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Inverse problem for Sturm–Liouville differential operators with two constant delays

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Abstract: In this manuscript, we study nonself-adjoint second-order differential operators with two constant delays. We investigate the properties of the spectral characteristics and the inverse problem of recovering operators from their spectra. An inverse spectral problem is studied of recovering the potential from spectra of two boundary value problems with one common boundary condition. The uniqueness theorem is proved for this inverse problem.

Key words: Sturm–Liouville problems, inverse problems, constant delay, asymptotic form of solution

1. Introduction

The method of separation of variables for solving PDEs with two constant delays naturally led to ODEs with two constant delays inside of the interval, which often appear in mathematics, physics, mechanics, geophysics, electronics, meteorology, etc. The inverse spectral problem consists of recovering operators from their spectral characteristics. The inverse spectral Sturm–Liouville problem can be regarded as three aspects: existence, uniqueness, and reconstruction of the coefficients of given specific properties of eigenvalues and eigenfunctions (see [2, 6, 9, 11, 17] and the references therein).

The interest in differential equations with delay started intensively growing in the twentieth century, stimulated by the appearance of various applications in natural sciences and engineering, including the theory of automatic control, the theory of self-oscillating systems, long-term forecasting in the economy, biophysics, etc. For general background on functional differential equations, we refer to the monographs [8, 12, 16] and the references therein.

There exist a number of results revealing spectral properties of differential operators with delay (see, e.g., [13] and the references therein). At the same time, concerning the inverse spectral theory, its classical methods do not work for such operators as well as for other classes of nonlocal operators, and therefore there are only a few separate results in this direction, which do not form a comprehensive scheme. However, some aspects of inverse problems for differential operators with a constant delay were studied in [3, 7, 10, 14, 15, 18, 19].

Recently, Freiling and Yurko in [7] proved that if the spectra of the problems $L_j(q)$, $j = 0, 1$, coincide with the spectra of $L_j(0)$, $j = 0, 1$, respectively, then $q(x) = 0$ a.e. on $(0, \pi)$. Pikula et al. in [15] and Vladicic and Pikula in [18] studied the reconstruction of the potential function $q(x)$ and the delay point a from the two spectra if $a \in (\pi/2, \pi)$. Buterin and Yurko in [4] and Buterin et al. in [3] studied the inverse Sturm–Liuoville

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differential operator with constant delay. Also, the necessary and sufficient conditions for the solvability of the inverse problem in terms of asymptotics have been proved. More recently, Shahriari et al.* studied the inverse delay Sturm–Liouville problems with transmission conditions inside the interval.

In the present paper, we study an inverse problem of Sturm–Liouville differential operators. In inverse spectral problems, the task is to find a coefficient in the equation using the spectral data. In this note, we discuss the uniqueness of the spectral problem by developing the result of [7] for the inverse Sturm–Liouville problem with two delay constants inside the interval. For this purpose, we study the asymptotic form of solutions, eigenvalues, and eigenfunctions of the problems. We investigate the inverse spectral problem of recovering operators from their two spectra in the Dirichlet–Dirichlet and Dirichlet–Neumann boundary conditions with two constant delays inside the interval.

2. Asymptotic form of solutions and eigenvalues

We consider the boundary value problem $L_j := L_j(q_1(x); q_2(x); a_1; a_2)$, $j = 0, 1$, of the form

$$\ell y := -y''(x) + q_1(x)y(x - a_1) + q_2(x)y(x - a_2) = \lambda y(x), \qquad x \in (0, \pi), \tag{2.1}$$

with the boundary conditions

$$y(0) = y^{(j)}(\pi) = 0, \tag{2.2}$$

where $q_1(x) \in L(a_1, \pi)$, $q_2(x) \in L(a_2, \pi)$, $q_1(x) = 0$ for $x < a_1$, and $q_2(x) = 0$ for $x < a_2$ are complex-valued functions. The coefficients a_1, $a_2 \in [0, \pi)$ are real and assumed to be known a priori and fixed and $a_1 < a_2$. Let $\varphi(x, \lambda)$ be the solution of Eq. (2.1) with $a_1 N < \pi \leq a_1(N + 1)$ and $a_2 M < \pi \leq a_2(M + 1)$ under the initial conditions $\varphi(0, \lambda) = 0$, $\varphi'(0, \lambda) = 1$. For each fixed x, and $j = 0, 1$, the functions $\varphi^j(x, \lambda)$ are entire in λ of order $1/2$. The function $\varphi(x, \lambda)$ is the unique solution of the integral equation

$$\varphi(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_0^x \frac{\sin \rho(x - t)}{\rho} \left(q_1(t)\varphi(t - a_1, \lambda) + q_2(t)\varphi(t - a_2, \lambda)\right) dt, \tag{2.3}$$

with $\rho^2 = \lambda$ and $\rho = \sigma + i\tau$. Solving (2.3) by the method of successive approximations, we get

$$\varphi(x, \lambda) = \varphi_0(x, \lambda) + \varphi_1(x, \lambda) + \cdots + \varphi_N(x, \lambda). \tag{2.4}$$

Thus, we have

$$\varphi_0(x, \lambda) = \frac{\sin \rho x}{\rho}, \tag{2.5}$$

$$\varphi_k(x, \lambda) = \begin{cases} 0, & x \leq ka_1, \\ \int_{ka_1}^x \frac{\sin \rho(x-t)}{\rho} q_1(t)\varphi_{k-1}(t - a_1, \lambda)dt, & ka_1 \leq x \leq ka_2, \\ \int_{ka_1}^x \frac{\sin \rho(x-t)}{\rho} q_1(t)\varphi_{k-1}(t - a_1, \lambda)dt & \\ \quad + \int_{ka_2}^x \frac{\sin \rho(x-t)}{\rho} q_2(t)\varphi_{k-1}(t - a_2, \lambda)dt, & x \geq ka_2, \end{cases} \tag{2.6}$$

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$$\varphi'_k(x,\lambda)=\begin{cases} 0, & x\le ka_1,\\ \int_{ka_1}^{x}\cos\rho(x-t)q_1(t)\varphi_{k-1}(t-a_1,\lambda)dt, & ka_1\le x\le ka_2,\\ \int_{ka_1}^{x}\cos\rho(x-t)q_1(t)\varphi_{k-1}(t-a_1,\lambda)dt \\ \quad +\int_{ka_2}^{x}\cos\rho(x-t)q_2(t)\varphi_{k-1}(t-a_2,\lambda)dt, & x\ge ka_2, \end{cases} \tag{2.7}$$

for $k=1,2,\dots,M$ and

$$\varphi_k(x,\lambda)=\begin{cases} 0, & x\le ka_1,\\ \int_{ka_1}^{x}\frac{\sin\rho(x-t)}{\rho}q_1(t)\varphi_{k-1}(t-a_1,\lambda)dt, & x\ge ka_1, \end{cases} \tag{2.8}$$

$$\varphi'_k(x,\lambda)=\begin{cases} 0, & x\le ka_1,\\ \int_{ka_1}^{x}\cos\rho(x-t)q_1(t)\varphi_{k-1}(t-a_1,\lambda)dt, & x\ge ka_1, \end{cases} \tag{2.9}$$

for $k=M+1,M+2,\dots,N$. Then for $k\ge 1$, by using the formulas (2.5)–(2.7), we calculate

$$\begin{aligned}\varphi_1(x,\lambda)&=\begin{cases} 0, & x\le a_1,\\ \int_{a_1}^{x}\frac{\sin\rho(x-t)}{\rho}q_1(t)\varphi_0(t-a_1,\lambda)dt, & a_1\le x\le a_2,\\ \int_{a_1}^{x}\frac{\sin\rho(x-t)}{\rho}q_1(t)\varphi_0(t-a_1,\lambda)dt \\ \quad +\int_{a_2}^{x}\frac{\sin\rho(x-t)}{\rho}q_2(t)\varphi_0(t-a_2,\lambda)dt, & x\ge a_2, \end{cases}\\ &=\begin{cases} 0, & x\le a_1,\\ \int_{a_1}^{x}\frac{\sin\rho(x-t)}{\rho}q_1(t)\frac{\sin\rho(t-a_1)}{\rho}dt, & a_1\le x\le a_2,\\ \int_{a_1}^{x}\frac{\sin\rho(x-t)}{\rho}q_1(t)\frac{\sin\rho(t-a_1)}{\rho}dt \\ \quad +\int_{a_2}^{x}\frac{\sin\rho(x-t)}{\rho}q_2(t)\frac{\sin\rho(t-a_2)}{\rho}dt, & x\ge a_2, \end{cases}\\ &=\begin{cases} 0, & x\le a_1,\\ \frac{1}{2\rho^2}\left(-\cos\rho(x-a_1)\int_{a_1}^{x}q_1(t)dt+\int_{a_1}^{x}\cos\rho(2t-x-a_1)q_1(t)dt\right), & a_1\le x\le a_2,\\ \frac{1}{2\rho^2}\left(-\cos\rho(x-a_1)\int_{a_1}^{x}q_1(t)dt-\cos\rho(x-a_2)\int_{a_2}^{x}q_2(t)dt\right. \\ \quad \left.+\int_{a_1}^{x}\cos\rho(2t-x-a_1)q_1(t)dt+\int_{a_2}^{x}\cos\rho(2t-x-a_2)q_2(t)dt\right), & x\ge a_2, \end{cases}\end{aligned} \tag{2.10}$$

and

$$\varphi'_1(x,\lambda)=\begin{cases} 0, & x\le a_1,\\ \frac{1}{2\rho}\left(\sin\rho(x-a_1)\int_{a_1}^{x}q_1(t)dt+\int_{a_1}^{x}\sin\rho(2t-x-a_1)q_1(t)dt\right), & a_1\le x\le a_2,\\ \frac{1}{2\rho}\left(\sin\rho(x-a_1)\int_{a_1}^{x}q_1(t)dt+\sin\rho(x-a_2)\int_{a_2}^{x}q_2(t)dt\right. \\ \quad \left.+\int_{a_1}^{x}\sin\rho(2t-x-a_1)q_1(t)dt+\int_{a_2}^{x}\sin\rho(2t-x-a_2)q_2(t)dt\right), & x\ge a_2. \end{cases} \tag{2.11}$$

For $k=2$ from (2.7)–(2.11), we get

$$\varphi_2(x,\lambda)=\begin{cases} 0, & x\le 2a_1,\\ \int_{2a_1}^{x}\frac{\sin\rho(x-t)}{\rho}q_1(t)\varphi_1(t-a_1,\lambda)dt, & 2a_1\le x\le 2a_2,\\ \int_{2a_1}^{x}\frac{\sin\rho(x-t)}{\rho}q_1(t)\varphi_1(t-a_1,\lambda)dt \\ \quad+\int_{2a_2}^{x}\frac{\sin\rho(x-t)}{\rho}q_2(t)\varphi_1(t-a_2,\lambda)dt, & x\ge 2a_2,\end{cases} \tag{2.12}$$

and

$$\varphi_2'(x,\lambda)=\begin{cases} 0, & x\le 2a_1,\\ \int_{2a_1}^{x}\cos\rho(x-t)q_1(t)\varphi_1(t-a_1,\lambda)dt, & 2a_1\le x\le 2a_2,\\ \int_{2a_1}^{x}\cos\rho(x-t)q_1(t)\varphi_1(t-a_1,\lambda)dt \\ \quad+\int_{2a_2}^{x}\cos\rho(x-t)q_2(t)\varphi_1(t-a_2,\lambda)dt, & x\ge 2a_2.\end{cases} \tag{2.13}$$

From Eqs. (2.10)–(2.13) with a simple calculation, we obtain

$$\varphi_2^{(j)}(x,\lambda)=\begin{cases} 0, & x\le 2a_1,\\ O(\rho^{j-3}\exp(|\tau|(x-2a_1))), & x\ge 2a_1,\end{cases}\qquad |\rho|\to\infty, \tag{2.14}$$

where $\tau=\mathrm{Im}\rho$. Using Eqs.(2.5)–(2.11) by induction, it is easy to show that

$$\varphi_k^{(j)}(x,\lambda)=\begin{cases} 0, & x\le ka_1,\\ O(\rho^{j-k-1}\exp(|\tau|(x-ka_1))), & x\ge ka_1,\end{cases}\qquad |\rho|\to\infty. \tag{2.15}$$

Denote $\Delta_j(\lambda):=\varphi^{(j)}(\pi,\lambda)$, $j=0,1$. The functions $\Delta_j(\lambda)$ are entire functions in λ of order $\frac{1}{2}$ and the zeros of $\Delta_j(\lambda)$ coincide with the eigenvalues λ_{nj} of $L_j(q)$. Thus, the function $\Delta_j(\lambda)$ is called the characteristic function for $L_j(q)$. From Eqs. (2.4)–(2.5), (2.10)–(2.11), and (2.13), we obtain the following asymptotic formula for $|\rho|\to\infty$:

$$\begin{aligned}\Delta_0(\lambda)=&\varphi(\pi,\lambda) \qquad (2.16)\\ =&\frac{\sin\rho\pi}{\rho}+\frac{1}{2\rho^2}\Big[-\cos\rho(\pi-a_1)w_1-\cos\rho(\pi-a_2)w_2\\ &+\int_{a_1}^{\pi}\cos\rho(2t-\pi-a_1)q_1(t)dt+\int_{a_2}^{\pi}\cos\rho(2t-\pi-a_2)q_2(t)dt\Big]\\ &+O\left(\frac{\exp(|\tau|(\pi-a_1))}{\rho^3}\right)\end{aligned}$$

and

$$\begin{aligned}\Delta_1(\lambda) =&\varphi'(\pi,\lambda) \\ =&\cos\rho\pi + \frac{1}{2\rho}\Big[\sin\rho(\pi-a_1)w_1 + \sin\rho(\pi-a_2)w_2 \\ &+ \int_{a_1}^{\pi}\sin\rho(2t-\pi-a_1)q_1(t)dt + \int_{a_2}^{\pi}\sin\rho(2t-\pi-a_2)q_2(t)dt\Big] \\ &+ O\left(\frac{\exp(|\tau|(\pi-a_1))}{\rho^2}\right),\end{aligned} \tag{2.17}$$

where $w_1 := \int_{a_1}^{\pi} q_1(t)dt$ and $w_2 := \int_{a_2}^{\pi} q_2(t)dt$. Using (2.16) and (2.17), by the well-known method used in [6], we get the asymptotic formula for the eigenvalues $\lambda_{nj} = \rho_{nj}^2$ as $n \to \infty$:

$$\begin{aligned}\rho_{n0} &= n + \frac{1}{2\pi n}\left[w_1\cos na_1 + w_2\cos na_2\right] + o\left(\frac{1}{n}\right), \\ \rho_{n1} &= n - \frac{1}{2} + \frac{1}{2\pi n}\left[w_1\cos\left(n-\frac{1}{2}\right)a_1 + w_2\cos\left(n-\frac{1}{2}\right)a_2\right] + o\left(\frac{1}{n}\right).\end{aligned} \tag{2.18}$$

Lemma 2.1 *The specification of the spectrum* $\{\lambda_{nj}\}$, $n \geq 1$ *and* $j = 0,1$, *uniquely determines the characteristic function* $\Delta_j(\lambda)$ *by the formulas*

$$\Delta_0(\lambda) = \pi\prod_{n=1}^{\infty}\frac{\lambda_{n0}-\lambda}{n^2} \text{ and } \Delta_1(\lambda) = \prod_{n=1}^{\infty}\frac{\lambda_{n1}-\lambda}{(n-\frac{1}{2})^2}. \tag{2.19}$$

Proof By Hadamard's factorization theorem [5, p. 289], $\Delta_0(\lambda)$ is uniquely determined up to a multiplicative constant by its zeros:

$$\Delta_0(\lambda) = C\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{\lambda_{n0}}\right) \tag{2.20}$$

(the case when $\Delta_0(0) = 0$ requires minor modifications). Consider the function

$$\tilde{\Delta}_0(\lambda) := \frac{\sin\rho\pi}{\rho} = \pi\prod_{n=1}^{\infty}\left(1-\frac{\lambda}{n^2}\right);$$

then

$$\frac{\Delta_0(\lambda)}{\tilde{\Delta}_0(\lambda)} = \frac{C}{\pi}\prod_{n=1}^{\infty}\frac{n^2}{\lambda_{n0}}\prod_{n=1}^{\infty}\left(1+\frac{\lambda_{n0}-n^2}{n^2-\lambda}\right).$$

Taking (2.16) and (2.18) into account, we calculate

$$\lim_{\lambda\to-\infty}\frac{\Delta_0(\lambda)}{\tilde{\Delta}_0(\lambda)} = 1, \qquad \lim_{\lambda\to-\infty}\left(1+\frac{\lambda_{n0}-n^2}{n^2-\lambda}\right) = 1$$

and hence

$$C = \pi\prod_{n=1}^{\infty}\frac{\lambda_{n0}}{n^2}.$$

Substituting this into (2.20), we arrive at (2.19). The proof of $\Delta_1(\lambda)$ is the same as that of $\Delta_0(\lambda)$. □

Denote

$$\mathcal{L}(\rho) := \Delta_1(\lambda) + i\rho\Delta_0(\lambda).$$

The function $\mathcal{L}(\rho)$ is entire in ρ, and $\mathcal{L}(\rho)$ is the characteristic function for the Regge-type boundary value problem (2.1) with the boundary conditions $y(0) = y'(\pi) + i\rho y(\pi) = 0$. It follows from (2.4) that

$$\mathcal{L}(\rho) = \mathcal{L}_0(\rho) + \mathcal{L}_1(\rho) + \mathcal{L}_2(\rho) + \cdots + \mathcal{L}_N(\rho), \tag{2.21}$$

where $\mathcal{L}_k(\rho) = \varphi'_k(\pi,\lambda) + i\rho\varphi_k(\pi,\lambda)$. In particular, $\mathcal{L}_0(\rho) = \exp(i\rho\pi)$. Using (2.6) and (2.7), we get

$$\mathcal{L}_k(\rho) = \begin{cases} \int_{ka_1}^{\pi} \exp(i\rho(\pi-t))q_1(t)\varphi_{k-1}(t-a_1,\lambda)dt \\ \quad + \int_{ka_2}^{\pi} \exp(i\rho(\pi-t))q_2(t)\varphi_{k-1}(t-a_2,\lambda)dt, & \text{for } k = 1,2,\ldots,M, \\ \int_{ka_1}^{\pi} \exp(i\rho(\pi-t))q_1(t)\varphi_{k-1}(t-a_1,\lambda)dt, & \text{for } k = M+1, M+2,\ldots,N. \end{cases} \tag{2.22}$$

Moreover, it follows from (2.10) and (2.11) that

$$\begin{aligned} \mathcal{L}_1(\rho) = {} & \frac{\exp(i\rho(\pi-a_1))}{2i\rho}w_1 + \frac{\exp(i\rho(\pi-a_2))}{2i\rho}w_2 \\ & - \frac{\exp(i\rho(\pi+a_1))}{2i\rho}\int_{a_1}^{\pi}\exp(-2i\rho t)q_1(t)dt - \frac{\exp(i\rho(\pi+a_2))}{2i\rho}\int_{a_2}^{\pi}\exp(-2i\rho t)q_2(t)dt. \end{aligned} \tag{2.23}$$

Taking (2.15) and (2.22) into account, we get

$$\begin{aligned} \mathcal{L}_k(\rho) = O\Bigg(& \frac{\exp i\rho(\pi+(k-1)a_1)}{\rho^k}\Bigg[\int_{ka_1}^{\pi}\exp(-i\rho(2t-a_1))q_1(t)dt \\ & + \int_{ka_2}^{\pi}\exp(-i\rho(2t-a_2))q_2(t)dt\Bigg]\Bigg), \end{aligned} \tag{2.24}$$

for $\mathrm{Im}\rho > 0$, $|\rho| \to \infty$, and $k \geq 1$.

3. The uniqueness theorem

Let $\{\tilde{\lambda}_{nj}\}_{n\geq 1}$, $j = 0,1$, be the eigenvalues of the boundary value problems $\tilde{L}_j := L_j(\tilde{q}_1;\tilde{q}_2;a_1;a_2)$ with $\tilde{q}_1(x) = 0$ and $\tilde{q}_2(x) = 0$. Then $\tilde{\lambda}_{n0} = n^2$ and $\tilde{\lambda}_{n1} = {\rho_n^{\circ}}^2 = \left(n-\frac{1}{2}\right)^2$. Denote by $\tilde{\mathcal{L}}(\rho)$ the characteristic function of $\tilde{L} := L(\tilde{q}_1;\tilde{q}_2;a_1;a_2)$. Clearly, $\tilde{\mathcal{L}}(\rho) = \exp(i\rho\pi)$.

Theorem 3.1 (Main theorem) *If $\tilde{\lambda}_{nj} = \lambda_{nj}$ for all $n \geq 1$ and $j = 0,1$, then $q_1(x) = 0$ a.e. on (a_1,π) and $q_2(x) = 0$ a.e. on $(a_2,\pi]$.*

Proof (1) By virtue of Lemma 2.1, one has

$$\Delta_0(\lambda) = \frac{\sin\rho\pi}{\rho}, \qquad \Delta_1(\lambda) = \cos\rho\pi$$

and consequently $\mathcal{L}(\rho) = \exp(i\rho\pi)$. Using (2.21), we get

$$\mathcal{L}_1(\rho) = -\mathcal{L}^+(\rho), \tag{3.1}$$

where

$$\mathcal{L}^{+}(\rho)=\sum_{k=2}^{N}\mathcal{L}_k(\rho),\quad \text{for } k\geq 2 \qquad \text{and } \mathcal{L}^{+}(\rho)=0,\quad \text{for } k=1.$$

It follows from (2.18) that $w_1\cos na_1+w_2\cos na_2=0$. The functions $\cos na_1$ and $\cos na_2$, $(a_1>a_2)$ are linear independent. We get $w_1=0$ and $w_2=0$. Together with (2.23) this yields

$$\begin{aligned}\mathcal{L}_1(\rho)=&-\frac{\exp(i\rho\pi)}{2i\rho}\int_{a_1}^{\pi}\exp(-i\rho(2t-a_1))q_1(t)dt\\&-\frac{\exp(i\rho\pi)}{2i\rho}\int_{a_2}^{\pi}\exp(-i\rho(2t-a_2))q_2(t)dt.\end{aligned}\tag{3.2}$$

(2) Let $N=1$, i.e. $a_1\in(\frac{\pi}{2},\pi)$, and then $\mathcal{L}^{+}(\rho)=0$. From (3.1), we see that $\mathcal{L}_1(\rho)=0$. Using (3.2), we get

$$\frac{\exp(i\rho\pi)}{2i\rho}\left[\int_{a_1}^{\pi}\exp(-i\rho(2t-a_1))q_1(t)dt+\int_{a_2}^{\pi}\exp(-i\rho(2t-a_2))q_2(t)dt\right]=0.$$

By rewriting Eq. (3.2) and with the assumption of $q_2(x)$, we obtain

$$\frac{\exp(i\rho\pi)}{2i\rho}\left[\int_{a_1}^{\pi}\exp\left(-2i\rho t\right)\exp(i\rho a_1)q_1(t)dt+\int_{a_1}^{\pi}\exp\left(-2i\rho t\right)\exp(i\rho a_2)q_2(t)dt\right]=0$$

or

$$\int_{a_1}^{\pi}\exp\left(-2i\rho t\right)\left(\exp(i\rho a_1)q_1(t)+\exp(i\rho a_2)q_2(t)\right)dt=0.$$

From the completeness of $\exp\left(-2i\rho t\right)$ on (a_1,π), we obtain

$$\exp(i\rho a_1)q_1(t)+\exp(i\rho(a_2))q_2(t)=0\qquad\text{a.e. on } (a_1,\pi).$$

The functions $\exp(i\rho a_1)$ and $\exp(i\rho a_2)$ are linear independent in ρ, so we get $q_1(x)=0$ and $q_2(x)=0$ a.e. on (a_1,π). Thus, Theorem 3.1 is proved for $N=1$. □

Below, we will assume that $N\geq 2$.

Lemma 3.2 *If $q_1(x)=0$ and $q_2(x)=0$ a.e. on $(2a_1,\pi)$, then $q_1(x)=0$ and $q_2(x)=0$ a.e. on (a_1,π).*

Proof For the proof, we consider two cases:

- $2a_1\leq a_2$. The proof is similar to [7, Lemma 2].
- $2a_1>a_2$. By virtue of (2.24), for $q_1(x)=0$ and $q_2(x)=0$ a.e. on $(2a_1,\pi)$, we get $\mathcal{L}_k(\rho)=0$ for $k\geq 2$ and hence $\mathcal{L}^{+}(\rho)=0$. From Eq. (3.1), we get $\mathcal{L}_1(\rho)=0$; consequently, $q_1(x)=0$ and $q_2(x)=0$ a.e. on (a_1,π).

□

(3) For definiteness, we assume that $N=2S+1$, $S\geq 1$, i.e. N is odd. (The case when N is even requires minor technical modifications.)

Lemma 3.3 *Fix* $\nu = \overline{0, 2S-1}$. *If* $q_1(x)+q_2(x) = 0$ *a.e. on the interval* $(\pi - \nu a_1/2, \pi)$, *then* $q_1(x)+q_2(x) = 0$ *a.e. on the interval* $(\pi - (\nu+1)a_1/2, \pi)$.

Proof Since $\pi - \nu a_1/2 > 2a_1$, from (2.24) we have

$$\mathcal{L}_2(\rho) = \begin{cases} O\left(\frac{\exp(i\rho(\pi+a_1))}{\rho^2} \int_{2a_1}^{\pi-\nu a_1/2} \exp(-i\rho(2t-a_1))q_1(t)dt\right), & \pi - \nu a_1/2 < 2a_2; \\ O\left(\frac{\exp(i\rho(\pi+a_1))}{\rho^2} \left[\int_{2a_1}^{\pi-\nu a_1/2} \exp(-i\rho(2t-a_1))q_1(t)dt \right.\right. & \\ \qquad \left.\left. + \int_{2a_2}^{\pi-\nu a_1/2} \exp(-i\rho(2t-a_2))q_2(t)dt\right]\right), & \pi - \nu a_1/2 > 2a_2. \end{cases} \tag{3.3}$$

In the integrals $2t - \pi - 2a_1 \in (2a_1 - \pi, \pi - (\nu+2)a_1)$, where $\pi - (\nu+2)a_1 \geq \pi - Na_1 > 0$. This yields

$$\mathcal{L}_2(\rho) = O\left(\frac{1}{\rho^2}\exp(-i\rho(\pi-(\nu+2)a_1))\right), \qquad \text{Im}\rho \geq 0, \quad |\rho| \to \infty. \tag{3.4}$$

For $k \geq 2$, the functions $\mathcal{L}_k(\rho)$ have less growth than the right-hand side in (3.4). This means that

$$\mathcal{L}^+(\rho) = O\left(\frac{1}{\rho^2}\exp(-i\rho(\pi-(\nu+2)a_1))\right), \qquad \text{Im}\rho \geq 0, \quad |\rho| \to \infty. \tag{3.5}$$

It follows from (3.1), (3.2), and (3.5) that

$$\begin{aligned} \mathcal{L}_1(\rho) = &- \frac{\exp(i\rho(\pi+a_1))}{2i\rho} \int_{a_1}^{\pi-\nu a_1/2} \exp(-2i\rho t)q_1(t)dt \\ &- \frac{\exp(i\rho(\pi+a_2))}{2i\rho} \int_{a_2}^{\pi-\nu a_1/2} \exp(-2i\rho t)q_2(t)dt \\ =& O\left(\frac{1}{\rho}\exp(-i\rho(\pi-(\nu+2)a_1))\right), \quad \text{Im}\rho \geq 0, \ |\rho| \to \infty. \end{aligned}$$

It follows from the above equation that

$$\begin{aligned} &\exp(i\rho(2\pi-(\nu+1)a_1)) \int_{a_1}^{\pi-\nu a_1/2} \exp(-2i\rho t)\exp(i\rho a_1)q_1(t)dt \\ &\quad + \exp(i\rho(2\pi-(\nu+1)a_1)) \int_{a_2}^{\pi-\nu a_1/2} \exp(-2i\rho t)\exp(i\rho a_2)q_2(t)dt \\ &= O\left(\frac{1}{\rho}\right), \quad \text{Im}\rho \geq 0, \ |\rho| \to \infty. \end{aligned}$$

We have

$$\begin{aligned} &\int_{a_1}^{\pi-\nu a_1/2} \exp(-2i\rho t)\exp(i\rho a_1)q_1(t)dt + \int_{a_2}^{\pi-\nu a_1/2} \exp(-2i\rho t)\exp(i\rho a_2)q_2(t)dt \\ &= O\left(\exp(i\rho(-2\pi+(\nu+1)a_1))\right), \quad \text{Im}\rho \geq 0, \ |\rho| \to \infty. \end{aligned} \tag{3.6}$$

Let us define the function

$$F(\rho) := \exp(i\rho(2\pi-(\nu+2)a_1)) \int_{\pi-(\nu+1)a_1/2}^{\pi-\nu a_1/2} \exp(-2i\rho t)\big[\exp(i\rho a_1)q_1(t) + \exp(i\rho a_2)q_2(t)\big]dt. \tag{3.7}$$

The function $F(\rho)$ is entire in ρ. Clearly, $F(\rho) = O(1)$ for $\text{Im}\rho \le 0$. From (3.6) and (3.7), we obtain $F(\rho) = O(1)$ for $\text{Im}\rho \ge 0$. Using Liouville's theorem (see [5, p. 77]), $F(\rho) = C$-const. Since $F(\rho) = o(1)$ for real ρ, $|\rho| \to \infty$. We get $F(\rho) = 0$. From (3.7), we obtain

$$\int_{\pi-(\nu+1)a_1/2}^{\pi-\nu a_1/2} \exp(-2i\rho t)\big[\exp(i\rho a_1)q_1(t) + \exp(i\rho a_2)q_2(t)\big]dt = 0.$$

The completeness of $\exp(-2i\rho t)$ in the interval $(\pi-(\nu+1)a_1/2, \pi-\nu a_1/2)$ concludes that $\exp(i\rho a_1)q_1(x) + \exp(i\rho a_2)q_2(x) = 0$ a.e. on the interval. Thus, we get $q_1(x) = 0$ and $q_2(x) = 0$ a.e. in $(\pi-(\nu+1)a_1/2, \pi-\nu a_1/2)$. □

(4) Applying Lemma 3.3 successively for $\nu = 0, 1, \ldots, 2S-1$, we obtain $q_1(x) = 0$ and $q_2(x) = 0$ a.e. on the interval $(\pi - Sa_1, \pi)$. We note that it is not possible to use Lemma 3.3 for $\nu = 2S$, and we need the following lemma for this fact.

Lemma 3.4 *If $q_1(x) = 0$ and $q_2(x) = 0$ a.e. on the interval $(\pi - Sa_1, \pi)$, then $q_1(x) = 0$ and $q_2(x) = 0$ a.e. on the interval $((S+2)a_1/2, \pi)$.*

Proof For $k = S+2$, we have $\pi - Sa_1 - ka_1 \le \pi - (N+1)a_1 \le 0$. Consequently, $\mathcal{L}_k(\rho) = 0$ for $k \ge S+2$. According to (2.24), for $k = \overline{2, S+1}$, we get

$$\mathcal{L}_k(\rho) = O\left(\frac{\exp i\rho(\pi-(k-1)a_1)}{\rho^k}\left[\int_{ka_1}^{\pi-Sa_1} \exp(-i\rho(2t-a_1))q_1(t)dt + \int_{ka_2}^{\pi} \exp(-i\rho(2t-a_2))q_2(t)dt\right]\right), \tag{3.8}$$

and we note that $2t - \pi - ka_1 \le 0$. It follows that

$$\mathcal{L}_k(\rho) = O\left(\frac{1}{\rho^k}\exp(i\rho(\pi - ka_1))\right), \quad \text{Im}\rho \ge 0,\ |\rho| \to \infty\ k = \overline{2, S+1}, \tag{3.9}$$

and hence

$$\mathcal{L}^+(\rho) = O\left(\frac{1}{\rho^2}\exp(i\rho(\pi - (S+1)a_1))\right), \quad \text{Im}\rho \ge 0,\ |\rho| \to \infty. \tag{3.10}$$

Applying (3.1), (3.2), and (3.10), we have

$$\exp(i\rho(\pi + a_1)\left[\int_{ka_1}^{\pi-Sa_1} \exp(-i\rho(2t-a_1))q_1(t)dt + \int_{ka_2}^{\pi} \exp(-i\rho(2t-a_2))q_2(t)dt\right] = O\left(\frac{1}{\rho}\exp(i\rho(\pi-(S+1)a_1))\right), \quad \text{Im}\rho \ge 0,\ |\rho| \to \infty.$$

Rewrite the equation as follows:

$$\exp(i\rho((S+1)a_1)\int_{a_1}^{\pi-Sa_1} \exp(-2i\rho t)\big[\exp(i\rho a_1)q_1(t) + \exp(i\rho a_2)q_2(t)\big]dt = O\left(\frac{1}{\rho}\right), \quad \text{Im}\rho \ge 0,\ |\rho| \to \infty. \tag{3.11}$$

We thus get the equation

$$\int_{a_1}^{\pi-Sa_1} \exp(-2i\rho t)\big[\exp(i\rho a_1)q_1(t)+\exp(i\rho a_2)q_2(t)\big]dt$$
$$= O\left(\frac{\exp(-i\rho((S+1)a_1)}{\rho}\right), \quad \mathrm{Im}\rho \geq 0,\ |\rho| \to \infty. \tag{3.12}$$

Denote

$$F_1(\rho) := \exp(i\rho((S+1)a_1) \int_{(S+2)a_1/2}^{\pi-Sa_1} \exp(-2i\rho t)\big[\exp(i\rho a_1)q_1(t)+\exp(i\rho a_2)q_2(t)\big]dt.$$

The function $F_1(\rho)$ is entire in ρ, and $F_1(\rho) = O(1)$ for $\mathrm{Im}\rho \leq 0$. By referring to and reviewing (3.11) and (3.12), we get $F_1(\rho) = O(1)$ for $\mathrm{Im}\rho \geq 0$. Therefore, $F_1(\rho) = C$. Since $F_1(\rho) = o(1)$ for real ρ, as $|\rho| \to \infty$, it follows that $F_1(\rho) = 0$, i.e.

$$\int_{(S+2)a_1/2}^{\pi-Sa_1} \exp(-2i\rho t)(\exp(i\rho a_1)q_1(t)+\exp(i\rho a_2)q_2(t))dt = 0.$$

From the completeness of $\exp(-2i\rho t)$ on $((S+2)a_1/2, \pi - Sa_1)$, we get $\exp(i\rho a_1)q_1(t)+\exp(i\rho a_2)q_2(t) = 0$ a.e. on the interval. Then $q_1(x) = 0$ and $q_2(x) = 0$ a.e. on (a_1, π). □

(5) If $S = 1$ or $S = 2$, then from Lemmas 3.3 and 3.4, we have proved that $q_1(x) = 0$ and $q_2(x) = 0$ a.e. on (a_1, π) a.e. on $(2a_1, \pi)$. According to Lemma 3.2, we conclude that $q_1(x) = 0$ and $q_2(x) = 0$ a.e. on (a_1, π). Thus, Theorem 3.1 is proved for $S = 1$ and $S = 2$.

Let $S \geq 3$. Fix $\nu = \overline{5, S+2}$. Denote $u := [(\nu+1)/2]$. Clearly, $u < \nu$.

Lemma 3.5 *If $q_1(x) = 0$ and $q_2(x) = 0$ a.e. on (a_1, π) on the interval $(\nu a_1/2, \pi)$, then $q_1(x) = 0$ and $q_2(x) = 0$ a.e. on (a_1, π) on the interval $(ua_1/2, \pi)$.*

Proof Since $\nu/2 - k \leq \nu/2 - u \leq 0$ for $k > u$, it follows that $\mathcal{L}_k(\rho) = 0$ for $k > u$. From Eq. (2.24) and assumption of this lemma,

$$\mathcal{L}_k(\rho) = O\left(\frac{1}{\rho^k}\int_{ka_1}^{\nu a_1/2} \exp(-i\rho(2t-\pi-(k-1)a_1))\big[\exp(i\rho a_1)q_1(t)+\exp(i\rho a_2)q_2(t)\big]dt\right),$$
$$\mathrm{Im}\rho \geq 0,\ |\rho| \to \infty.$$

Since $2t - \pi - ka_1 \leq 0$, we get

$$\mathcal{L}_k(\rho) = O\left(\frac{1}{\rho^k}\exp(i\rho(\pi - ka_1))\right), \quad \mathrm{Im}\rho \geq 0,\ |\rho| \to \infty,\ k = \overline{2, u-1},$$

and hence

$$\mathcal{L}^+(\rho) = O\left(\frac{1}{\rho^2}\exp(i\rho(\pi-(u-1)a_1))\right), \quad \mathrm{Im}\rho \geq 0,\ |\rho| \to \infty. \tag{3.13}$$

From Eqs. (3.1), (3.2), and (3.13), we obtain

$$\exp(i\rho(u-1)a_1)\int_{a_1}^{\nu a_1/2}\exp(-2i\rho t)(\exp(i\rho a_1)q_1(t)+\exp(i\rho a_2)q_2(t))dt$$
$$=O\left(\frac{1}{\rho}\right),\quad \text{Im}\rho\geq 0,\ |\rho|\to\infty. \tag{3.14}$$

Moreover,

$$\exp(i\rho(u-1)a_1)\int_{a_1}^{ua_1/2}\exp(-2i\rho t)(\exp(i\rho a_1)q_1(t)+\exp(i\rho a_2)q_2(t))dt$$
$$=O\left(\exp(-i\rho ua_1)\right)\quad \text{Im}\rho\geq 0,\ |\rho|\to\infty. \tag{3.15}$$

If $ua_1/2<a_2$, then $q_1(t)+q_2(t)=q_1(t)$. Denote

$$F_2(\rho):=\exp(i\rho(u-1)a_1)\int_{ua_1/2}^{\nu a_1/2}\exp(-2i\rho t)(\exp(i\rho a_1)q_1(t)+\exp(i\rho a_2)q_2(t))dt.$$

The function $F_2(\rho)$ is entire in ρ, and $F_2(\rho)=O(1)$ for $\text{Im}\rho\leq 0$. By referring to and reviewing (3.14) and (3.15), we get $F_2(\rho)=O(1)$ for $\text{Im}\rho\geq 0$. Therefore, $F_2(\rho)=C$. Since $F_2(\rho)=o(1)$ for real ρ, as $|\rho|\to\infty$, it follows that $F_2(\rho)=0$, i.e. consequently $\exp(i\rho a_1)q_1(t)+\exp(i\rho a_2)q_2(t)=0$ a.e. on the interval $(ua_1/2,\nu a_1/2)$. Thus, we get $q_1(x)=0$ and $q_2(x)=0$ a.e. on the interval. □

Applying Lemma 3.5 several times successively starting from $\nu=S+2$, we obtain the relation $q_1(x)=0$ and $q_2(x)=0$ a.e. on $(2a_1,\pi)$. Then, by virtue of Lemma 3.2, $q_1(x)=0$ and $q_2(x)=0$ a.e. on the interval (a_1,π).

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References

[1] Bondarenko N, Yurko VA. On recovering Sturm-Liouville differential operators with deviating argument. 2018 arXiv preprint, arXiv:1802.02321.

[2] Borg G. Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. Acta Mathematica 1945; 78: 1-96 (in German).

[3] Buterin SA, Pikula M, Yurko VA. Sturm-Liouville differential operators with deviating argument. Tamkang Journal of Mathematics 2017; 48 (1): 61-71.

[4] Buterin SA, Yurko VA. An inverse spectral problem for Sturm–Liouville operators with a large constant delay. Analysis and Mathematical Physics 2017; 71 (3): 1521–1529. doi: 10.1007/s13324-017-0176-6

[5] Conway JB. Functions of One Complex Variable, Vol. 1. 2nd ed. New York, NY, USA: Springer-Verlag, 1995.

[6] Freiling G, Yurko VA. Inverse Sturm–Liouville Problems and Their Applications. New York, NY, USA: NOVA Science Publishers, 2001.

[7] Freiling G, Yurko VA. Inverse problems for Sturm–Liouville differential operators with a constant delay. Applied Mathematics Letters 2012; 25: 1999-2004.

[8] Hale J. Theory of Functional-Differential Equations. New York, NY, USA: Springer-Verlag, 1977.

[9] Levitan BM. Inverse Sturm-Liouville Problems. Utrecht, the Netherlands: VNU Science Press, 1987.

[10] Mosazadeh S. On the solution of an inverse Sturm–Liouville problem with a delay and eigenparameter-dependent boundary conditions. Turkish Journal of Mathematics 2018; 42: 3090-3100.

[11] Mosazadeh S, Jodayree Akbarfam A. On Hochstadt–Lieberman theorem for impulsive Sturm–Liouville problems with boundary conditions polynomially dependent on the spectral parameter. Turkish Journal of Mathematics 2018; 42: 3002-3009.

[12] Myshkis AD. Linear Differential Equations with a Delay Argument. Moscow, USSR: Nauka, 1972 (in Russian).

[13] Norkin SB. Second Order Differential Equations with a Delay Argument. Moscow, USSR: Nauka, 1965 (in Russian).

[14] Pikula M. Determination of a Sturm–Liouville-type differential operator with delay argument from two spectra. Matematicki Vesnik 1991; 43 (3-4): 159-171.

[15] Pikula M, Vladicic V, Markovic O. A solution to the inverse problem for the Sturm–Liouville-type equation with a delay. Filomat 2013; 27 (7): 1237-1245.

[16] Smith H. An Introduction to Delay Differential Equations with Sciences Applications to the Life. New York, NY, USA: Springer, 2011.

[17] Teschl G. Mathematical Methods in Quantum Mechanics, with Applications to Schrödinger Operators. Graduate Studies in Mathematics. Providence, RI, USA: American Mathematical Society, 2009.

[18] Vladicic V, Pikula M. An inverse problems for Sturm–Liouville-type differential equation with a constant delay. Sarajevo Journals of Mathematics 2016; 12 (24) : 83-88.

[19] Yang CF. Trace and inverse problem of a discontinuous Sturm–Liouville operator with retarded argument. Journal of Mathematical Analysis and Applications 2012; 395: 30-41.

[20] Yurko VA, Yang CF. Recovering differential operators with nonlocal boundary conditions. Analysis and Mathematical Physics 2016; 6 (4): 315-326.