Study on the existence of solutions to two specific types of differential-difference equations

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Abstract: This paper concerns the description of the entire or meromorphic solutions to two certain types of differential-difference equations under some certain conditions. The significance of our results lies in that we find the entire solutions of the second type equation with the form $f = Ae^{Bz}$, where A, B are constants that are completely determined only by coefficients and correlated indices. Our results are accurate in a certain sense and are supplemented by an example. In particular, our results generalize and improve a result of Zhang and Huang, and they are closely related to recent results by Dong and Liao.

Key words: Differential-difference equations, hyperorder, meromorphic solutions, Nevanlinna theory

1. Introduction

In 1964, Hayman ([11], Theorem 3.9) considered the following nonlinear differential equation:

$$f^n(z) + P_d(f(z)) = g(z), \tag{1.1}$$

where $P_d(f)$ is a differential polynomial in f with degree d, and proved the following result.

Theorem 1.1 *Suppose that $f(z)$ is a nonconstant meromorphic function, $d \leq n - 1$, and f, g satisfy $N(r, f) + N(r, 1/g) = S(r, f)$ in (1.1). Then we have $g(z) = (f(z) + \gamma(z))^n$, where $\gamma(z)$ is meromorphic and a small function of $f(z)$.*

$p_1, p_2, \alpha_1, \alpha_2$ are small functions of f, and g is replaced by $p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}$ for the following equation:

$$f^n(z) + P_d(f(z)) = p_1(z)e^{\alpha_1(z)} + p_2(z)e^{\alpha_2(z)}. \tag{1.2}$$

Li and Yang ([18], Theorem 1) observed that there are no transcendental entire solutions when $n \geq 4, d \leq n-3$; later, Li ([17], Theorem 2) and Liao et al. ([21], Theorem 1) obtained entire or meromorphic solutions to the new equations under conditions $n \geq 2 \ (resp. n \geq 3), d \leq n - 2$. Meanwhile, for some other works related to (1.1), readers can refer to [5, 15, 16, 19–22, 30].

Based on the above results, one sees that the right side of these equations has one or two items. Hence, a natural question is: Can we characterize all entire or meromorphic solutions of these equations if the right side is replaced by multiple items? In 2018, Zhang and Huang ([31], Theorem 1.1) investigated this case and got a result of the nonlinear difference equation of the following form:

$$f^n(z) + p(z)f(z+c) = \beta_1 e^{\alpha_1 z} + \beta_2 e^{\alpha_2 z} + \cdots + \beta_s e^{\alpha_s z}, \qquad (1.3)$$

under $n \geq 2 + s$ and some given assumptions. They proved that any meromorphic solution f on $\mathbb{C}$ of the functional equation (1.3) must satisfy $\sigma_2(f) \geq 1$, where $\sigma_2(f)$ is the hyperorder of f.

Take positive integers t and k. For $t+1$ complex numbers $c_0(=0), c_1, ..., c_t$, we define the following differential-difference equations in the complex plane $\mathbb{C}$:

$$P(f) = \sum_{\mathbf{I} \in \mathcal{I}} a_{\mathbf{I}} \left(\mathbf{f}^{(\mathbf{k})} \right)^{\mathbf{I}} = \sum_{\mathbf{I}} a_{\mathbf{I}} \prod_{l=0}^{t} \left(f_{c_l}^{(\mathbf{k})} \right)^{I_l} = 0, \qquad (1.4)$$

where $\mathbf{k} = (0, 1, ..., k)$; $\mathbf{I} = (I_0, ..., I_t)$, $I_l = (i_{l0}, i_{l1}, ..., i_{lk})$ are multiindices of nonnegative integers $\mathbb{Z}_+$; $\mathcal{I}$ is a finite set of $\mathbb{Z}_+^{(t+1)(k+1)}$; $\mathbf{f} = (f_{c_0}, ..., f_{c_t})$, in which f_{c_l} is defined by $f_{c_l}(z) = f(z+c_l)$; $f_{c_l}^{(\mathbf{k})} = \left(f_{c_l}, f'_{c_l}, ..., f_{c_l}^{(k)} \right)$; and $\left(f_{c_l}^{(\mathbf{k})} \right)^{I_l} = f_{c_l}^{i_{l0}} \left(f'_{c_l} \right)^{i_{l1}} \cdots \left(f_{c_l}^{(k)} \right)^{i_{lk}}$ and $a_{\mathbf{I}}$ are nonzero meromorphic functions in $\mathbb{C}$. With the development of difference analogues of Nevanlinna theory, especially the difference analogue of the logarithmic derivative lemma given by Chiang and Feng [4] and Halburd and Korhonen [8], respectively, differential-difference equations have been studied rapidly [1, 6, 9, 10, 21, 23, 26, 27, 29].

Our first result generalizes the result of Zhang and Huang mentioned above. We consider the following differential-difference equation on f:

$$\sum_{i=1}^{p} f^{n_i}(z) f^{(k)}(z) + \sum_{j=1}^{q} p_j(z) f(z+c_j) = \sum_{l=1}^{s} \beta_l e^{\alpha_l z}, \qquad (1.5)$$

under the condition **(A)**: Suppose that $p_1(z), ..., p_q(z)$ are polynomials, $c_1, c_2, ..., c_q, \beta_1, \beta_2, ..., \beta_s$ are all nonzero constants, and take positive integers $n_i(i = 1, 2, ..., p)$, s, k with $n_p > n_{p-1} > \cdots n_1 \geq s+2$ and $\sum_{j=1}^{q} p_j(z) f(z+c_j) \not\equiv 0$. Let $\alpha_1, \alpha_2, ..., \alpha_s$ be distinct nonzero constants satisfying $\frac{\alpha_i}{\alpha_j} \neq n_t + 1$ for all $i, j \in \{1, 2, ..., s\}$ and $t \in \{1, ..., p\}$.

We obtain the first result as follows:

Theorem 1.2 *The differential-difference equation* (1.5) *under the assumption* **(A)** *does not have any polynomial solutions, transcendental entire solutions with finite order satisfying* $\lambda(f) < \sigma(f)$, *and meromorphic solutions with at least one pole satisfying* $\sigma_2(f) < 1$.

We take $n_1 = 5, n_2 = 6, s = 3$ in Theorem 1.2 and give the following example to illustrate the correctness of our conditions.

Example 1.3 *The transcendental entire solution* $f(z) = e^{iz}$ *is a solution of*

$$(f^6 + f^5)f' - f(z + \frac{\pi}{2}) = -ie^{iz} + ie^{6iz} + ie^{7iz}. \qquad (1.6)$$

We can easily find that the condition $\frac{\alpha_i}{\alpha_j} \neq n_t + 1$ for all $i, j \in \{1, 2, ..., s\}$ and $t \in \{1, ..., p\}$ is necessary in Theorem 1.2. However, we raise the following question.

Problem 1.4 *Can the conclusion $\lambda(f) < \sigma(f)$ in Theorem 1.2 be omitted or not?*

In fact, our goal is to find and characterize the existence and general form of the entire solutions of (1.5) under some certain conditions. Hence, we will discuss the more general equation as follows:

$$\sum_{i=1}^{p} f^{n_i}(z) f^{(k)}(z) + \sum_{j=1}^{q} p_j(z) f^{(m_j)}(z + c_j) = \sum_{l=1}^{s} \beta_l e^{\alpha_l z}, \qquad (1.7)$$

under the assumption **(B)**: Suppose that $p_1(z), ..., p_q(z)$ are polynomials, $c_1, c_2, ..., c_q, \beta_1, \ldots, \beta_s$ are all nonzero constants, and take nonnegative integers s, k, $n_1, \ldots, n_p$, $m_1, \ldots, m_q$ with $m_q > \cdots > m_1 \geq 0$ and $n_p > n_{p-1} > \cdots > n_1 \geq s + 2$ and $\sum_{j=1}^{q} p_j(z) f^{(m_j)}(z + c_j) \not\equiv 0$. Let $\alpha_1, \alpha_2, ..., \alpha_s$ be distinct nonzero constants, and we prove the following result.

Theorem 1.5 *If f is an entire solution of the differential-difference equation (1.7) with finite order and $\lambda(f) < \sigma(f)$ under the assumption **(B)**, then f has only the following form:*

$$f(z) = De^{\alpha_k z},$$

where D is a constant and $\alpha_k \in \{\alpha_1, \ldots, \alpha_s\}$. Moreover, we obtain $s = p+1$ and let the sequence $\{1, \cdots, p+1\}$ be rearranged to coincide with $\{\hat{j}_1, \cdots, \hat{j}_{p+1}\}$ such that $|\alpha_{\hat{j}_1}| < \cdots < |\alpha_{\hat{j}_{p+1}}|$. Then for $i = 1, 2, \ldots, p$, the following three conclusions hold:

(i) $\alpha_k = \alpha_{\hat{j}_1}$, $\alpha_{\hat{j}_{i+1}} = (n_i + 1)\alpha_{\hat{j}_1}$,

(ii) $\sum_{j=1}^{q} \alpha_{\hat{j}_1}^{m_j} e^{\alpha_{\hat{j}_1} c_j} p_j(z) = \beta_{\hat{j}_1} (\frac{\beta_{\hat{j}_{i+1}}}{\alpha_{\hat{j}_1}^k})^{\frac{-1}{n_i+1}}$,

(iii) D is completely determined by $\alpha_{\hat{j}_1}, \beta_l, n_i, k$ such that $D = (\frac{\beta_{\hat{j}_{i+1}}}{\alpha_{\hat{j}_1}^k})^{\frac{1}{n_i+1}}$.

Example 1.3 given above shows the sharpness of our results of Theorem 1.5. According to example 1.3, we can fix $p_1(z) = -1$, $k = q = 1$, $m_1 = 0$, $c_1 = \frac{\pi}{2}$, and $n_1 = 5, n_2 = 6, \alpha_{\hat{j}_1} = \beta_{\hat{j}_2} = \beta_{\hat{j}_3} = i, \alpha_{\hat{j}_2} = 6i, \alpha_{\hat{j}_3} = 7i$ in (1.7). Via Theorem 1.5, we can observe that $D = (\frac{\beta_{\hat{j}_2}}{\alpha_{\hat{j}_1}^k})^{\frac{1}{n_1+1}} = (\frac{\beta_{\hat{j}_3}}{\alpha_{\hat{j}_1}^k})^{\frac{1}{n_2+1}} = 1$ is consistent with $f(z) = e^{iz}$ being a solution of (1.6) and $\sum_{j=1}^{q} \alpha_{\hat{j}_1}^{m_j} e^{\alpha_{\hat{j}_1} c_j} p_j(z) = \beta_{\hat{j}_1} (\frac{\beta_{\hat{j}_{i+1}}}{\alpha_{\hat{j}_1}^k})^{\frac{-1}{n_i+1}}$ is true. Moreover, $\alpha_{\hat{j}_2} = 6i, \alpha_{\hat{j}_3} = 7i$ also imply that $\alpha_{\hat{j}_{i+1}} = (n_i + 1)\alpha_{\hat{j}_1}$ holds.

One notices that our Theorem 1.5 above actually provides a deep extension to the main results of Dong and Liao [7] when we remove $f^{(k)}(z)$ from equation (1.7). In particular, if we take $m_j = 0$ $(j = 1, 2, \ldots, q)$ in (1.7), this is just (1.5). We thus find the entire solutions under some certain conditions, and we also get the following corollary.

Corollary 1.6 *If f is an entire solution of the differential-difference equation* (1.5) *with finite order and* $\lambda(f) < \sigma(f)$ *under the assumption* (B), *then f has only form $f(z) = De^{\alpha_k z}$, satisfying* $(i), (iii)$ *above, and*

$$\sum_{j=1}^{q} e^{\alpha_{\hat{j}_1} c_j} p_j(z) = \beta_{\hat{j}_1} \left(\frac{\beta_{\hat{j}_{i+1}}}{\alpha_{\hat{j}_1}^k} \right)^{\frac{-1}{n_i+1}}.$$

2. Preliminaries

We assume that the reader is familiar with Nevanlinna theory [13, 25] of meromorphic functions f in $\mathbb{C}$, such as the first main theorem of f, the second main theorem of f, the characteristic function $T(r, f)$, the proximity function $m(r, f)$, the counting functions $N(r, f)$, $\bar{N}(r, f)$, and $S(r, f)$, where as usual $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside a possible exceptional set of finite logarithmic measure. Furthermore, recall that the order of f is defined by

$$\sigma(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}.$$

The hyperorder of f is defined by

$$\sigma_2(f) = \limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r}.$$

The exponent of convergence of zeros of f is defined by

$$\lambda(f) = \limsup_{r \to \infty} \frac{\log N(r, \frac{1}{f})}{\log r} = \limsup_{r \to \infty} \frac{\log n(r, \frac{1}{f})}{\log r}.$$

The hyperexponent of convergence of poles of f is defined by

$$\lambda_2 \left(\frac{1}{f} \right) = \limsup_{r \to \infty} \frac{\log \log N(r, f)}{\log r} = \limsup_{r \to \infty} \frac{\log \log n(r, f)}{\log r}.$$

Take complex numbers $d_0(=0), d_1, ..., d_t$. Let $R(f)$ be a differential-difference polynomial of f defined by

$$R(f) = \sum_{\mathbf{J} \in \mathcal{J}} b_{\mathbf{J}} \prod_{l=0}^{t} \left(f_{d_l}^{(\mathbf{k})} \right)^{J_l}, \tag{2.1}$$

where $\mathbf{k} = (0, 1, ..., k)$; $\mathbf{J} = (J_0, ..., J_t)$, $J_l = (j_{l0}, j_{l1}, ..., j_{lk})$ are multiindices of nonnegative integers $\mathbb{Z}_+$; $\mathcal{J}$ is a finite set of $\mathbb{Z}_+^{(t+1)(k+1)}$; and $b_{\mathbf{J}}$ are nonzero small functions of f. For complex numbers $e_0(=0), e_1, ..., e_t$, we use $Q(f)$ to denote a difference polynomial of f as follows:

$$Q(f) = \sum_{\mathbf{K} \in \mathcal{K}} C_{\mathbf{K}} f_{e_0}^{K_0} \cdots f_{e_t}^{K_t}, \tag{2.2}$$

where $\mathbf{K} = (K_0, ..., K_t)$ are multiindices of nonnegative integers $\mathbb{Z}_+$; $\mathcal{K}$ is a finite set of $\mathbb{Z}_+^{t+1}$; and $C_{\mathbf{K}}$ are nonzero small functions of f. Next we consider the following equation:

$$R(f)Q(f) = P(f), \tag{2.3}$$

where $P(f)$ is a differential-difference polynomial defined by the left side of (1.4).

The first lemma is a variant of the result due to Laine and Yang [14], and this lemma is Lemma 2.2, which was proved in [12].

Lemma 2.1 *Let f be a transcendental meromorphic solution of hyperorder $\sigma_2(f) < 1$ of equation (2.3) with $\deg P(f) \le \deg Q(f)$. Assume that there is only unique monomial of degree $\deg Q(f)$ in $Q(f)$. Then,*

$$m(r, R(f)) = S(r, f)$$

holds possibly outside an exceptional set of finite logarithmic measure.

The following lemma is referred to [28, Theorem 1.52].

Lemma 2.2 *If $f_j(z)$ $(1 \le j \le n)$ and $g_j(z)(1 \le j \le n)(n \ge 2)$ are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j}(z) \equiv 0$,

(ii) the orders of f_j are less than that of $e^{g_h(z) - g_k(z)}$ for $1 \le j \le n$, $1 \le h < k \le n$,
then $f_j(z) \equiv 0$ for $1 \le j \le n$.

The third lemma is referred to [3].

Lemma 2.3 *Let $f(z)$ be an entire function of finite order p. Then*

$$f(z) = z^m e^{h(z)} \prod_{n \ge 1} (1 - \frac{z}{z_n}) e^{(z + \frac{z^2}{2} + \cdots + \frac{z^q}{q})},$$

where $h(z)$ is a polynomial of degree not greater than p, and $\{z_n\}_{n=1}^{\infty}$ from the family of zeros of f distinct from $z = 0$.

Setting $E(z) = \prod_{n \ge 1} (1 - \frac{z}{z_n}) e^{(z + \frac{z^2}{2} + \cdots + \frac{z^q}{q})}$, a well-known fact about Lemma 2.3 asserts that $\sigma(E) = \lambda(f) \le \sigma(f)$, and $\sigma(f) = \sigma(e^h)$ when $\lambda(f) < \sigma(f)$.

3. Proof of Theorem 1.2

We will distinguish two cases to prove Theorem 1.2.

Case 1. f has at least one pole with $\sigma_2(f) < 1$.

Assuming that z_0 is a pole of f with multiplicity m_0 (≥ 1), we can denote $f(z_0) = \infty^{m_0}$. From (1.5), we know that there exists $c_{n_1} \in \{c_1, c_2, ..., c_q\}$ such that $z_0 + c_{n_1}$ is also a pole of f, and it follows that

$$f(z_0 + c_{n_1}) = \infty^{m_1},$$

where $m_1 \ge (n_p + 1)m_0 + k$. Substituting z by $z + c_{n_1}$ into (1.5), we get

$$\sum_{i=1}^{p} f^{n_i}(z + c_{n_1}) f^{(k)}(z + c_{n_1}) + \sum_{j=1}^{q} p_j(z + c_{n_1}) f(z + c_j + c_{n_1}) = \sum_{l=1}^{s} \beta_l e^{\alpha_l(z + c_{n_1})}. \tag{3.1}$$

Then, seeing (3.1) and the fact that $z_0 + c_{n_1}$ is a pole of $f^{n_p} f^{(k)}$ with a multiplicity $\geq (n_p+1)^2 m_0 + k(n_p+1) + k$, it yields that there exists $c_{n_2}, n_2 \in \{1, 2, ..., q\}$ such that $z_0 + c_{n_1} + c_{n_2}$ is a pole of f with

$$f(z_0 + c_{n_1} + c_{n_2}) = \infty^{m_2},$$

where $m_2 \geq (n_p+1)^2 m_0 + k(n_p+1) + k$. According to the above argument, it is easily seen that for any $j \geq 1$, there exist some $n_1, n_2, ..., n_j$ such that $z_0 + \Sigma_{l=1}^{j} c_{n_l}$ satisfy

$$f(z_0 + \Sigma_{l=1}^{j} c_{n_l}) = \infty^{m_j},$$

where $m_j \geq (n_p+1)^j m_0 + k[(n_p+1)^{j-1} + (n_p+1)^{j-2} + \cdots + 1]$. We can estimate the counting $n(r, f)$ of poles of f in the disc $|z| \leq r$. Letting $r_t = t \max\{|c_1|, ...|c_q|\} + |z_0| + 1$, we have for each integer $t \geq 1$ that

$$n(r_t, f) \geq m_0 + \sum_{j=1}^{t} (n_p+1)^j m_0 + k\left[(n_p+1)^{j-1} + (n_p+1)^{j-2} + \cdots + 1\right].$$

Because $n_p > n_{p-1} > \cdots n_1 \geq 2 + s \geq 3$ by assumption, one then has

$$\sigma_2(f) \geq \lambda_2\left(\frac{1}{f}\right) = \limsup_{r \to \infty} \frac{\log \log n(r, f)}{\log r} \geq \limsup_{t \to \infty} \frac{\log \log n(r_t, f)}{\log r_t}$$

$$\geq \limsup_{t \to \infty} \frac{\log \log (n_q + 1)^t}{\log t} = 1.$$

It contradicts our assumption that $\sigma_2(f) < 1$.

Case 2. f is an entire function.

If f is a polynomial, we can easily yield a contradiction from (1.5). Hence, f is transcendental. Now, suppose that $H(z) = \sum_{i=1}^{p} f^{n_i}(z) f^{(k)}(z) + \sum_{j=1}^{q} p_j(z) f(z + c_j)$. Differentiating both sides of the equation (1.5), we have

$$H'(z) = \sum_{l=1}^{s} \alpha_l \beta_l e^{\alpha_l z}. \tag{3.2}$$

Eliminating $e^{\alpha_1 z}$ from (1.5) and (3.2), we obtain

$$\alpha_1 H(z) - H'(z) = \sum_{l=2}^{s} \beta_l(\alpha_1 - \alpha_l) e^{\alpha_l z}. \tag{3.3}$$

Differentiating (3.3), we have

$$\alpha_1 H'(z) - H''(z) = \sum_{l=2}^{s} \beta_l(\alpha_1 - \alpha_l)\alpha_l e^{\alpha_l z}. \tag{3.4}$$

Then (3.3) and (3.4) further lead to

$$\alpha_1 \alpha_2 H(z) - (\alpha_1 + \alpha_2)H'(z) + H''(z) = \sum_{l=3}^{s} \beta_l(\alpha_1 - \alpha_l)(\alpha_2 - \alpha_l) e^{\alpha_l z}. \tag{3.5}$$

Repeating the above steps, we can deduce inductively that

$$\sum_{j=0}^{s}(-1)^j e_{s-j}(\alpha_1,...,\alpha_s)H^{(j)} = 0. \tag{3.6}$$

Here, $e_j(\alpha_1,...,\alpha_s), j = 0,...,s$ are the elementary symmetric polynomials [24] in s variables. $\alpha_1,...,\alpha_s$ and $e_k(\alpha_1,...,\alpha_s)$ for $k = 0,1,\ldots,s$ are defined by

$$e_0(\alpha_1,...,\alpha_s) = 1, e_1(\alpha_1,...,\alpha_s) = \sum_{1\leq j\leq s}\alpha_j, e_2(\alpha_1,...,\alpha_s) = \sum_{1\leq i<j\leq s}\alpha_i\alpha_j,$$

$$e_3(\alpha_1,...,\alpha_s) = \sum_{1\leq i<j<l\leq s}\alpha_i\alpha_j\alpha_l,$$

and so forth, ending with $e_s(\alpha_1,...,\alpha_s) = \alpha_1\alpha_2\cdots\alpha_s$.

Let $L(w)$ be a linear differential operator defined by

$$L(w) = \sum_{j=0}^{s}(-1)^j e_{s-j}(\alpha_1,...,\alpha_s)w^{(j)}. \tag{3.7}$$

Applying this operator above, we can rewrite (3.6) as

$$L(\sum_{i=1}^{p} f^{n_i}(z)f^{(k)}(z)) = -L(\sum_{j=1}^{q} p_j(z)f(z+c_j)). \tag{3.8}$$

In addition,

$$\left(f^n f^{(k)}\right)^{(l)} = \sum_{i=0}^{l}\binom{l}{i}(f^n)^{(i)}\left(f^{(k)}\right)^{(l-i)} = \sum_{i=1}^{l}\binom{l}{i}\left(f^{(k)}\right)^{(l-i)}$$

$$\cdot\left[nf^{n-1}f^{(i)} + \sum_{j=2}^{i-1}\sum_{\lambda}\gamma_{j\lambda}f^{n-j}(f')^{\lambda_{j1}}(f'')^{\lambda_{j2}}\cdots\left(f^{(i-1)}\right)^{\lambda_{j,i-1}}\right. \tag{3.9}$$

$$\left. + n(n-1)\cdots(n-(i-1))f^{n-i}(f')^i\right] + f^n f^{(k+l)}$$

for $l = 0,1,...,s$, where $\gamma_{j\lambda}$ are positive integers, $\lambda_{j1},\lambda_{j2},...,\lambda_{j,i-1}$ are nonnegative integers, and sum $\sum_{\lambda}$ is carried out such that $\lambda_{j1}+\lambda_{j2}+\cdots+\lambda_{j,i-1} = j$ and $\lambda_{j1}+2\lambda_{j2}+\cdots+(i-1)\lambda_{j,i-1} = i$.

Noting that $n_q > \cdots > n_1$, then by (3.7) and (3.9), we get

$$L(\sum_{i=1}^{p} f^{n_i}(z)f^{(k)}(z)) = f^{n_1-s}\varphi, \tag{3.10}$$

where φ is a differential polynomial in f of degree $n_p - n_1 + s + 1$ with constant coefficients.

If $\varphi \neq 0$, noting that $L(\sum_{j=1}^{q} p_j(z)f(z+c_j))$ is a difference polynomial in f of degree 1 with polynomial coefficients and $n_1 \geq s+2$, applying Lemma 2.1, we observe from (3.8) and (3.10) that

$$T(r,\varphi) = m(r,\varphi) = S(r,f), \quad T(r,f\varphi) = m(r,f\varphi) = S(r,f), \tag{3.11}$$

which immediately leads to

$$T(r, f) \leq T(r, f\varphi) + T\left(r, \frac{1}{\varphi}\right) = S(r, f).$$

This is a contradiction.

If $\varphi = 0$, then the expression of (3.8) yields $L(\sum_{i=1}^{p} f^{n_i}(z)f^{(k)}(z)) = 0$ and $L(\sum_{j=1}^{q} p_j(z)f(z+c_j)) = 0$. By using the definition of operator L, we get

$$L(\sum_{j=1}^{q} p_j(z)f(z+c_j)) = \sum_{j=0}^{s} (-1)^j e_{s-j}(\sum_{j=1}^{q} p_j(z)f(z+c_j))^{(j)} = 0.$$

We get the characteristic equation as follows:

$$(-1)^s \lambda^s + (-1)^{s-1} e_1 \lambda^{s-1} + \cdots + (-1)^{s-j} e_j \lambda^{s-j} + \cdots + e_s = 0. \tag{3.12}$$

It follows from (3.12) that $\alpha_1, \alpha_2, ..., \alpha_s$ are s distinct roots, and then

$$\sum_{j=1}^{q} p_j(z)f(z+c_j) = b_1 e^{\alpha_1 z} + b_2 e^{\alpha_2 z} + \cdots + b_s e^{\alpha_s z}, \tag{3.13}$$

where b_l $(l = 1, 2, ..., s)$ are constants. Noting that $f(z)$ is a transcendental entire function with finite order and $\lambda(f) < \sigma(f)$, by Lemma 2.3, we can factorize $f(z)$ as

$$f(z) = h(z)e^{g(z)}, \tag{3.14}$$

where $h(z)$ is the canonical product formed by zeros of $f(z)$ and $\sigma(h) = \lambda(f) < \sigma(f)$, and $g(z)$ is a polynomial. Supposing that $g(z)$ is a polynomial of degree 0, we get a contradiction immediately by comparing the order of the left side and the right side of (1.5). Hence, $g(z)$ is a polynomial of degree not less than 1. Therefore, substituting (3.14) into (3.13), it yields that

$$\sum_{j=1}^{q} G_j(z)e^{g(z+c_j)} = \sum_{l=1}^{s} b_l e^{\alpha_l z}, \tag{3.15}$$

where $G_j(z) = p_j(z)h(z+c_j)$ and $T(r, G_j) = o(T(r, f)), j = 1, \cdots, q$.

Now we confirm the fact that $g(z)$ is a polynomial of degree 1.

Claim. Otherwise, we suppose that $g(z)$ is a polynomial of degree $n(\geq 2)$. Write $g(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ with $a_n \neq 0$, and then we can rewrite (3.15) as

$$e^{a_n z^n} \sum_{j=1}^{q} \widetilde{G}_j(z) - \sum_{l=1}^{s} b_l e^{\alpha_l z} = 0, \tag{3.16}$$

where $\widetilde{G}_j(z) = G_j(z)e^{g(z+c_j)-a_n z^n}$, and $\sigma(\widetilde{G}_j(z)) \leq n-1$. If $\sum_{j=1}^{q} \widetilde{G}_j(z) \not\equiv 0$, then we can get

$$e^{a_n z^n} = \frac{\sum_{l=1}^{s} b_l e^{\alpha_l z}}{\sum_{j=1}^{q} \widetilde{G}_j(z)}. \tag{3.17}$$

Notice the order of the right-hand side of (3.17) is $\leq n-1$ and the order of the left-hand side of (3.17) is n. This is a contradiction. Hence, $\sum_{j=1}^{q} \widetilde{G}_j(z) \equiv 0$, and then we get $\sum_{l=1}^{s} b_l e^{\alpha_l z} \equiv 0$ in (3.16). Noting that $\lambda(f) < \sigma(f)$ and invoking Lemma 2.2 to $\sum_{l=1}^{s} b_l e^{\alpha_l z} = 0$, it follows that $b_l = 0 \ (l = 1, \cdots, s)$, which is impossible. Thus, the claim is proved.

Then we can rewrite f as follows:

$$f(z) = h(z)e^{az+b} = \widetilde{h}(z)e^{az}, a(\neq 0), b \in \mathbb{C}, \tag{3.18}$$

where $\widetilde{h}(z) = e^b h(z)$. Then (3.15) becomes

$$\sum_{j=1}^{q} \hat{G}_j(z)e^{a(z+c_j)} - \sum_{l=1}^{s} b_l e^{\alpha_l z} = 0, \tag{3.19}$$

where $\hat{G}_j(z) = p_j(z)\widetilde{h}(z + c_j)$.

If $a = \alpha_k$, for some $k \in \{1, \cdots, s\}$, then we can rewrite (3.19) in the following form:

$$(\sum_{j=1}^{q} \hat{G}_j(z)e^{\alpha_k c_j} - b_k)e^{\alpha_k z} - \sum_{1 \leq l(\neq k) \leq s} b_l e^{\alpha_l z} = 0. \tag{3.20}$$

Using Lemma 2.2 again, we find $b_l = 0$ for $l \neq k$, and $b_k = \sum_{j=1}^{q} \hat{G}_j(z)e^{\alpha_k c_j}$. Substituting $b_k \neq 0$ into (3.13), it follows that

$$\sum_{j=1}^{q} p_j(z)f(z + c_j) = b_k e^{\alpha_k z}. \tag{3.21}$$

On the other hand, (3.18) implies

$$f^{(k)}(z) = \tau_k(z)e^{\alpha_k z}, \tag{3.22}$$

where τ_k is a polynomial in h and their derivatives. Substituting (3.18), (3.21), and (3.22) into (1.5), we get

$$\sum_{i=1}^{p} \widetilde{h}^{n_i}(z)\tau_k(z)e^{(n_i+1)\alpha_k z} + (b_k - \beta_k)e^{\alpha_k z} = \sum_{1 \leq l(\neq k) \leq s} \beta_l e^{\alpha_l z}. \tag{3.23}$$

Note that $\frac{\alpha_i}{\alpha_j} \neq n_t + 1$ for all $i, j \in \{1, 2, ..., s\}$ and $t \in \{1, ..., p\}$. Then we discuss the following two cases.

Subcase 2.1. $s > 1$. Then by using Lemma 2.2, the above equation immediately yields $\beta_l = 0(1 \leq l \neq k \leq s)$, and we obtain a contradiction.

Subcase 2.2. $s = 1$. Now equation (1.5) becomes

$$\sum_{i=1}^{p} f^{n_i}(z)f^{(k)}(z) + \sum_{j=1}^{q} p_j(z)f(z + c_j) = \beta_1 e^{\alpha_1 z}. \tag{3.24}$$

Differentiating both sides of (3.24), we get

$$\sum_{i=1}^{p} \left[n_i f^{n_i-1}(z)f'(z)f^{(k)}(z) + f^{n_i}(z)f^{(k+1)}(z) \right] + \sum_{j=1}^{q}(p_j(z)f(z+c_j))' = \alpha_1\beta_1 e^{\alpha_1 z}.$$

Combining this equation with (3.24), we get

$$f^{n_1-1}F = \alpha_1 \sum_{j=1}^{q} p_j(z)f(z+c_j) - \sum_{j=1}^{q}(p_j(z)f(z+c_j))', \qquad (3.25)$$

where $F = \sum_{i=1}^{p} \left[n_i f^{n_i-n_1}(z)f'(z)f^{(k)}(z) + f^{n_i-n_1+1}(z)f^{(k+1)}(z) - \alpha_1 f^{n_i-n_1+1}(z)f^{(k)}(z) \right].$

If $F \neq 0$, it follows from (3.25) and Lemma 2.1 that

$$T(r,F) = m(r,F) = m\left(r, \frac{\alpha_1 \sum_{j=1}^{q} p_j(z)f(z+c_j) - \sum_{j=1}^{q}(p_j(z)f(z+c_j))'}{f^{n_1-1}} \right) = S(r,f), \qquad (3.26)$$

$$T(r,fF) = m(r,fF) = m\left(r, \frac{\alpha_1 \sum_{j=1}^{q} p_j(z)f(z+c_j) - \sum_{j=1}^{q}(p_j(z)f(z+c_j))'}{f^{n_1-2}} \right) = S(r,f), \qquad (3.27)$$

since $n_1 \geq 2 + s = 3$. Combining (3.26) with (3.27), we get

$$T(r,f) \leq T(r,fF) + T\left(r, \frac{1}{F} \right) = T(r,fF) + T(r,F) + O(1) = S(r,f).$$

This is a contradiction.

When $F = 0$, or equivalently

$$\frac{\sum_{i=1}^{p} n_i f^{n_i-n_1}(z)f'(z)}{\sum_{i=1}^{p} f^{n_i-n_1+1}(z)} + \frac{f^{(k+1)}(z)}{f^{(k)}(z)} = \alpha_1;$$

that is,

$$\frac{\sum_{i=1}^{p} n_i f^{n_i-1}(z)f'(z)}{\sum_{i=1}^{p} f^{n_i}(z)} + \frac{f^{(k+1)}(z)}{f^{(k)}(z)} = \alpha_1;$$

then by integrating, it follows that

$$\sum_{i=1}^{p} f^{n_i}(z)f^{(k)}(z) = \tau_1 e^{\alpha_1 z},$$

where τ_1 is a nonzero constant.

It is easy to see $k = 1$ from $s = 1$, so $a = \alpha_k = \alpha_1$, and then from (3.18) we know that

$$f(z) = \widetilde{h}(z)e^{\alpha_1 z}. \tag{3.28}$$

It contradicts with $\sum_{i=1}^{p} f^{n_i}(z)f^{(k)}(z) = \tau_1 e^{\alpha_1 z}$ since $k \geq 1, n_p > n_{p-1} > \cdots > n_1 \geq s + 2 \geq 3$.

If $a \neq \alpha_l (l = 1, \cdots, s)$, noting that $\lambda(f) < \sigma(f)$ and applying Lemma 2.2, we have $b_l = 0$ $(l = 1, \cdots, s)$, a contradiction. Thus, the proof of Theorem 1.2 is completed.

4. Proof of Theorem 1.5

If f is an entire solution with finite order of the differetial-difference equation (1.7), then f must be transcendental. Otherwise, if f is a polynomial, it follows from (1.7), a contradiction. Noting that $\lambda(f) < \sigma(f)$, then we can factorize $f(z)$ as

$$f(z) = \gamma(z)e^{g(z)}, \tag{4.1}$$

where $\gamma(z)$ is the canonical product formed by zeros of $f(z)$ such that $\sigma(\gamma) = \lambda(f) < \sigma(f)$, and we can obtain that $g(z)$ is a polynomial of degree not less than 1 according to the same arguments as shown between (3.13) and (3.15).

Now, returning to (1.7), we assume $H = \sum_{i=1}^{p} f^{n_i}(z)f^{(k)}(z) + \sum_{j=1}^{q} p_j(z)f^{(m_j)}(z + c_j)$. According to an analogous argument as in (3.2)–(3.7) of Case 2 of Theorem 1.2, we immediately obtain

$$L(\sum_{i=1}^{p} f^{n_i}(z)f^{(k)}(z)) = -L(\sum_{j=1}^{q} p_j(z)f^{(m_j)}(z + c_j)). \tag{4.2}$$

Similarly, using parallel analysis as described in (3.10)–(3.18), we have

$$\sum_{j=1}^{q} p_j(z)f^{(m_j)}(z + c_j) = \sum_{l=1}^{s} t_l e^{\alpha_l z}, \tag{4.3}$$

where t_l $(l = 1, 2, ..., s)$ are constants, and $g(z)$ is a polynomial of degree 1. It follows from (4.1) that

$$f(z) = \gamma(z)e^{dz}, \tag{4.4}$$

where d is a nonzero constant. Then, from (4.4), we can obtain

$$f^{(m_j)}(z + c_j) = e^{dc_j}e^{dz} \sum_{i=0}^{m_j} \xi_i \gamma^{(i)}(z + c_j), \tag{4.5}$$

where $\xi_i = \frac{m_j! d^{m_j - i}}{i!(m_j - i)!}$. Combining with (4.3), it follows that

$$e^{dz} \sum_{j=1}^{q} e^{dc_j} p_j(z) \sum_{i=0}^{m_j} \xi_i \gamma^{(i)}(z + c_j) - \sum_{l=1}^{s} t_l e^{\alpha_l z} = 0. \tag{4.6}$$

We shall distinguish the following two cases for discussion:

Case 1. If $d \neq \alpha_l (l = 1, \cdots, s)$, noticing that $\sigma(\gamma) = \lambda(f) < \sigma(f)$ and using Lemma 2.2, we have $t_l = 0$ $(l = 1, \cdots, s)$, a contradiction.

Case 2. If $d = \alpha_k$, for some $k \in \{1, \cdots, s\}$, by (4.6), we have the following form:

$$(\sum_{j=1}^{q} e^{dc_j} p_j(z) \sum_{i=0}^{m_j} \xi_i \gamma^{(i)}(z + c_j) - t_k)e^{\alpha_k z} - \sum_{1 \leq l(\neq k) \leq s} t_l e^{\alpha_l z} = 0. \tag{4.7}$$

Obviously, we obtain $t_l = 0$ for $l \neq k$, and $t_k = \sum_{j=1}^{q} e^{dc_j} p_j(z) \sum_{i=0}^{m_j} \xi_i \gamma^{(i)}(z + c_j)$, where Lemma 2.2 was applied. Since $t_k \neq 0$, (4.3) implies

$$\sum_{j=1}^{q} p_j(z) f^{(m_j)}(z + c_j) = t_k e^{\alpha_k z}. \tag{4.8}$$

On the other hand, (4.4) implies

$$f^{(k)} = e^{dz} \sum_{i=0}^{k} \hat{\xi}_i \gamma^{(i)}(z), \tag{4.9}$$

where $\hat{\xi}_i = \frac{k! d^{k-i}}{i!(k-i)!}$. Substituting (4.4), (4.8), and (4.9) into (1.7), we get

$$t_k e^{\alpha_k z} + \sum_{i=1}^{p} \gamma^{n_i}(z) e^{(n_i+1)\alpha_k z} \sum_{i=0}^{k} \hat{\xi}_i \gamma^{(i)}(z) = \sum_{l=1}^{s} \beta_l e^{\alpha_l z}. \tag{4.10}$$

In fact, we have a claim that $s = p + 1$. Otherwise, if $s > p + 1$, noting that $T(r, \gamma(z)) = S(r, e^z)$ and the assumption $s + 2 \leq n_1 < \cdots < n_p$ implies $(n_1 + 1)\alpha_k, \cdots, (n_p + 1)\alpha_k$ are different from each other. Thus, we can obtain $\beta_l = 0$ for some l by Lemma 2.2, which is a contradiction by our assumption that $\beta_1, \beta_2, \ldots, \beta_s$ are nonzero different constants.

If $s < p + 1$, we can obtain $t_k = 0$ or $\gamma^{n_i}(z) \sum_{i=0}^{k} \hat{\xi}_i \gamma^{(i)}(z) = 0$ for some $i \in \{1, ..., p\}$ according to the above argument. Noting $t_k \neq 0$, then we get

$$\gamma^{n_i}(z) \sum_{i=0}^{k} \hat{\xi}_i \gamma^{(i)}(z) = 0.$$

Since $\gamma(z) \neq 0$, we have $\sum_{i=0}^{k} \hat{\xi}_i \gamma^{(i)}(z) = 0$, and then we get $f^{(k)} = 0$ from (4.9). It indicates that $\sigma(f) = \sigma(f^{(k)}) = 0$, which implies that the order of the left side of (1.7) is 0. However, the order of the right side of (1.7) is 1. It is a contradiction. Hence, $s = p + 1$.

Next, noting both sides of (4.10), we can rearrange the sequence $\{1, \cdots, p+1\}$ to become $(\hat{j}_1, \cdots, \hat{j}_{p+1})$ such that

$$\alpha_{\hat{j}_1} = \alpha_k, \quad \alpha_{\hat{j}_{i+1}} = (n_i + 1)\alpha_k, \ i = 1, \cdots, p. \tag{4.11}$$

Then we have $|\alpha_{\hat{j}_1}| < \cdots < |\alpha_{\hat{j}_{p+1}}|$ since the assumption $s + 2 \leq n_1 < \cdots < n_p$ implies that $|\alpha_1|, \cdots, |\alpha_{p+1}|$ are different constants from (4.11). Let $\alpha_{\hat{j}_1} = \alpha_k$ be the minimum modulus. Thus, we can rewrite (4.10) as follows:

$$\sum_{i=1}^{p}(\gamma^{n_i}(z)\sum_{i=0}^{k}\hat{\xi}_i\gamma^{(i)}(z) - \beta_{\hat{j}_{i+1}})e^{\alpha_{\hat{j}_{i+1}}z} + (t_k - \beta_{\hat{j}_1})e^{\alpha_{\hat{j}_1}z} = 0. \tag{4.12}$$

Applying Lemma 2.2 again, we have

$$t_k = \beta_{\hat{j}_1}, \quad \gamma^{n_i}(z)\sum_{i=0}^{k}\hat{\xi}_i\gamma^{(i)}(z) = \beta_{\hat{j}_{i+1}}, \ i = 1, \cdots, p, \tag{4.13}$$

which means that $\gamma(z)$ reduces to a constant, say γ, and $\sum_{i=0}^{k}\hat{\xi}_i\gamma^{(i)}(z) = d^k\gamma$. Noting the relation $d = \alpha_k = \alpha_{\hat{j}_1}$ and (4.13), we have

$$\gamma = (\frac{\beta_{\hat{j}_{i+1}}}{\alpha_{\hat{j}_1}^{k}})^{\frac{1}{n_i+1}}. \tag{4.14}$$

Further, substituting (4.5), (4.13), and (4.14) into (4.8), we can obtain

$$\sum_{j=1}^{q}e^{\alpha_{\hat{j}_1}c_j}p_j(z)\alpha_{\hat{j}_1}^{m_j} = \beta_{\hat{j}_1}(\frac{\beta_{\hat{j}_{i+1}}}{\alpha_{\hat{j}_1}^{k}})^{\frac{-1}{n_i+1}}, i = 1, \ldots, p.$$

Noting that $d = \alpha_k = \alpha_{\hat{j}_1}$, (4.4), and (4.14), we have solutions of (1.7):

$$f = (\frac{\beta_{\hat{j}_{i+1}}}{\alpha_{\hat{j}_1}^{k}})^{\frac{1}{n_i+1}}e^{\alpha_{\hat{j}_1}z}, i = 1, \ldots, p.$$

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