mathcal{SP}_h^{2|1}) = \mathbb{K}\{x, \theta, y\}/I_h$$

is called a quantum symplectic superspace.

According to Definition 3.5, we have

$$x\theta = \theta x, \quad xy = yx + hx^2, \quad \theta y = y\theta + hx\theta, \quad \theta^2 = \tfrac{h}{2}\,x^2. \tag{3.10}$$

If we use the relations (3.3), we obtain the relations

$$x_+\theta = \theta x_+, \quad \theta x_- = x_-\theta + hx_+\theta, \quad x_+x_- = x_-x_+ + 2\theta^2. \tag{3.11}$$

We see that the central element of the algebra $\mathcal{O}(\mathcal{SP}_h^{2|1})$ is

$$r_h = \tfrac{h}{2}\,x_+^2 - \theta^2, \tag{3.12}$$

which can be obtained from (3.4).

Definition 3.6 *Let* $\Lambda(\mathcal{SP}_h^{2|1}) := \mathcal{O}(\mathcal{SP}_h^{1|2})$ *be the algebra with the generators* ξ, z, *and* η *satisfying the relations*

$$\xi z = z\xi, \quad z\eta = \eta z - h\xi z, \quad \xi\eta = -\eta\xi, \quad \xi^2 = 0, \quad \eta^2 = h\left(\tfrac{1}{2}\,z^2 + \xi\eta\right), \tag{3.13}$$

where $\tau(\xi) = 1 = \tau(\eta)$ *and* $\tau(z) = 0$. *We call* $\Lambda(\mathcal{SP}_h^{2|1})$ *the quantum exterior algebra of the quantum superspace* $\mathcal{SP}_h^{2|1}$.

Remark 3.7 *Obviously, in the limit* $h \to 0$ *the algebra* $\mathcal{O}(\mathcal{SP}_h^{2|1})$ *is the* $\mathbb{Z}_2$*-graded polynomial algebra in three supercommuting indeterminates and the algebra* $\Lambda(\mathcal{SP}^{2|1})$ *is the exterior algebra of* $\mathcal{SP}^{2|1}$.

4. The quantum symplectic supergroup $\mathrm{SP}_h(2|1)$

In this section, we will consider the $(2+1) \times (2+1)$-supermatrices acting on the quantum symplectic superspaces $SP_h^{2|1}$ and $SP_h^{1|2}$.

4.1. The super-Hopf algebra $\mathcal{O}(\mathrm{SP}_h(2|1))$

Let $a, b, c, d, e, \gamma, \alpha, \delta, \beta$ be elements of a supercommutative algebra $\mathcal{A}$ where Latin letters are of degree 0 and Greek letters are of degree 1. Let $\mathcal{O}(\mathrm{SP}(2|1)))$ be defined as the polynomial algebra $\mathbb{K}[a, b, c, d, e, \alpha, \beta, \gamma, \delta]$. It is convenient and more illustrative to write a point $(a, b, c, d, e, \alpha, \beta, \gamma, \delta)$ of $\mathcal{O}(\mathrm{SP}(2|1))$ in the matrix form

$$T = \begin{pmatrix} a & \alpha & b \\ \gamma & e & \beta \\ c & \delta & d \end{pmatrix} = (t_{ij}). \tag{4.1}$$

Using the orthosymplectic conditions

$$T^{st}C_h T = DC_h, \quad TC_h^{-1}T^{st} = DC_h^{-1} \tag{4.2}$$

where T^{st} denotes the super transposition of T and

$$C_h = \lim_{q \to 1}\left[g^{st}C_q g\right] = \begin{pmatrix} \tfrac{h}{2} & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C_q = \begin{pmatrix} 0 & 0 & -q^{-1/2} \\ 0 & 1 & 0 \\ q^{1/2} & 0 & 0 \end{pmatrix},$$

we get the quantum superdeterminant as

$$D_h = ad - bc - \alpha\delta + \frac{h}{2}\,bd = da - cb + \delta\alpha - \frac{h}{2}\,db.$$

Remark 4.1 *In the classical case, it is well known that the matrix T given in (4.1) defines the linear transformation $T : \mathcal{SP}^{2|1} \longrightarrow \mathcal{SP}^{2|1}$. In the quantum case, the matrix T also defines the linear transformation $T : \mathcal{SP}_h^{2|1} \longrightarrow \mathcal{SP}_h^{2|1}$. As a result of this we have $TX = \tilde{X} \in \mathcal{SP}_h^{2|1}$, where $X = (x,\theta,y)^t$. Therefore, we will use such transforms while obtaining h-deformed commutation relations between the matrix elements of T.*

We now assume that the generators of $\mathcal{O}(\mathrm{SP}(2|1))$ supercommute with the generators of the algebras $\mathcal{O}(\mathcal{SP}_h^{2|1})$ and $\mathcal{O}(\mathcal{SP}_h^{1|2})$ and define the couples $(\tilde{x},\tilde{\theta},\tilde{y})$ and $(\tilde{\xi},\tilde{z},\tilde{\eta})$ by the following matrix equalities:

$$\begin{pmatrix} \tilde{x} \\ \tilde{\theta} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} a & \alpha & b \\ \gamma & e & \beta \\ c & \delta & d \end{pmatrix} \begin{pmatrix} x \\ \theta \\ y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{\xi} \\ \tilde{z} \\ \tilde{\eta} \end{pmatrix} = \begin{pmatrix} a & \alpha & b \\ \gamma & e & \beta \\ c & \delta & d \end{pmatrix} \begin{pmatrix} \xi \\ z \\ \eta \end{pmatrix}. \tag{4.3}$$

Then we can give the following theorem, which we can prove by making straightforward calculations. Some relations in the theorem below are arranged using the orthosymplectic conditions.

Theorem 4.2 *If the couples $(\tilde{x},\tilde{\theta},\tilde{y})$ and $(\tilde{\xi},\tilde{z},\tilde{\eta})$ in (4.3) satisfy the relations (3.10) and (3.13), respectively, then the generators of $\mathcal{O}(\mathrm{OSP}_h(2|1))$ satisfy the following relations:*

$$ab = ba - hb^2, \quad ac = ca + h(a^2 - \mathbf{1}), \quad ad = da + h(ba - db - hb^2),$$

$$bc = cb + h(ab + db + hb^2), \quad bd = db + hb^2, \quad cd = dc + h(\mathbf{1} - d^2), \quad a\alpha = \alpha a - h\alpha b,$$

$$a\delta = \delta a + h\alpha(a - d), \quad b\alpha = \alpha b, \quad b\delta = \delta b + h\alpha b, \quad c\alpha = \alpha c - h(\alpha a + \delta b),$$

$$c\delta = \delta c - h\delta d, \quad d\alpha = \alpha d - h\alpha b, \quad d\delta = \delta d,$$

$$\alpha^2 = -\frac{h}{2}\,b^2, \quad \delta^2 = \frac{h}{2}\,(\mathbf{1} - d^2), \tag{4.4}$$

$$ae = ea + h\alpha\beta, \quad be = eb, \quad ce = ec + h(\alpha\gamma + \delta\beta), \quad de = ed + h\alpha\beta,$$

$$a\beta = \beta a - h\beta b, \quad a\gamma = \gamma a, \quad b\beta = \beta b, \quad b\gamma = \gamma b + h\beta b, \quad c\beta = \beta c - h(\beta d + \gamma b),$$

$$c\gamma = \gamma c - h\gamma a, \quad d\beta = \beta d - h\beta b, \quad d\gamma = \gamma d + h\beta(d - a), \quad e\alpha = \alpha e - h\beta b,$$

$$e\beta = \beta e - h\alpha b, \quad e\gamma = \gamma e - h(\alpha a + \beta e - h\alpha b), \quad e\delta = \delta e - h(\alpha e + \beta d - h\beta b),$$

$$\alpha\beta = -\beta\alpha, \quad \alpha\gamma = -\gamma\alpha - hbe, \quad \alpha\delta = -\delta\alpha + h(\alpha^2 - db), \quad \beta\gamma = -\gamma\beta + h(\beta^2 + ab),$$

$$\beta\delta = -\delta\beta + hbe, \quad \gamma\delta = -\delta\gamma + h(ae - de), \quad \beta^2 = \frac{h}{2}\,b^2, \quad \gamma^2 = \frac{h}{2}\,(a^2 - \mathbf{1}).$$

Remark 4.3 *The element D_h is a central element of the algebra $\mathcal{O}(\mathrm{SP}_h(2|1))$; that is, it commutes with all generators of $\mathcal{O}(\mathrm{SP}_h(2|1))$. For this reason, we use $D_h = \mathbf{1}$ when writing some relations in (4.4).*

Remark 4.4 *In [10], similar relations to the relations (4.4) have been obtained via a matrix R_h using the standard FRT construction [8] (RTT-relations) and the h-orthosymplectic condition.*

It can be shown that the super algebra $\mathcal{O}(\mathrm{OSP}_h(2|1))$ is a super-Hopf algebra. For this, the requests of Definition 2.9 must be satisfied. The operations are quite long, but can be obtained by direct calculations.

Theorem 4.5 *There exists a unique super-Hopf algebra structure on the superalgebra* $\mathcal{O}(\mathrm{OSP}_h(2|1))$ *with comaps* Δ*,* ϵ*, and* S *such that*

$$\Delta(t_{ij}) = \sum_{k=1}^{3} t_{ik} \otimes t_{kj}, \quad \epsilon(t_{ij}) = \delta_{ij}, \quad S(T) = T^{-1},$$

where

$$T^{-1} = \begin{pmatrix} d + \frac{h}{2}\,b & \beta & -b \\ -(\delta + \frac{h}{2}\,\alpha) & e & \alpha \\ -c + \frac{h}{2}\,(d - a + \frac{h}{2}\,b) & -\gamma + \frac{h}{2}\,\beta & a - \frac{h}{2}\,b \end{pmatrix}.$$

Definition 4.6 *The super-Hopf algebra* $\mathcal{O}(\mathrm{OSP}_h(2|1))$ *is called the coordinate algebra of the quantum supergroup* $\mathrm{OSP}_h(2|1)$.

4.2. Coactions on the quantum symplectic superspaces

Let H be a super-Hopf algebra and $\mathcal{X}$ a supervector space. Then a left coaction of H on $\mathcal{X}$ is a linear map $\delta_L : \mathcal{X} \longrightarrow H \otimes \mathcal{X}$ obeying the identities

$$(\mathrm{id} \otimes \delta_L) \circ \delta_L = (\Delta \otimes \mathrm{id}) \circ \delta_L \quad \text{and} \quad (\epsilon \otimes \mathrm{id}) \circ \delta_L = \mathrm{id}.$$

The supervector space $\mathcal{X}$ with a left coaction δ_L of H is called an H-comodule.

Definition 4.7 *If* $\mathcal{X}$ *is a superalgebra and the coaction* $\delta_L : \mathcal{X} \longrightarrow H \otimes \mathcal{X}$ *satisfies, for all* $x, y \in \mathcal{X}$,

$$\delta_L(xy) = \delta_L(x)\delta_L(y) \quad \text{and} \quad \delta_L(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1},$$

then $\mathcal{X}$ *is called a left quantum superspace for* H *or a left* H*-comodule superalgebra.*

Theorem 4.8 *(i) The superalgebras* $\mathcal{O}(\mathcal{SP}_h^{2|1})$ *and* $\mathcal{O}(\mathcal{SP}_h^{1|2})$ *are both left comodule algebras of the Hopf superalgebra* $\mathcal{O}(\mathrm{OSP}_h(2|1))$ *with left coactions*

$$\delta_L : \mathcal{O}(\mathcal{SP}_h^{2|1}) \longrightarrow \mathcal{O}(\mathrm{OSP}_h(2|1)) \otimes \mathcal{O}(\mathcal{SP}_h^{2|1}), \quad \delta_L(X_i) = \sum_{k=1}^{3} t_{ik} \otimes X_k \tag{4.5}$$

and

$$\tilde{\delta}_L : \mathcal{O}(\mathcal{SP}_h^{1|2}) \longrightarrow \mathcal{O}(\mathrm{OSP}_h(2|1)) \otimes \mathcal{O}(\mathcal{SP}_h^{1|2}), \quad \tilde{\delta}_L(\hat{X}_i) = \sum_{k=1}^{3} t_{ik} \otimes \hat{X}_k. \tag{4.6}$$

(ii) For the element r_h *of the superalgebra* $\mathcal{O}(\mathcal{SP}_h^{2|1})$*, we have* $\delta_L(r_h) = \mathbf{1} \otimes r_h$.

Proof   (i): These assertions can be obtained from the relations in (3.11) with (4.3).

(ii): To see that $\delta_L(r_h) = \mathbf{1} \otimes r_h$ we use the definition of δ_L in (4.5) and the relations (3.11) with $D_h = \mathbf{1}$. $\square$

5. The quantum de Rham complex on $\mathcal{SP}_h^{2|1}$

The de Rham complex of a smooth manifold $\mathcal{X}$ is the cochain complex that in degree $n \in \mathbb{N}$ has the vector space $\Omega^n(\mathcal{X})$ of n-degree differential forms on $\mathcal{X}$. Under the wedge product, the de Rham complex becomes a differential graded algebra. In this section, we set up a quantum de Rham complex, a $\mathbb{Z}_2$-graded differential calculus, on the symplectic superspace $\mathcal{SP}_h^{2|1}$.

Definition 5.1 *A first-order $\mathbb{Z}_2$-graded differential calculus over a superalgebra $\mathcal{A}$ is a pair (Ω, d), where Ω is an $\mathcal{A}$-bimodule and $\mathsf{d} : \mathcal{A} \longrightarrow \Omega$ is a linear map obeying the $\mathbb{Z}_2$-graded Leibniz rule*

$$\mathsf{d}(a \cdot b) = (\mathsf{d}a) \cdot b + (-1)^{\tau(a)} a \cdot (\mathsf{d}b), \quad \forall a, b \in \mathcal{A}$$

such that $\Omega = \mathrm{span}\{a \cdot \mathsf{d}b \cdot c : a, b, c \in \mathcal{A}\}$. The elements of (Ω, d) are said to be 1-forms.

Definition 5.2 *Let H be a super-Hopf algebra and $\mathcal{X}$ a left H-comodule superalgebra. A first-order $\mathbb{Z}_2$-graded differential calculus over $\mathcal{X}$ is called left covariant if Ω is a left H-comodule with left coaction $\delta_L : \Omega \longrightarrow H \otimes \Omega$ such that*

$$\delta_L(x \, \mathsf{d}y) = \delta_L(x)(\tau' \otimes \mathsf{d})\delta_L(y),$$

where $\tau' : \Omega \longrightarrow \Omega$, $\tau'(u) = (-1)^{\tau(u)} u$.

It is possible to construct noncommutative analogues of the de Rham complex of a C^∞-manifold with the property of covariance, going beyond first-order calculus. To construct a quantum de Rham complex on the quantum superspace $\mathcal{SP}_h^{2|1}$ we choose the cotangent space or differential 1-forms. Since we want to multiply forms by functions from the left and right, we demand that this must be an Ω-bimodule.

A $\mathbb{Z}_2$-graded differential algebra over $\mathcal{A}$ is a $\mathbb{Z}_2$-graded algebra $\Omega = \bigoplus_{n \geq 0} \Omega^n$, with the linear map d of degree 1 such that $\mathsf{d} \circ \mathsf{d} := \mathsf{d}^2 = 0$ and the $\mathbb{Z}_2$-graded Leibniz rule holds. Here we assume that $\Omega^0 := \mathcal{A}$ and $\Omega^{<0} = 0$.

5.1. The relations of coordinates with differentials

To set up a $\mathbb{Z}_2$-graded differential calculus on $\mathcal{O}(\mathcal{SP}_h^{2|1})$, we first introduce the first-order differentials of the generators of $\mathcal{O}(\mathcal{SP}_h^{2|1})$ as $\mathsf{d}x = \xi$, $\mathsf{d}\theta = z$ and $\mathsf{d}y = \eta$. Then the differential d is uniquely defined by the conditions in Definition 5.1, and the h-deformed commutation relations between the differentials have the form

$$\mathsf{d}x \wedge \mathsf{d}x = 0, \quad \mathsf{d}x \wedge \mathsf{d}\theta = \mathsf{d}\theta \wedge \mathsf{d}x, \quad \mathsf{d}x \wedge \mathsf{d}y = -\mathsf{d}y \wedge \mathsf{d}x,$$

$$\mathsf{d}\theta \wedge \mathsf{d}y = \mathsf{d}y \wedge \mathsf{d}\theta - h \, \mathsf{d}x \wedge \mathsf{d}\theta, \quad \mathsf{d}y \wedge \mathsf{d}y = h \, (\mathsf{d}x \wedge \mathsf{d}y + \tfrac{1}{2} \, \mathsf{d}\theta \wedge \mathsf{d}\theta). \tag{5.1}$$

Theorem 5.3 *There exists unique $\mathbb{Z}_2$-graded differential calculus $\Omega(\mathcal{SP}_h^{2|1})$ on $\mathcal{O}(\mathcal{SP}_h^{2|1})$, which is left covariant with respect to the super-Hopf algebra $\mathcal{O}(\mathrm{SP}_h(2|1))$ such that $\{\mathsf{d}x, \mathsf{d}\theta, \mathsf{d}y\}$ is a free right $\mathcal{O}(\mathcal{SP}_h^{2|1})$-module basis of $\Omega(\mathcal{SP}_h^{2|1})$. The corresponding formulas describing $\Omega(\mathcal{SP}_h^{2|1})$ are:*

$$x \, \mathsf{d}x = \mathsf{d}x \, x, \quad x \, \mathsf{d}\theta = \mathsf{d}\theta \, x, \quad x \, \mathsf{d}y = \mathsf{d}y \, x + h \, \mathsf{d}x \, x, \quad \theta \, \mathsf{d}x = -\mathsf{d}x \, \theta,$$

$$\theta \, \mathsf{d}\theta = \mathsf{d}\theta \, \theta - h \, \mathsf{d}x \, x, \quad \theta \, \mathsf{d}y = -\mathsf{d}y \, \theta - h \, \mathsf{d}\theta \, x, \quad y \, \mathsf{d}x = \mathsf{d}x \, y - h \, \mathsf{d}x \, x, \tag{5.2}$$

$$y \, \mathsf{d}\theta = \mathsf{d}\theta \, y - h \, \mathsf{d}x \, \theta, \quad y \, \mathsf{d}y = \mathsf{d}y \, y + h \, (\tfrac{h}{2} \, \mathsf{d}x \, x - \mathsf{d}x \, y + \mathsf{d}\theta \, \theta + \mathsf{d}y \, x).$$

Proof To find the desired commutation relations, let us begin by writing the expressions $x \, \mathsf{d}x$, $x \, \mathsf{d}\theta$, etc. in terms of $\mathsf{d}x \, x$, $\mathsf{d}\theta \, x$, etc. in a compact form:

$$X \otimes \mathsf{d}X = B \, (\mathsf{d}X \otimes X), \quad X = (x, \theta, y)^t, \tag{5.3}$$

where $B = (B_{ij}^{kl})$ is a 9×9-matrix with constant entries. Here, there seem to be 81 indeterminate constants $(B_{11}^{11}, B_{11}^{12},$ etc.) when the sum is explicitly written, but in fact we have 41 indeterminate constants due to consistency and we can determine them in four steps.

(1) If we apply the differential d from the left to the relations (3.10) and use the corresponding cross-commutation relations (5.3), then we see that 18 of the constants are eliminated. For example,

$$B_{21}^{12} = -1 - B_{12}^{12}, \quad B_{22}^{11} = -\frac{h}{2} \, (1 + B_{11}^{11}), \quad B_{31}^{11} = B_{13}^{11} - h \, (1 + B_{11}^{11}).$$

(2) If we apply the differential d from the left to relations (5.3) and compare them with the relations (5.1), then 10 of those coefficients are eliminated. For instance,

$$B_{11}^{22} = \frac{h}{2} \, B_{11}^{33}, \quad B_{12}^{21} = 1 + B_{12}^{12} + h \, B_{12}^{32}, \quad B_{23}^{21} = B_{23}^{12} + h \, B_{23}^{32}.$$

(3) We first note that compatibility with the left coaction of $\mathcal{O}(\mathrm{SP}_h(2|1))$ means that (4.5) defines a graded differential algebra homomorphism $\Omega(\mathcal{SP}_h^{2|1}) \longrightarrow \mathcal{O}(\mathrm{SP}_h(2|1)) \otimes \Omega(\mathcal{SP}_h^{2|1})$. We now use compatibility with the left coaction of $\mathcal{O}(\mathrm{SP}_h(2|1))$. This leaves one free parameter and the relations (5.3) are in the following form:

$x \, \mathsf{d}x = B_{11}^{11} \, \mathsf{d}x \, x,$

$x \, \mathsf{d}\theta = \frac{1}{2} \, (B_{11}^{11} - 1) \, \mathsf{d}x \, \theta + \frac{1}{2} \, (B_{11}^{11} + 1) \, \mathsf{d}\theta \, x,$

$x \, \mathsf{d}y = \frac{h}{2} \, (B_{11}^{11} + 1) \, \mathsf{d}x \, x + \frac{1}{2} \, (B_{11}^{11} - 1) \, \mathsf{d}x \, y + \frac{1}{2} \, (B_{11}^{11} + 1) \, \mathsf{d}y \, x,$

$\theta \, \mathsf{d}x = -\frac{1}{2} \, (B_{11}^{11} + 1) \, \mathsf{d}x \, \theta + \frac{1}{2} \, (1 - B_{11}^{11}) \, \mathsf{d}\theta \, x,$

$\theta \, \mathsf{d}\theta = -\frac{h}{2} \, (B_{11}^{11} + 1) \, \mathsf{d}x \, x + \mathsf{d}\theta \, \theta,$ \hfill (5.4)

$\theta \, \mathsf{d}y = -\frac{h}{2} \, (B_{11}^{11} + 1) \, \mathsf{d}\theta \, x + \frac{1}{2} \, (1 - B_{11}^{11}) \, \mathsf{d}\theta \, y - \frac{1}{2} \, (B_{11}^{11} + 1) \, \mathsf{d}y \, \theta,$

$y \, \mathsf{d}x = -\frac{h}{2} \, (B_{11}^{11} + 1) \, \mathsf{d}x \, x + \frac{1}{2} \, (B_{11}^{11} + 1) \, \mathsf{d}x \, y + \frac{1}{2} \, (B_{11}^{11} - 1) \, \mathsf{d}y \, x,$

$y \, \mathsf{d}\theta = -\frac{h}{2} \, (B_{11}^{11} + 1) \, \mathsf{d}x \, \theta + \frac{1}{2} \, (B_{11}^{11} + 1) \, \mathsf{d}\theta \, y + \frac{1}{2} \, (B_{11}^{11} - 1) \, \mathsf{d}y \, \theta,$

$y \, \mathsf{d}y = \frac{h^2}{4} \, (B_{11}^{11} + 1) \, \mathsf{d}x \, x + [\frac{1}{2} \, (B_{11}^{11} - 1) - h \, B_{11}^{11}] \, \mathsf{d}x \, y + \frac{h}{2} \, (B_{11}^{11} + 1) \, \mathsf{d}\theta \, \theta + \frac{1}{2} \, (B_{11}^{11} - 1 + 2h) \, \mathsf{d}y \, x + B_{11}^{11} \, \mathsf{d}y \, y.$

(4) The parameter B_{11}^{11} is fixed by checking the associativity of the cubics. For instance, if we use that $(x \, \mathsf{d}\theta) \wedge \mathsf{d}y$ is equal to $x \, (\mathsf{d}\theta \wedge \mathsf{d}y)$, we find that the parameter B_{11}^{11} should be equal to 1.

Now the relations (5.4) with $B_{11}^{11} = 1$ are equivalent to the relations (5.2). $\qquad \square$

The quantum de Rham complex of the quantum superspace $\mathcal{SP}_h^{2|1}$ is the differential graded algebra $\Omega(\mathcal{SP}_h^{2|1})$ where the superalgebras $\mathcal{O}(\mathcal{SP}_h^{2|1})$ and $\Lambda(\mathcal{SP}_h^{2|1})$ are parts of $\Omega(\mathcal{SP}_h^{2|1})$. The complete definition is given below.

Definition 5.4 *The quantum de Rham complex* $\Omega(\mathcal{SP}_h^{2|1})$ *is a* $\mathbb{Z}_2$*-graded differential algebra generated by the elements of the set* $\{x, \theta, y, \mathrm{d}x, \mathrm{d}\theta, \mathrm{d}y\}$ *and the relations (3.10), (5.1), and (5.2).*

Note 3. The relations in (5.2) and (5.1) with the matrix $\hat{R}$ can be written in the forms

$$(-1)^{\tau(x_i)} x_i \, \mathrm{d}x_j = \sum_{k,l} \hat{R}_{ij}^{kl} \, \mathrm{d}x_k \, x_l, \quad (-1)^{\tau(x_i)} \mathrm{d}x_i \wedge \mathrm{d}x_j = \sum_{k,l} (-1)^{\tau(\mathrm{d}x_k)} \hat{R}_{ij}^{kl} \, \mathrm{d}x_k \wedge \mathrm{d}x_l,$$

where

$$\hat{R} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ h & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h & 0 & 0 & 0 & 1 & 0 \\ -h & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -h & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{h^2}{2} & 0 & -h & 0 & h & 0 & h & 0 & 1 \end{pmatrix} = (\hat{R}_{ij}^{kl}).$$

The matrix $R = P\hat{R}$ coincides with transposition of the matrix B in [2] (or [10] with $h = p$, therein) where P is the superpermutation matrix.

5.2. The relations with partial derivatives

Let $\mathcal{A}$ be any associative unital algebra and we denote the ring of 3×3-matrices with entries in $\mathcal{A}$ by $M_3(\mathcal{A})$. Let Ω be an $\mathcal{A}$-bimodule with free module structure. Since the ring of all endomorphisms $\Omega \longrightarrow \Omega$ of rank 3 is isomorphic to the ring $M_3(\mathcal{A})$, there exists an algebra homomorphism $\mu : \mathcal{A} \longrightarrow M_3(\mathcal{A})$.

From Theorem 5.3 it is obvious that a free right $\mathcal{O}(\mathcal{SP}_h^{2|1})$-module basis of $\mathcal{O}(\mathcal{SP}_h^{2|1})$-bimodule $\Omega(\mathcal{SP}_h^{2|1})$ is the set $\{\mathrm{d}x, \mathrm{d}\theta, \mathrm{d}y\}$ and the relations (5.2) hold. We now consider the left module structure of $\mathcal{O}(\mathcal{SP}_h^{2|1})$-bimodule $\Omega(\mathcal{SP}_h^{2|1})$ and interpret the left product $u \cdot \mathrm{d}v$ as an endomorphism of the right module $\Omega(\mathcal{SP}_h^{2|1})$. Then we can define an algebra homomorphism $\mu : \mathcal{O}(\mathcal{SP}_h^{2|1}) \to M_3(\mathcal{O}(\mathcal{SP}_h^{2|1}))$ such that

$$u \cdot \mathrm{d}x_i = \sum_{j=1}^{3} \mathrm{d}x_j \cdot \mu(u)_j^i \qquad (x_1 = x, \ x_2 = \theta, \ x_3 = y) \tag{5.5}$$

for all $u \in \mathcal{O}(\mathcal{SP}_h^{2|1})$. Relations (5.5) are equivalent to the relations (5.2), where

$$\mu(x) = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ hx & 0 & x \end{pmatrix}, \quad \mu(\theta) = \begin{pmatrix} -\theta & 0 & 0 \\ -hx & \theta & 0 \\ 0 & -hx & -\theta \end{pmatrix}, \quad \mu(y) = \begin{pmatrix} y - hx & 0 & 0 \\ -h\theta & y & 0 \\ h(\frac{h}{2}x - y) & h\theta & y + hx \end{pmatrix}. \tag{5.6}$$

Remark 5.5 *It is easy to see that the relations (3.10) are preserved under the action of the map* μ*.*

To obtain the commutation relations of the generators of $\mathcal{O}(\mathcal{SP}_h^{2|1})$ with derivatives, we first introduce the partial derivatives of the generators of the algebra.

Definition 5.6 *A family of quantum vector fields* $\partial_x, \partial_\theta, \partial_y : \mathcal{O}(\mathcal{SP}_h^{2|1}) \to \mathcal{O}(\mathcal{SP}_h^{2|1})$ *dual to* $\{\mathrm{d}x, \mathrm{d}\theta, \mathrm{d}y\}$ *is defined by*

$$\mathrm{d}f = \mathrm{d}x\,\partial_x(f) + \mathrm{d}\theta\,\partial_\theta(f) + \mathrm{d}y\,\partial_y(f), \quad f \in \mathcal{O}(\mathcal{SP}_h^{2|1}), \tag{5.7}$$

where $\tau(\partial_\theta) = 1$*, for consistency. These vector fields are called the partial derivatives of the calculus* (Ω, d)*.*

Clearly, the partial derivatives are left linear and satisfy the property $\partial_i(x_j) = \delta_{ij}$.

The next theorem gives the relations between the elements of $\mathcal{O}(\mathcal{SP}_h^{2|1})$ and their partial derivatives.

Theorem 5.7 *The commutation relations of the partial derivatives with the generators of* $\mathcal{O}(\mathcal{SP}_h^{2|1})$ *are*

$$\partial_x x = 1 + x\partial_x + h\,x\partial_y, \quad \partial_x \theta = \theta\partial_x + h\,x\partial_\theta, \quad \partial_x y = (y - hx)\,\partial_x - h\,\theta\partial_\theta + h(\tfrac{h}{2}\,x - y)\,\partial_y,$$

$$\partial_\theta x = x\partial_\theta, \quad \partial_\theta \theta = 1 - \theta\partial_\theta + h\,x\partial_y, \quad \partial_\theta y = y\partial_\theta + h\,\theta\partial_y, \tag{5.8}$$

$$\partial_y x = x\partial_y, \quad \partial_y \theta = \theta\partial_y, \quad \partial_y y = 1 + (y + hx)\,\partial_y.$$

Proof Let us denote the partial derivatives ∂_x, ∂_θ, and ∂_y by ∂_i for $i = 1, 2, 3$, respectively. Then it can be shown that the derivatives ∂_i and the homomorphism μ are related by

$$\partial_j(f \cdot g) = \partial_j(f) \cdot g + (-1)^{\tau(f)} \sum_{i=1}^{3} \mu(f)_j^i \cdot \partial_i(g) \tag{5.9}$$

for all $f, g \in \mathcal{O}(SP_h^{1|2})$. Now the required relations follow from (5.9) with (5.6). □

Theorem 5.8 *The partial derivatives satisfy the following commutation relations:*

$$\partial_x \partial_\theta = \partial_\theta \partial_x + h\,\partial_\theta \partial_y, \quad \partial_\theta \partial_\theta = \tfrac{h}{2}\,\partial_y \partial_y, \quad \partial_\theta \partial_y = \partial_y \partial_\theta,$$

$$\partial_x \partial_y = \partial_y \partial_x + h\,\partial_y \partial_y. \tag{5.10}$$

Proof From (5.7) we see that

$$0 = \mathrm{d}^2 f = -\mathrm{d}x \wedge (\mathrm{d}x\,\partial_x + \mathrm{d}\theta\,\partial_\theta + \mathrm{d}y\,\partial_y)\partial_x f + \mathrm{d}\theta \wedge (\mathrm{d}x\,\partial_x + \mathrm{d}\theta\,\partial_\theta + \mathrm{d}y\,\partial_y)\partial_\theta f$$

$$+ \mathrm{d}y \wedge (\mathrm{d}x\,\partial_x + \mathrm{d}\theta\,\partial_\theta + \mathrm{d}y\,\partial_y)\partial_y f$$

for $f \in \mathcal{O}(SP_h^{1|2})$. Now the desired relations are obtained by use of the relations (5.1). □

References

[1] Aghamohammadi A, Khorrami M, Shariati A. Jordanian deformation of SL(2) as a contraction of its Drinfeld-Jimbo deformation. Journal of Physics A: Mathematical and General 1995; 28 (8): L225-L231.

[2] Aizawa N, Chakrabarti R. Noncommutative geometry of super-Jordanian $OSP_h(2|1)$ covariant quantum space. Journal of Mathematical Physics 2004; 45 (4): 1623-1638. doi: 10.1063/1.1650538

[3] Aizawa N, Chakrabarti R. Noncommutative superspaces covariant under $OSP_q(1|2)$ algebra. In: 6th International Workshop on Lie Theory and Its Applications in Physics; Varna, Bulgaria. Sofia, Bulgaria: Heron Press, 2006, pp. 97-108.

[4] Celik S. Covariant differential calculi on quantum symplectic superspace $SP_q^{1|2}$. Journal of Mathematical Physics 2017; 58 (2): 023508. doi: 10.1063/1.4975924

[5] Chaichian M, Kulish PP. Quantum group covariant systems. In: Jancewicz B, Sobczyk J (editors). From Field Theory to Quantum Groups. Wroclaw, Poland: World Scientific Publishing Co. Pte. Ltd., 1996, pp. 99-112.

[6] Connes A. Non-commutative differential geometry. Publications Mathematiques de l'Institut des Hautes Etudes Scientifiques 1985; 62 (1): 41-144.

[7] Demidov EE, Manin YI, Mukhin EE, Zhdanovich DV. Nonstandard quantum deformations of GL(n) and constant solutions of the Yang-Baxter equation. Progress of Theoretical Physics Supplement 1990; 102: 203-218.

[8] Faddeev LD, Reshetikhin NY, Takhtajan LA. Quantization of Lie groups and Lie algebras. Leningrad Mathematical Journal 1990; 1 (1): 193-225.

[9] Inonu E, Wigner EP. On the contraction of groups and their representations. Proceedings of the National Academy of Science of the United States of America 1953; 39 (6): 510-524.

[10] Juszczak C, Sobczyck J. New quantum deformation of $OSP_h(1|2)$. Czechoslovak Journal of Physics 1998; 48 (11): 1375-1383. doi: 10.1023/A:1021601105140

[11] Kobayashi T, Uematsu T. Differential calculus on the quantum superspace and deformation of phase space. Zeitschrift für Physik C: Particles and Fields 1992; 56 (2): 193-199.

[12] Manin YI. Quantum Groups and Noncommutative Geometry. Montreal, Canada: Centre de Recherches Mathematiques, Universite de Montreal, 1988.

[13] Manin YI. Multiparametric quantum deformation of the general linear supergroup. Communications in Mathematical Physics 1989; 123 (1): 163-175.

[14] Wess J, Zumino B. Covariant differential calculus on the quantum hyperplane. Nuclear Physics B-Proceedings Supplements 1990, 18 (2): 302-312. doi: 10.1016/0920-5632(91)90143-3

[15] Woronowicz SL. Compact matrix pseudogroups (quantum groups). Communications in Mathematical Physics 1987; 111 (4): 613-665.

[16] Woronowicz SL. Differential calculus on compact matrix pseudogroups. Communications in Mathematical Physics 1989; 122 (1): 125-170.