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Fatou, Julia, and escaping sets in holomorphic (sub)semigroup dynamics

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Abstract: We investigate under what conditions the Fatou, Julia, and escaping sets of a transcendental semigroup are respectively equal to the Fatou, Julia, and escaping sets of their subsemigroups. We define the partial fundamental set and fundamental set of a holomorphic semigroup, and on the basis of these sets, we prove that the Fatou and escaping sets of a transcendental semigroup S are nonempty.

Key words: Transcendental semigroup, escaping set, finite index, cofinite index, Rees index, partial fundamental set, fundamental set

1. Introduction

We confine our study to the Fatou, Julia, and escaping sets of a holomorphic semigroup and its subsemigroup. A semigroup S is a very classical algebraic structure with a binary composition that satisfies the associative law. Semigroups arose naturally from the general mappings of a set into itself. Hence, a set of holomorphic functions on complex plane $\mathbb{C}$ or Riemann sphere $\mathbb{C}_\infty$ naturally forms a semigroup. Here, we take a set A of holomorphic functions and construct a semigroup S that consists of all elements that can be expressed as a finite composition of elements in A. We call such a semigroup S the *holomorphic semigroup* generated by the set A. A nonempty subset T of a holomorphic semigroup S is a *subsemigroup* of S if $f \circ g \in T$ for all $f,\ g \in T$.

For simplicity, we denote the class of all rational functions on $\mathbb{C}_\infty$ by $\mathscr{R}$ and the class of all transcendental entire functions on $\mathbb{C}$ by $\mathscr{E}$. Let $\mathscr{F} = \{f_\alpha : \alpha \in \Delta\} \subseteq \mathscr{R}$ or $\mathscr{E}$. The *holomorphic semigroup* generated by $\mathscr{F}$ is denoted by

$$S = \langle f_\alpha \rangle.$$

The index set Δ is allowed to be infinite in general unless otherwise stated. It is easy to see that S is a collection of holomorphic functions, and is closed under functional composition. S is called a *rational semigroup* or a *transcendental semigroup* depending on whether $\mathscr{F} \subseteq \mathscr{R}$ or $\mathscr{F} \subseteq \mathscr{E}$. A holomorphic semigroup S is *abelian* if $f_\alpha \circ f_\beta = f_\beta \circ f_\alpha$ for all generators f_α and f_β of S.

A semigroup generated by finitely many holomorphic functions $f_i, (i = 1, 2, \ldots, n)$ is called a *finitely generated holomorphic semigroup*, and we write $S = \langle f_1, f_2, \ldots, f_n \rangle$. If S is generated by only one holomorphic function f, then S is called a *cyclic semigroup*, and we write $S = \langle f \rangle$. In this case, each $g \in S$ can be written

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2010 *AMS Mathematics Subject Classification:* 37F10, 30D05 as $g = f^n$, where f^n is the nth iterate of f with itself. We say that $S = \langle f \rangle$ is a *trivial semigroup*. By the definition of holomorphic semigroup, we at once get the following result.

Proposition 1.1 *Let $S = \langle f_\alpha \rangle$ be a holomorphic semigroup. Then for every $f \in S$, f^m (for all $m \in \mathbb{N}$) can be written as $f^m = f_{\alpha_1} \circ f_{\alpha_2} \circ f_{\alpha_3} \circ \cdots \circ f_{\alpha_p}$, where $\alpha_i \in \{\alpha : \alpha \in \Delta\}$, for some $p \in \mathbb{N}$.*

A family $\mathscr{F}$ of holomorphic functions forms a *normal family* in a domain D if every sequence $(f_\alpha) \subseteq \mathscr{F}$ has a subsequence (f_{α_k}) which is uniformly convergent or divergent on all compact subsets of D. If there is a neighborhood U of a point $z \in \mathbb{C}$ such that $\mathscr{F}$ is a normal family in U, then we say that $\mathscr{F}$ is normal at z. We say that a holomorphic function f is *iteratively divergent* at $z \in \mathbb{C}$ if

$$f^n(z) \to \infty \text{ as } n \to \infty.$$

A semigroup S is *iteratively divergent* at z if every $f \in S$ is iteratively divergent at z. A semigroup S is said to be *iteratively bounded* at z if there is an element $f \in S$ which is not iteratively divergent at z.

Like in classical complex dynamics (that is, based on the Fatou-Julia-Eremenko theory of a holomorphic function), the Fatou, Julia, and escaping sets in the settings of a holomorphic semigroup are defined as follows:

Definition 1.1 (Fatou, Julia, and escaping sets) *The Fatou set of the holomorphic semigroup S is defined by*

$$F(S) = \{z \in \mathbb{C} : S \text{ is normal at } z\},$$

and the Julia set $J(S)$ of S is the complement of $F(S)$. If S is a transcendental semigroup, the escaping set of S is defined by

$$I(S) = \{z \in \mathbb{C} : S \text{ is iteratively divergent at } z\}.$$

We call each point of the set $I(S)$ an escaping point.

It is obvious that $F(S)$ is the largest open subset (of $\mathbb{C}$ or $\mathbb{C}_\infty$) on which the semigroup S is normal. And its complement $J(S)$ is a closed set for any semigroup S. However, the escaping set $I(S)$ is neither an open nor a closed set (if it is nonempty) for any transcendental semigroup S. Any maximally connected subset U of the Fatou set $F(S)$ is called a *Fatou component*. If $S = \langle f \rangle$, then the Fatou, Julia, and escaping sets are respectively denoted by $F(f), J(f)$, and $I(f)$.

It is possible that the Fatou, Julia, or escaping set of a holomorphic semigroup may be equal, respectively, to the Fatou, Julia, or escaping set of a proper subsemigroup.

Definition 1.2 (Finite index and cofinite index) *A subsemigroup T of a holomorphic semigroup S is said to be of finite index if there exists a finite collection $\{f_1, f_2, \ldots, f_n\}$ of elements of S^1, where $S^1 = S \cup \{Identity\}$, such that*

$$S = (f_1 \circ T) \cup (f_2 \circ T) \cup \ldots \cup (f_n \circ T). \tag{1.1}$$

The smallest n that satisfies 1.1 is called the index of T in S. Similarly, a subsemigroup T of a holomorphic semigroup S is said to be of cofinite index if there exists finite collection $\{f_1, f_2, \ldots, f_n\}$ of elements of S^1 such that for any $f \in S$, there is $i \in \{1, 2, \ldots, n\}$ such that

$$f_i \circ f \in T. \tag{1.2}$$

The smallest n that satisfies 1.2 is called the cofinite index of T in S.

Note that the size of a subsemigroup T of a semigroup S is measured in terms of index. If a subsemigroup T has a finite index or cofinite index in the semigroup S, then we say T is a finite indexed subsemigroup or a cofinite indexed subsemigroup, respectively.

For any holomorphic function f,

$$CV(f) = \{w \in \mathbb{C} : w = f(z) \text{ for some } z \text{ such that } f'(z) = 0\}$$

(where $f^{'}$ represents derivative of f with respect to z) is the set of *critical values* of f. The set $AV(f)$ consisting of all $w \in \mathbb{C}$ such that there exists a curve $\Gamma : [0, \infty) \to \mathbb{C}$ so that $\Gamma(t) \to \infty$ and $f(\Gamma(t)) \to w$ as $t \to \infty$ is the set of *asymptotic values* of f and

$$SV(f) = \overline{(CV(f) \cup AV(f))}$$

is the set of *singular values* of f. If $SV(f)$ is finite, then f is said to be of *finite type*. If $SV(f)$ is bounded, then f is said to be of *bounded type*. The sets

$$\mathscr{S} = \{f : f \text{ is of finite type}\}$$

and

$$\mathscr{B} = \{f : f \text{ is of bounded type}\}$$

are respectively known as *Speiser class* and *Eremenko-Lyubich class*.

In [8, Theorem 5.1], Poon proved that the Fatou and Julia sets of a finitely generated abelian transcendental semigroup S is the same as the Fatou and Julia sets of each of its particular functions if the semigroup S is generated by finite type transcendental entire functions. In [13, Theorems 3.3], we proved that the escaping set of a transcendental semigroup S is the same as the escaping set of each of its particular functions if the semigroup S is generated by finite type transcendental entire functions. In this paper, we prove the following assertion:

Theorem 1.1 *If a subsemigroup T has finite index or cofinite index in an abelian transcendental semigroup S, then $I(S) = I(T), J(S) = J(T)$ and $F(S) = F(T)$.*

In Section 2, we define Rees index in semigroups. We then prove Theorem 1.1 for a subsemigroup T having finite Rees index.

From [11, Theorem 3.1 (1) and (3)], we can say that Fatou and escaping sets of holomorphic semigroup may be empty. The result [8, Theorem 5.1] is one of the case of nonempty Fatou set and that of [13, Theorem 3.3] is a case of the nonempty escaping set of transcendental semigroup. We obtain another case of nonempty Fatou and escaping sets on the basis of the following definition.

Definition 1.3 (Partial fundamental set and fundamental set) *A set U is called a partial fundamental set for the semigroup S if*

1. *$U \neq \emptyset$,*
2. *$U \subset R(S)$,*
3. *$f(U) \cap U = \emptyset$ for all $f \in S$.*

If in addition to (1)*,* (2)*, and* (3)*, U satisfies the property*

4. $\bigcup_{f\in S} f(U) = R(S)$*,*

then U is called a fundamental set for S.

The set $R(S)$ is defined and discussed in Remark 4.1 of Section 4. On the basis of Definition 1.3, we obtain the following result.

Theorem 1.2 *Let S be a holomorphic semigroup and U a partial fundamental set for S. Then $U \subset F(S)$. If, in addition, S is a transcendental semigroup and U is a fundamental set, then $U \subset I(S)$.*

The organization of this paper is as follows: In Section 2, we briefly review the notion of finite index subsemigroups and cofinite index subsemigroups with suitable examples, we review some results from rational (sub)semigroup dynamics, and we extend the same in transcendental (sub)semigroup dynamics. We introduce the Rees index of a subsemigroup, and we prove the dynamical similarity of a holomorphic semigroup and its subsemigroup. In Section 3, we prove Theorem 1.1, and we also prove it without the abelian condition for the subsemigroup having finite Rees index. In Section 4, we define discontinuous semigroups, and on the basis of this notion, we discuss partial fundamental sets and fundamental sets, and then we prove Theorem 1.2.

2. Results from general holomorphic (sub)semigroup dynamics

There are various notions of how large a substructure is inside of an algebraic object in order that the two structures share certain properties. One such a notion is *index*, and it plays an important role in general group theory and semigroup theory. It is used to measure the difference between a group (semigroup) and a subgroup (subsemigroup). It occurs in many important theorems of the group theory and semigroup theory. The notions of finite index, cofinite index and Rees index of subsemigroup have been used to gauge the size of subsemigroup. If the subsemigroup T is big enough in semigroup S, then S and T share many properties. In this context, Theorem 1.1 states that if T has finite index or cofinte index in S, then both S and T share the same Fatou, Julia, and escaping sets. In the semigroup theory, the cofinite index is also known as *Grigorochuk index*, and this index was introduced by Grigorochuk [3] in 1988. Maltcev and Ruskuc [7, Theorem 3.1] proved that for every element f of a finitely generated semigroup S and every proper cofinite indexed subsemigroup T, one has $f \circ T \neq S$. Note that if the semigroup is a group, the notion of finite index and cofinite index coincide. The subsemigroup T of a finitely generated semigroup S consisting of all words of finite length (compositions of a finite number of holomorphic functions) has a finite index and a cofinite index in S.

From Definition 1.2, the finite index and cofinite index of subsemigroups of the following examples will be clear.

Example 2.1 *A subsemigroup*

$$T = \langle \sin\sin z,\ \cos\cos z,\ \sin\cos z,\ \cos\sin z\rangle$$

of the transcendental semigroup $S = \langle \sin z, \cos z\rangle$ has finite index 3 and cofinite index 2.

Example 2.2 *A subset $T = \{$words (compositions) begining with $f\}$ of a holomorphic semigroup $S = \langle f, g\rangle$ is clearly a subsemigroup of S. Then T has an infinite index but cofinite index 1 in S.*

Note that in Example 2.2, S is finitely generated but T is not. Since any generating set of T must contain $\{f \circ g^n : n \geq 1\}$. The only cofinite subsemigroup of T is T itself. Hence, T has cofinite index 1 in S.

Example 2.3 *Let $S = \langle f \rangle$ where f is a holomorphic function. Then the subsemigroup $T = \langle f^n : n \in \mathbb{N} \rangle$. has finite index n in S and cofinite index 1 in S.*

Note that in Example 2.3, the subsemigroup T has n different translates in S, which are $T, f \circ T, \ldots, f^{n-1} \circ T$. Here, the only cofinite subsemigroup of T is T itself. If we choose the subsemigroup of S to be S itself, then there are infinitely many translates of S, namely, $h \circ S = h \circ \langle f \rangle$ for all $h \in S$. So S has an infinite index in itself. Again, it has cofinite index 1 in itself.

Using Theorem 3.1 of [11], we can prove the following assertion:

Lemma 2.1 *For any subsemigroup T of a holomorphic semigroup S, we have $F(S) \subset F(T), J(S) \supset J(T)$.*

Proof We prove that $F(S) \subset F(T)$. The next inclusion follows taking the complements. By Theorem 3.1 of [11], $F(S) \subset \cap_{f \in S} F(f)$, and $F(T) \subset \cap_{g \in T} F(g)$ for any subsemigroup T of the semigroup S. Since any $g \in T$ is also in S; thus, by the same Theorem 3.1 of [11], we also have $F(S) \subset F(g)$ for all $g \in T$ and hence, $F(S) \subset \cap_{g \in T} F(g)$. Now for any $z \in F(S)$, we have $z \in \cap_{g \in T} F(g)$ for all $g \in T$. This implies that $z \in F(g)$ for all $g \in T$. This proves $z \in F(T)$ and hence, $F(S) \subset F(T)$. □

Hinkannen and Martin [4, Theorem 2.4] proved that if a subsemigroup T has a finite index or a cofinite index in the rational semigroup S, then $F(S) = F(T)$ and $J(S) = J(T)$. In the following theorem, we prove the same result in the case of a general holomorphic semigroup. Note that by a general holomorphic semigroup, we mean either a rational semigroup or a transcendental semigroup.

Theorem 2.1 *If a subsemigroup T has a finite index or a cofinite index in the holomorphic semigroup S, then $F(S) = F(T)$ and $J(S) = J(T)$.*

Proof From Lemma 2.1, $F(S) \subset F(T)$ for any holomorphic semigroup S. If S is a rational semigroup, the result follows from [4, Theorem 2.4]. We prove the reverse inclusion, if S is a transcendental semigroup.

Let the subsemigroup T of a semigroup S has finite index n. Then by Definition 1.2, there exists a finite collection $\{f_1, f_2, \ldots, f_n\}$ of elements of S^1 such that

$$S = f_1 \circ T \cup f_2 \circ T \cup \ldots \cup f_n \circ T.$$

Then for any $g \in S$, there is an $h \in T$ such that $g = f_i \circ h$. Choose a sequence $(g_j)_{j \in \mathbb{N}}$ in S. Then each g_j is of the form $g_j = f_i \circ h_j$, where $h_j \in T$ and $1 \leq i \leq n$. Here, we may assume the same i for all j. Hence, without loss of generality, we may choose a subsequence (g_{j_k}) of (g_j) such that $g_{j_k} = f_i \circ h_{j_k}$ for particular f_i, where (h_{j_k}) is a subsequence of (h_j) in T. Since on $F(T)$, the sequence (h_{j_k}) has a convergent subsequence so do the sequences (g_{j_k}) and (g_j) in $F(S)$. This proves that $F(T) \subset F(S)$.

Let the subsemigroup T of a semigroup S have cofinite index n. Then by Definition 1.2, there exists a finite collection $\{f_1, f_2, \ldots, f_n\}$ of elements of S^1 such that for every $f \in S$, there is $i \in \{1, 2, \ldots, n\}$ such that $f_i \circ f \in T$. Let us choose a sequence $(g_j)_{j \in \mathbb{N}}$ in S. Then, for each j, there is an i with $1 \leq i \leq n$ such that $f_i \circ g_j = h_j \in T$. Let $z \in F(T)$. Then the sequence (h_j) has a convergent subsequence in T, and hence so does the sequence (g_j) in $F(S)$. This proves that $F(T) \subset F(S)$. □

Next, we see a special subsemigroup of a holomorphic semigroup that yields a cofinite index.

Definition 2.1 (Stablizer, wandering component and stable domains) *For a holomorphic semigroup S, let U be a component of the Fatou set $F(S)$ and U_f be a component of the Fatou set containing $f(U)$ for some $f \in S$. The set of the form*

$$S_U = \{f \in S : U_f = U\}$$

is called the stabilizer of U on S. If S_U is nonempty, we say that a component U satisfying $U_f = U$ is a stable basin for S. The component U of $F(S)$ is said to be wandering if the set $\{U_f : f \in S\}$ contains infinitely many elements. That is, U is a wandering domain if there is sequence $(f_i)_{i \in \mathbb{N}}$ of elements of S such that $U_{f_i} \neq U_{f_j}$ for $i \neq j$.

Note that for any rational function f, we always have $U_f = U$. So U_S is nonempty for a rational semigroup S. However, if f is transcendental, it is possible that $U_f \neq U$. Hence, S_U may be empty for a transcendental semigroup S. Bergweiler and Rohde [1] proved that $U_f - U$ contains at most one point which is an asymptotic value of f if f is an entire function.

Lemma 2.2 *Let S be a holomorphic semigroup. Then the stabilizer S_U (if it is nonempty) is a subsemigroup of S and $F(S) \subset F(S_U)$, $J(S) \supset J(S_U)$.*

Proof Let $f, g \in S_U$. Then by Definition 2.1, $U_f = U$ and $U_g = U$, where U_f and U_g are components of the Fatou sets containing $f(U)$ and $g(U)$, respectively. Then $f(U) \subseteq U_f = U$ and $g(U) \subseteq U_g = U \Longrightarrow (f \circ g)(U) = f(g(U)) \subseteq f(U_g) = f(U) \subseteq U_f = U$. Since $(f \circ g)(U) \subseteq U_{f \circ g}$, so either $U_{f \circ g} \subseteq U$ or $U \subseteq U_{f \circ g}$. The only possibility in this case is $U_{f \circ g} = U$. Hence, $f \circ g \in S_U$, which proves that S_U is a subsemigroup of S. The proofs of $F(S) \subset F(S_U)$, $J(S) \supset J(S_U)$ follow from Lemma 2.1. □

There may be a connection between having no wandering domains and the stable basins of cofinite index. We have established the connection in the following theorem for a general holomorphic semigroup S.

Theorem 2.2 *Let S be a holomorphic semigroup with no wandering domains. Let U be any component of the Fatou set. Then the forward orbit $\{U_f : f \in S\}$ of U under S contains a stabilizer of U of cofinite index.*

Proof If S is a rational semigroup, see, for instance, the proof of [4, Theorem 6.1]. If S is a transcendental semigroup, we sketch our proof in the following way.

We are given that U is a nonwandering component of the Fatou set $F(S)$. So U has a finite forward orbit $U_1, U_2, \ldots, U_n$ (say) with $U_1 = U$.

Case (i): If for every $i = 1, 2, \ldots n$, there is $f_i \in S$ such that $f_i(U_i) \subseteq U_1$, then by Lemma 2.2 the stabilizer $S_{U_1} = \{f \in S : U_{1f} = U_1\}$ is a subsemigroup of S. For any $f \in S$, there is f_i for each $i = 1, 2, \ldots, n$ such that $U_{1_{f_i \circ f}} = U_1$. This shows that $f_i \circ f \in S_{U_1}$. Therefore, U_1 is a required stable basin such that the stabilizer S_{U_1} has a cofinite index in S.

Case (ii): If, for every $j = 2, \ldots n$, there is $f_j \in S$ such that $f_j(U_j) \subseteq V$, where $V = U_j$ such that $j \geq 2$, then the number of components of forward orbits of V is strictly less than that of U. In this way, we can find a component $W = U_i$ for some $i \leq n$ whose forward orbit has fewest components. For every component W_g of the forward orbit of W, there is $f \in S$ such that $f(W_g) \subseteq W$. That is, $W_{g \circ f} = W$, and it follows that W is a required stable basin such that the stabilizer S_W has a cofinite index. □

Let S be a holomorphic semigroup and $f \in S$. Then $S \circ f$ and $f \circ S$ are subsemigroups of S. Note that $S \circ f$ and $f \circ S$ may not be finitely generated even if the semigroup S is. If $S \circ f = \langle f_1, f_2, \dots, f_n \rangle$ where $f_i \in S$ for $i = 1, 2, \dots n$, then $f_i = g_i \circ f$, where $g_i \in S$. For any $g \in S$, we have $g^n \circ f \in S \circ f$ for all $n \geq 1$ but not every $g^n \circ f \in \langle f_1, f_2, \dots, f_n \rangle$. From this fact, we came to know that the notion of cofinite index fails to preserve the basic finiteness (finitely generated) condition of a subsemigroup. That is, if T is a subsemigroup of cofinite index in semigroup S, then S being finitely generated may not always imply that T is finitely generated. There is another notion of index which preserves the finiteness condition of a subsemigroup.

Definition 2.2 (Rees index) *Let S be a semigroup and T be a subsemigroup. The Rees index of T in S is defined as $|S-T|+1$, where $|S-T|$ represents the cardinality of $S-T$. In this case, T is a large subsemigroup of S, and S is a small extension of T.*

The Rees index was first introduced by Jura [5] in the case where T is an ideal of the semigroup S. In such a case, the Rees index of T in S is the cardinality of factor semigroup S/T. From Definition 2.2, it is clear that the Rees index of T in S is the size of the complement $S - T$. For a subsemigroup to have finite Rees index in its parent semigroup is a fairly restrictive property, and it occurs naturally in semigroups (for instance, all ideals in the additive semigroup of positive integers are of finite Rees index). Note that Rees index does not generalize group index, and even the notion of finite Rees index does not generalize finite group index. That is, if G is an infinite group and H is a proper subgroup, the group index of H in G may be finite even though the Rees index is infinite. In fact, let G be an infinite group and H is a subgroup of G. Then H has finite Rees index in G if and only if $H = G$.

Next, we investigate how similar a semigroup S and its large subsemigroup T are. One basic similarity (proved first by Jura [5]) is the following result.

Lemma 2.3 *Let T be a large subsemigroup of a semigroup S. Then S is finitely generated if and only if T is finitely generated.*

Proof See for instance [10, Theorem 1.1]. □

On the basis of Lemma 2.3, we obtain the following dynamical similarity of a holomorphic semigroup and its subsemigroup.

Theorem 2.3 *Let T be a large subsemigroup of a finitely generated holomorphic semigroup S. Then $F(S) = F(T)$ and $J(S) = J(T)$.*

Proof We prove that $F(S) = F(T)$. The other equality follows by taking complements. By Lemma 2.1, it is clear that $F(S) \subset F(T)$. Hence, it is sufficient to prove that $F(T) \subset F(S)$. By Lemma 2.3, T is finitely generated. Let $X = \{f_1, f_2, \dots, f_n\} \subset S$ be a generating set of T. Clearly, S is generated by the set $Y = X \cup (S - T)$. Every sequence (f_i) in $F(T)$ (where $f_i = f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_n}$, and $i_n \in \{1, 2, \dots, n\}$) has a convergent subsequence. Now each element g_m of a sequence (g_m) in S can be written as $g_m = f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_n} \circ h_{j_1} \circ h_{j_2} \circ \dots \circ h_{j_k}$, where $S - T = \{h_1, h_2, \dots, h_k\} \subset S$ and $j_k \in \{1, 2, \dots, k\}$. Since $S - T$ is finite, a convergent sequence in $F(T)$ can be extended to a convergent sequence in $F(S)$. Thus, every sequence (g_m) in $F(S)$ has a convergent subsequence. Hence $F(T) \subset F(S)$. □

3. Proof of Theorem 1.1

We now prove a result analogous to Lemma 2.1 in the case of an escaping set of a transcendental semigroup.

Lemma 3.1 *For any subsemigroup T of a transcendental semigroup S, we have $I(S) \subset I(T)$.*

Proof By Theorem 3.1 of [11], $I(S) \subset \cap_{f \in S} I(f)$ and $I(T) \subset \cap_{g \in T} I(g)$ for any subsemigroup T of S. Since $T \subset S$, the same theorem implies that $I(S) \subset I(g)$ for all $g \in T$. Hence, $I(S) \subset \cap_{g \in T} I(g)$. Now for any $z \in I(S)$, we have $z \in \cap_{g \in T} I(g)$ for all $g \in T$. This implies that $z \in I(g)$ for all $g \in T$. By Definition 1.1, we have $g^n(z) \to \infty$ as $n \to \infty$ for all $g \in T$. This proves that $z \in I(T)$ and hence, $I(S) \subset I(T)$. □

Lemma 3.2 *Let S be a transcendental semigroup. Then*

1. *int.$(I(S)) \subset F(S)$ and ext.$(I(S)) \subset F(S)$, where int. and ext. respectively denote the interior and exterior of $I(S)$.*
2. *$\partial I(S) = J(S)$, where $\partial I(S)$ denotes the boundary of $I(S)$.*

Proof We refer to Lemma 4.2 and Theorem 4.3 of [6]. □

Note that Lemma 3.2 is an extension of Eremenko's result [2], $\partial I(f) = J(f)$, of classical transcendental dynamics to more general semigroup dynamics. We prove the following assertion which can be an alternative definition of escaping set.

Lemma 3.3 *If $z \in \mathbb{C}$ is an escaping point of a transcendental semigroup S, then every nonconvergent sequence in S has a divergent subsequence at z.*

Proof Let $z \in \mathbb{C}$ be an escaping point of a transcendental semigroup S. Let $f \in S$. Then by Definition 1.1, there is a sequence $(g_n)_{n \in \mathbb{N}}$ in S representing $g_1 = f,\ g_2 = f^2, \dots, g_n = f^n, \dots$(say) such that $g_n(z) \to \infty$ as $n \to \infty$ or there is a sequence in S which contains $(g_n)_{n \in \mathbb{N}}$ as a subsequence such that $g_n(z) \to \infty$ as $n \to \infty$. More generally, every nonconvergent sequence in S has a subsequence which diverges infinity at z. □

We are now ready to prove Theorem 1.1.

Proof [Proof of Theorem 1.1] We prove $I(S) = I(T)$. The fact that $J(S) = J(T)$ is obvious from Lemma 3.2 (2). That $F(S) = F(T)$ is also obvious. By Lemma 2.1, we always have $I(S) \subset I(T)$ for any subsemigroup T of S. For proving this theorem, it is enough to show the reverse inclusion $I(T) \subset I(S)$.

Let a subsemigroup T of a semigroup S have finite index n. Then, by Definition 1.2, there exists a finite collection $\{f_1, f_2, \dots, f_n\}$ of elements of S^1 such that

$$S = f_1 \circ T \cup f_2 \circ T \cup \ldots \cup f_n \circ T.$$

Then, for any $g \in S$, there is $h \in T$ such that $g = f_i \circ h$. Choose a sequence $(g_j)_{j \in \mathbb{N}}$ in S. Then each g_j is of the form $g_j = f_i \circ h_j$, where $h_j \in T$, $1 \leq i \leq n$. Here, we may assume the same i for all j. Let $z \in I(T)$. Then by Lemma 3.3, every nonconvergent sequence $(h_j)_{j \in \mathbb{N}}$ in T has a divergent subsequence $(h_{j_k})_{j_k \in \mathbb{N}}$ at the point z. That is, $h^n_{j_k}(z) \to \infty$ as $n \to \infty$ for all j_k. In this case, every sequence $(g_j)_{j \in \mathbb{N}}$ in S has a subsequence $(g_{j_k})_{k \in \mathbb{N}}$, where $g_{j_k} = f_i \circ h_{j_k}$ with $h^n_{j_k}(z) \to \infty$ as $n \to \infty$. Since S is an abelian transcendental semigroup, $g_{j_k} = f_i \circ h_{j_k} = h_{j_k} \circ f_i$. Thus, we may write $g^n_{j_k}(z) = h^n_{j_k}(f_i(z)) \to \infty$ as $n \to \infty$. This shows that $f_i(z) \in I(S)$. If f_i = identity for a particular i, we are done. If f_i is not identity, then it is an element of an abelian transcendental semigroup S, and in this case $I(S)$ is backward invariant by [12, Theorem 2.6]. Thus, we must have $z \in I(S)$. Therefore, $I(T) \subset I(S)$.

Let a subsemigroup T of a semigroup S have cofinite index n. Then by Definition 1.2, there exists a finite collection $\{f_1, f_2, \dots, f_n\}$ of elements of S^1 such that for every $f \in S$, there is $i \in \{1, 2, \dots, n\}$ such that $f_i \circ f \in T$. Let us choose a sequence $(g_j)_{j\in\mathbb{N}}$ in S. Then for each j, there is a i with $1 \leq i \leq n$ such that $f_i \circ g_j = h_j \in T$. Let $z \in I(T)$. Then by Lemma 3.3, every nonconvergent sequence $(h_j)_{j\in\mathbb{N}}$ in T has a divergent subsequence $(h_{j_k})_{j_k\in\mathbb{N}}$ at the point z. This follows that sequence $(f_i \circ g_j)$ has a divergent subsequence $(f_i \circ g_{j_k})$ (say) at z. Since S is abelian, we can write that $(f_i \circ g_{j_k})(z) = (g_{j_k} \circ f_i)(z) = g_{j_k}(f_i(z)) = h_{j_k}(z)$. Now for any $z \in I(T)$, $h_{j_k} \in T$, we must have $h^n_{j_k}(z) = g^n_{j_k}(f_i(z)) \to \infty$ as $n \to \infty$. This implies that $f_i(z) \in I(S)$. If f_i = identity for a particular i, we are done. If f_i is not the identity, then of it is an element of abelian transcendental semigroup S. Then as in the first part, we write that $I(T) \subset I(S)$. □

The abelian hypothesis can be deleted from Theorem 1.1 if we use the Rees index. Thus, we have the following generalization of Theorem 1.1.

Theorem 3.1 *If a subsemigroup T of a finitely generated transcendental semigroup S has a finite Rees index, then $I(S) = I(T), J(S) = J(T)$ and $F(S) = F(T)$.*

Proof If we prove $I(S) = I(T)$, then the equality $J(S) = J(T)$ will follow from Lemma 3.2 (2). The inclusion $I(S) \subset I(T)$ follows from Lemma 3.1. Thus, we prove $I(T) \subset I(S)$.

By Theorem 2.3, T is finitely generated. Let $X = \{f_1, f_2, \dots, f_n\} \subset S$ be a generating set of T. Clearly, S is generated by the set $Y = X \cup (S - T)$. By Lemma 3.3, every nonconvergent sequence (f_i) in T (where $f_i = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}$, and $i_n \in \{1, 2, \dots, n\}$) has a divergence subsequence (f_{n_k}) at each point of $I(T)$. Now each element g_m of the sequence (g_m) in S can be written as $g_m = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n} \circ h_{j_1} \circ h_{j_2} \circ \cdots \circ h_{j_k}$, where $S - T = \{h_1, h_2, \dots, h_k\} \subset S$ is a finite set and $j_k \in \{1, 2, \dots, k\}$. This shows that a divergent sequence in $I(T)$ can be extended to a divergent sequence in $I(S)$. Thus, every sequence (g_m) in $I(S)$ has a divergent subsequence. Hence $I(T) \subset I(S)$. □

4. Proof of Theorem 1.2

It is known that for certain holomorphic semigroups, the Fatou sets and the escaping sets might be empty. In this section, we discuss the notion of discontinuous semigroup. This notion yields a partial fundamental set and a fundamental set. We prove Theorem 1.2 by showing that a partial fundamental set is in the Fatou set $F(S)$ and that a fundamental set is in the escaping set $I(S)$.

Definition 4.1 (Discontinuous semigroup) *A semigroup S is said to be discontinuous at a point $z \in \mathbb{C}$ if there is a neighborhood U of z such that $f(U) \cap U = \emptyset$ for all $f \in S$ or equivalently, translates of U by distinct elements of S (S-translates) are disjoint. The neighborhood U of z is also called a nice neighborhood of z.*

Remark 4.1 *Given a holomorphic semigroup S, there are two natural subsets associated with S.*

1. *The regular set $R(S)$ that consists of points $z \in \mathbb{C}$ at which S is discontinuous.*

2. *The limit set* $L(S)$ *that consists of points* $z \in \mathbb{C}$ *for which there is a point* z_0, *and a sequence* (f_n) *of distinct elements of* S *such that* $f_n(z_0) \to z$ *as* $n \to \infty$.

A set $X \subset \mathbb{C}$ is S-invariant or invariant under S if $f(X) = X$ for all $f \in S$. It is clear that both of the sets $R(S)$ and $L(S)$ are S-invariant. If U is a nice neighborhood, then $U \subset R(S)$. Thus, $R(S)$ is an open set, whereas the set $L(S)$ a closed set, and $R(S) \cap L(S) = \emptyset$. Recall that a set U is a *partial fundamental set* for the semigroup S if (1) $U \neq \emptyset$, (2) $U \subset R(S)$, (3) $f(U) \cap U = \emptyset$ for all $f \in S$. If in addition to (1), (2), and (3), U satisfies the property (4) $\bigcup_{f \in S} f(U) = R(S)$, then U is called a *fundamental set* for S. We say that $x, y \in \mathbb{C}$ are S- equivalent if there is an $f \in S$ such that $f(x) = y$. Condition (3) asserts that no two points of U are S-equivalent under semigroup S, and condition (4) asserts that every point of $R(S)$ is equivalent to some point of U. Note that if we replace (3) by $f^{-1}(U) \cap U = \emptyset$ for all $f \in S$, we say U is a backward partial fundamental set for S; if, in addition, U satisfies $\bigcup_{f \in S} f^{-1}(U) = R(S)$, then we say U is a backward fundamental set. Note that Theorems 1.2 and 4.1 hold if we have given (partial) backward fundamental set in the statements. Similar to the results of Hinkkanen and Martin [4, Lemma 2.2] in the case of a rational semigroup, we prove the following in the case of transcendental semigroup S.

Proof [Proof of Theorem 1.2] Let S be a holomorphic semigroup. The set U is a nonempty open set, and $f(U) \cap U = \emptyset$ for all $f \in S$ by Definition 4.1. The statement $f(U) \cap U = \emptyset$ for all $f \in S$ implies that S omits U on U. Since U is open, it contains more than two points. Then by Montel's theorem, S is normal on U. Therefore, $U \subset F(S)$.

Let S be a transcendental semigroup. To prove $U \subset I(S)$, we have to show that $f^n(z) \to \infty$ as $n \to \infty$ for all $f \in S$ and for all $z \in U$. The condition $f(U) \cap U = \emptyset$ for all $f \in S$ implies that $f^n(U) \cap U = \emptyset$, since $f \in S$ implies $f^n \in S$. Also, U is a fundamental set, so by Definition 1.3 (4), we have $\bigcup_{f \in S} f(U) = R(S)$. By Remark 4.1(2), there are no points in U which appear as the limit points under distinct $(f_m)_{m \in \mathbb{N}}$ in S. That is, (f_m) has a divergent subsequence (f_{m_k}) at each point of U. Thus, by [11, Theorem 2.2], for any $z \in U$, $f^n(z) \to \infty$ as $n \to \infty$ for any $f \in (f_m)$. This shows that $U \subseteq I(S)$. □

Finally, we generalize Theorem 1.2 in the following form. We give a short sketch of the proof. For a more detailed proof, we refer to [9, Theorem 2.1].

Theorem 4.1 *Let* U_1 *and* U_2 *be two (partial) fundamental sets for transcendental semigroups* S_1 *and* S_2, *respectively. Suppose furthermore that* $\mathbb{C} \setminus U_1 \subset U_2$ *and* $\mathbb{C} \setminus U_2 \subset U_1$. *Then the semigroup* $S = \langle S_1, S_2 \rangle$ *is discontinuous, and* $U = U_1 \cap U_2$ *is a (partial) fundamental set for the semigroup* S.

Proof [Sketch of the proof] Let U_1, U_2 and S_1, S_2 be as given in the theorem. It is clear from Theorem 1.2 that $F(S_1) \neq \emptyset$, $F(S_2) \neq \emptyset$; also $I(S_1) \neq \emptyset$ and $I(S_2) \neq \emptyset$ if U_1 and U_2 are fundamental sets of S_1 and S_2 respectively. Note that $U \neq \emptyset$ by the assumption. Clearly, $f(U) \cap U = \emptyset$ for every $f \in S$. This proves S is discontinuous and that U is a (partial) fundamental set for S. □

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