Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 977 – 984
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doi:10.3906/mat-1811-35

On band operators

Bahri TURAN[1*], **Kazım ÖZCAN**[2]
[1]Department of Mathematics, Faculty of Science, Gazi University, Ankara, Turkey
[2]Gazi Lisesi, Ankara, Turkey

Received: 08.11.2018 • **Accepted/Published Online:** 19.02.2019 • **Final Version:** 27.03.2019

Abstract: Let G and H be Archimedean Riesz spaces. We study the properties of band operators and inverse band operators from G to H and investigate their relations to some well-known classes of operators. Then, we show that under some assumptions on the Riesz spaces G or H, if S is a bijective band operator from G into H then $S^{-1}: H \to G$ is a band operator. Additionally, we give some conditions under which a band operator becomes order bounded.

Key words: Band operator, inverse band operator, dijointness preserving operator

1. Introduction

All vector spaces are considered over the reals only. The linear subspace A of a Riesz space G is called an *ideal* whenever it follows from $x \in A$, $y \in G$ and $|y| \leq |x|$ that $y \in A$. In a Riesz space, a net (x_α) is said to be *order convergent to* x (in symbols, $x_\alpha \xrightarrow{o} x$) whenever there exists another net (y_α) satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for all α. A subset B of a Riesz space is said to be *order closed* whenever $(x_\alpha) \subseteq B$ and $x_\alpha \xrightarrow{o} x$ imply $x \in B$. An order closed ideal is referred to as a *band*. An ideal A is a band if and only if $(x_\alpha) \subseteq A$ and $0 \leq x_\alpha \uparrow x$ imply $x \in A$. Let A be a nonempty subset of a Riesz space G. Then the band generated by A is the smallest band that contains A. The bands generated by a set A and an element x will be denoted by B_A and B_x, respectively.

Let G and H be Riesz spaces. The set of all order bounded operators which map G into H, is denoted by $L_b(G,H)$. If H is equivalent to G, then $L_b(G,H)$ is denoted by $L_b(G)$. An operator S, between G and H, is said to be *Riesz homomorphism or lattice homomorphism* whenever $S(x \vee y) = Sx \vee Sy$ (or equivalently $S(x^+) = S(x)^+$ for all $x \in G$) holds for all $x, y \in G$. An operator $S: G \to H$ is called *disjointness preserving* if $Sx \perp Sy$ for all $x, y \in G$ satisfying $x \perp y$ (i.e. $|x| \wedge |y| = 0$). A positive operator $S: G \to H$ is disjointness preserving if and only if S is a Riesz homomorphism. An operator $\pi: G \to G$ on a Riesz space is said to be *band preserving* whenever $\pi(B) \subseteq B$ holds for each band B of G. π is a band preserving operator if and only if $\pi(x) \perp y$ whenever $x \perp y$ in G. A band preserving and order bounded operator π is called *orthomorphism* of G and the set of all orthomorphisms of G is denoted by $Orth(G)$. Every orthomorphism is disjointness preserving and order continuous. The order ideal generated by the identity operator I in $Orth(G)$ is called the *ideal centre* of G and is denoted by $Z(G)$.

In the problem section of [6], Abramovich raises the following question: if G, H are Riesz spaces and

*Correspondence: bturan@gazi.edu.tr
2010 *AMS Mathematics Subject Classification:* 47B60, 47B65, 46A40

$S : G \to H$ is invertible and disjointness preserving, is $S^{-1} : H \to G$ disjointness preserving as well? In [1, 2], Abramovich and Kitover constructed an invertible disjointness preserving operator S on a normed lattice such that S^{-1} is not disjointness preserving. Under some conditions, the Riesz spaces G or H, some positive solutions were given in [1, 2, 5, 7]. In [2], Abramovich and Kitover asked the following question: Let $S : G \to G$ be a bijective band preserving operator on a Riesz space. Is $S^{-1} : G \to G$ also band preserving? It was shown that this question has an affirmative answer in the following case: G has the principal projection property or G is relatively uniformly complete [7]. In this paper, we consider the definitions of band operator and inverse band operator and then, we study their properties. Next, we give relationship of the band operator with disjointness preserving operator and band preserving operator. Additionally, we investigate that the inverse of an invertible band operator is also a band operator and give some conditions under which band operator becomes order bounded.

We refer to [3, 9, 11, 15] for definitions and notations which are not explained here. All Riesz spaces in this paper are Archimedean.

2. Band Operators

Now, we recall the definitions of the band operator and inverse band operator which are similar definitions to the definitions of ideal operator and inverse ideal operator. The reader is referred to [13] for information about ideal operators and inverse ideal operators. Uyar introduced the definition of the band operator in [14]; with the help of it, she solved two problems in the theory of disjointnesss preserving operators. In [12], some properties were proved for band operators and inverse band operators, including the relations between band operators and disjointness preserving operators.

Definition 1 *Let S be an operator between Riesz spaces G and H.*
(i) S is called a band operator if $S(B)$ is a band in H for each band B in G.
(ii) S is called an inverse band operator if $S^{-1}(B)$ is a band in G for each band B in H.

Both the sets of band operators and inverse band operators are not vector spaces.

Example 2 *Consider the two operators I and $S : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $S(x, y) = (y, 0)$ and $I(x, y) = (x, y)$ for $(x, y) \in \mathbb{R}^2$. Then I and S are both band operators. But $I + S$ is not a band operator. Observe that $B = \{(0, y) : y \in \mathbb{R}\}$ is a band in $\mathbb{R}^2$, but $(I + S)(B) = \{(y, y) : y \in \mathbb{R}\}$ is not a band in $\mathbb{R}^2$.*

The operator $S : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $S(x, y) = (-y, 0)$ and the operator I are inverse band operators, but $I + S$ is not an inverse band operator. Observe that $B = \{(0, y) : y \in \mathbb{R}\}$ is a band in $\mathbb{R}^2$, but $(I + S)^{-1}(B) = \{(y, y) : y \in \mathbb{R}\}$ is not a band in $\mathbb{R}^2$.

Lemma 3 *Let S be an operator between Riesz spaces G and H. Then the following statements hold:*
(i) S is a band operator if and only if $B_{S(A)} \subseteq S(B_A)$ for every set $A \subseteq G$.
(ii) S is an inverse band operator if and only if $S(B_A) \subseteq B_{S(A)}$ for every set $A \subseteq G$.
In addition, let S be a bijective. Then,
(iii) A necessary and sufficient condition for S to be a band operator is that S^{-1} is an inverse band operator.
(iv) A necessary and sufficient condition for S to be an inverse band operator is that S^{-1} is a band operator.

Proof The proof follows from the definitions. □

Definition 4 *Let S be an operator between Riesz spaces G and H.*
(i) S is called a principal band operator if $B_{Sx} \subseteq S(B_x)$ for each $x \in G$.
(ii) S is called an inverse principal band operator if $S(B_x) \subseteq B_{Sx}$ for each $x \in G$.

If we take $A = \{x\}$ in the Lemma 3 we get the next result.

Corollary 5 *Let S be an operator between Riesz spaces G and H. Then the following statements hold:*
(i) If S is a band operator, then S is a principal band operator
(ii) If S is an inverse band operator, then S is an inverse principal band operator

Since $B_x = B_{|x|}$ and $B_{|x|} \cap B_{|y|} = B_{|x| \wedge |y|}$ for each elements x, y in a Riesz space G, we have the following lemma that was given in [12, Lemma 3.1].

Lemma 6 *Let G be a Riesz space and $x, y \in G$. Then, $x \perp y$ if and only if $B_x \cap B_y = \{0\}$.*

Proposition 7 [12, *Theorem 3.1*] *Let G, H be Riesz spaces and $S : G \to H$ be an operator then*
(i) If S is an injective principal band operator, then S is a disjointness preserving operator.
(ii) If S is a bijective inverse principal band operator, then S^{-1} is a disjointness preserving operator.

Proof (i) Let $x \perp y$ in G. Then, $B_x \cap B_y = \{0\}$ and $B_{Sx} \subseteq S(B_x)$, $B_{Sy} \subseteq S(B_y)$ as S principal band operator. Hence, $B_{Sx} \cap B_{Sy} \subseteq S(B_x) \cap S(B_y) = S(B_x \cap B_y) = \{0\}$ which yields $Sx \perp Sy$. (ii) It follows from (i) and Lemma 3 (iv). □

Corollary 8 *Let G, H be Riesz spaces with G has the projection property and $S : G \to H$ be an operator. If S is a surjective disjointness preserving operator then S is a band operator.*

Proof Let B be a band in G, and B^d the disjoint complement of B; thus, $G = B \oplus B^d$. Since S is disjointness preserving, we have $S(B^d) \subseteq S(B)^d$ which yields that $H = S(G) = S(B) + S(B^d) \subseteq S(B) + S(B)^d = S(B) \oplus S(B)^d$. From the Theorem 24.2 in [9], $S(B)$ is a band in H. □

Corollary 9 *Let G and H be Riesz spaces and $S : G \to H$ be a bijective operator. S and S^{-1} are disjointness preserving operators if and only if S and S^{-1} are band operators.*

Proof Let S and S^{-1} be disjointness preserving operators. We have $S(D^d) \subseteq S(D)^d$ and $S^{-1}(S(D)^d) \subseteq (S^{-1}S(D))^d = D^d$ for any subset D of G. It follows that $S(D)^d = S(D^d)$. If B is a band in G, then $S(B)^{dd} = S(B^d)^d = S(B^{dd}) = S(B)$ as $B = B^{dd}$. Therefore, S is a band operator. Similarly, it can be shown that S^{-1} is a band operator. If S and S^{-1} are band operators, then S and S^{-1} are disjointness preserving operators from the Proposition 7. □

Corollary 10 *Let S be a bijective operator between Riesz spaces G and H, and G has the projection property. Then the following statements are equivalent:*
(i) S is a disjointness preserving operator.

(ii) S *is a band operator.*

(iii) S *is a principal band operator.*

If we do not assume injectivity of S in the Proposition 7 (i), then S may not be disjointness preserving operator.

Example 11 *Let* $S:\mathbb{R}^2\to\mathbb{R}^2$ *be defined by* $S(x,y)=(0,x+y)$. *Then,* S *is a band operator. However,* S *is not a disjointness preserving operator as* $S(1,0)\perp S(0,1)$ *is not true whenever* $(1,0)\perp(0,1)$.

If we do not assume surjectivity of S in the Corollary 8, then for a band B in G its image $S(B)$ may fail to be a band in H.

Example 12 *Let* $G=c_0$ *be the Riesz space of all real sequences that converge to zero and* $H=l^\infty$ *be the Riesz space of all bounded real sequences. Since* G *is a Dedekind complete Riesz space,* G *has the projection property.The natural embedding operator* $j:G\to H$, $j(x)=x$ *is a disjointness preserving operator but it is not a band operator as* $j(G)=c_0$ *is not a band in* l^∞.

Example 13 *Not every ideal operator is a band operator. Let* G *and* H *be Riesz spaces in the Example 12. Since* c_0 *is an ideal in* l^∞, $j:G\to H$, $j(x)=x$ *is an ideal operator but it is not a band operator.*

Since any orthomorphism is disjointness preserving operator, we get the following corollary.

Corollary 14 *If* G *has the projection property, then each surjective orthomorphism* $\pi:G\to G$ *is a band operator.*

Corollary 15 *If* S *is a bijective band operator from a relatively uniformly complete vector lattice* G *into a normed lattice* H *then* $S^{-1}:H\to G$ *is a band operator.*

Proof Let S be a bijective band operator. According to the Proposition 7 (i), S is a disjointness preserving operator. By the Corollary 2.2 in [5], S^{-1} is a disjointness preserving operator. Considering the Corollary 8, we have S^{-1} is a band operator. □

We now give some properties of order bounded band operators and order bounded inverse band operators.

Theorem 16 *Let* G *and* H *be Riesz spaces,* $S:G\to H$ *be an order bounded injective principal band operator. Then,* S *is order continuous.*

Proof Let $u_0\geq u_\alpha\downarrow 0$ in G. There exists $y\in H_+$ satisfying $|Sx|\leq y$ for all $0\leq x\leq u_0$. Let $0<\epsilon<1$, then if $0\leq x\leq\epsilon u_0$ we have $|Sx|\leq\epsilon y$. Let z be such that $0\leq z\leq |Su_\alpha|$ for all α. Then

$$\begin{aligned}0 &\leq z\leq |Su_\alpha|=\left|S\left[(u_\alpha-\epsilon u_0)^+ +(u_\alpha\wedge\epsilon u_0)\right]\right|\\ &\leq \left|S(u_\alpha-\epsilon u_0)^+\right|+|S(u_\alpha\wedge\epsilon u_0)|\\ &\leq \left|S(u_\alpha-\epsilon u_0)^+\right|+\epsilon y\end{aligned}$$

Thus, $(z-\epsilon y)^+\leq|S(u_\alpha-\epsilon u_0)^+|$ holds for each α. Consequently, the hypothesis on S implies that

$$(z-\epsilon y)^+\in\cap B_{S[(u_\alpha-\epsilon u_0)^+]}\subseteq\cap S(B_{(u_\alpha-\epsilon u_0)^+})=S(\cap B_{(u_\alpha-\epsilon u_0)^+})$$

By the Lemma 139.3 in [15], $\cap B_{(u_\alpha - \epsilon u_0)^+} = \{0\}$, this yields $(z-\epsilon y)^+ = 0$ for each $0 < \epsilon < 1$. Thus, $z = 0$ as G is Archimedean. □

Corollary 17 *Let G and H be Riesz spaces and $S : G \to H$ be an order bounded injective band operator. Then, S is an order continuous operator.*

Proof From the Corollary 5 (i), the proof is clear. □

Theorem 18 *Let G and H be Riesz spaces and $S : G \to H$ be an order bounded disjointness preserving operator. If S is an order continuous operator, then S is an inverse band operator. If additionally S is surjective, then the converse is true.*

Proof Let S be an order continuous operator and let B be an arbitrary band in H. Let $z, y \in G$ be such that $|y| \leq |z|$ and $z \in S^{-1}(B)$. From the Proposition 1.2 in [5], $|Sy| \leq |Sz|$ holds. As $Sz \in B$ and B is a band, we get $Sy \in B$ which yields $y \in S^{-1}(B)$. Thus $S^{-1}(B)$ is an ideal in G. Let $(x_\alpha) \subseteq S^{-1}(B)$ and $x_\alpha \xrightarrow{o} x$ in G. Since S is order continuous and B is order closed we have $Sx \in B$. So $S^{-1}(B)$ is an order closed ideal showing that $S^{-1}(B)$ is a band. Conversely, let S be a surjective inverse band operator. Then, $S^{-1}(\{0\})$ is a band in G. From the Theorem 8.6 in [3], modulus of S exists and $|S|$ is a Riesz homomorphism with $|Sx| = |S|x|| = |S|\,|x|$. It is clear that $|S|$ is surjective and $Ker\,|S| = Ker S$ is a band in G. From the Theorem 7.9 in [3], $|S|$ is order continuous which yields that S is order continuous. □

Corollary 19 *Let G and H be Riesz spaces and $S : G \to H$ be a bijective order bounded operator. If S is a band operator then S^{-1} is a band operator.*

Proof Let S be a band operator. From the Corollary 17 and the Theorem 18, S is an inverse band operator. So S^{-1} is a band operator. □

Since any orthomorphism $\pi : G \to G$ is an order bounded disjointness preserving and order continuous operator, we have the following corollary.

Corollary 20 *Each orthomorphism is an inverse band operator.*

The converse of the Corollary 20 may not be true.

Example 21 *Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $S(x,y) = (0,x)$. S is an inverse band operator. Since $S(1,0) \perp (0,1)$ is not true whenever $(1,0) \perp (0,1)$, so S is not an orthomorphism.*

Example 22 *There is an operator π such that π is an orthomorphism but not a band operator. Let $G = C[0,1]$, $q : [0,1] \to \mathbb{R}$, $q(x) = x$. The operator $\pi_q : G \to G$ defined by $\pi_q(f) = qf$ is an orthomorphism. However, π_q is not band operator. To see this, define the function $w : [0,1] \to \mathbb{R}$ by*

$$w(x) = \begin{cases} \left|x \sin \frac{1}{x}\right| & : \quad x \in (0,1] \\ 0 & : \quad x = 0 \end{cases}$$

Then $w \in G$, $q \in \pi_q(G)$ and $0 \leq w \leq q$. On the other hand, observe that $w \notin \pi_q(G)$. This shows that $\pi_q(G)$ is not an order ideal in G. Thus, $\pi_q(G)$ is not a band in G.

Example 23 *Not every band operator is an orthomorphism. Let* S *be the operator in the Example 11 defined by* $S(x,y) = (0, x+y)$*. Then,* S *is a band operator but* S *is not an orthomorphism as* $S(1,0) \perp (0,1)$ *is not true whenever* $(1,0) \perp (0,1)$.

In general band operator is not order bounded. Now, we will get the necessary conditions for the band operator to be order bounded.

Example 24 *Let* $G = L_0[a,b]$ *be the Riesz space of all (equivalence classes of) measurable functions on* $[a,b]$*. We know that* G *is a Dedekind complete Riesz space. From the Corollary 13.9 in* [1]*, there exists a bijective disjointness preserving operator* $R : L_0[a,b] \to L_0[a,b]$ *such that* R^{-1} *is not disjointness preserving operator. Since* G *has the projection property and* R *is onto,* R *is a band operator. Assume now that* R *is order bounded. From the Corollary 19,* R^{-1} *is band operator. Thus, we have* R^{-1} *is disjointness preserving operator from the Proposition 7. This is a contradiction.*

We recall some definitions that we will use later on.

Definition 25 *A Riesz space* G *has rich center if for each* $x, y \in G$ *satisfying* $|x| \leq |y|$ *there exists a central operator* $s \in Z(G)$ *such that* $s(y) = x$ *and* $|s| \leq I$.

Every Dedekind σ-complete Riesz space has rich center.

Definition 26 *Let* G *be a Riesz space. If each band preserving operator on* G *is order bounded,* G *is said to have bounded band preserving property.*

Every Banach lattices and every Dedekind σ-complete normed vector lattices have bounded band preserving property [1, Theorem 2.6], [8, Corollary 9.10]. If G is a uniformly complete vector lattice on which there is defined a locally convex locally solid Hausdorff topology, then G has bounded band preserving property [10, Corollary 2.3]. There exists a Riesz space such that it does not have the bounded band preserving property [3, Example 8.4].

Lemma 27 [14, *Proposition 2*] *Let* G, H *be Riesz spaces, and* $R : G \to H$ *be a bijective operator. Then the following statements hold:*
(i) If R *is a band operator then* $R^{-1}sR$ *is a band preserving operator for each* $s \in Orth(H)$.
(ii) If R^{-1} *is a band operator then* RsR^{-1} *is a band preserving operator for each* $s \in Orth(G)$.

Proof (i) Let B be a band in G. Since R is a band operator, $R(B)$ is a band in H. Therefore, $sR(B) \subseteq R(B)$ for each $s \in Orth(H)$. It follows that $R^{-1}sR(B) \subseteq R^{-1}R(B) = B$. (ii) The proof is similar to (i). □

Using the Lemma 27, we can give the following lemma.

Lemma 28 *Let* G, H *be Riesz spaces with* G *having bounded band preserving property, and* $R : G \to H$ *be a bijective band operator. Then the mapping*

$$\Phi : Orth(H) \to Orth(G), \; s \to \Phi(s) = R^{-1}sR$$

is an algebraic one to one homomorphism. If H *is* ru*-complete, then* Φ *is a Riesz homomorphism.*

Proof For two arbitrary elements s, t from $Orth(H)$, we obviously have

$$\Phi(s)\Phi(t) = (R^{-1}sR)(R^{-1}tR) = R^{-1}stR = \Phi(st).$$

Let $s \in Orth(H)$ and $\Phi(s) = 0$ hold. For arbitrary element y from H, there is an element u in G such that $R(u) = y$. This implies that

$$R^{-1}s(y) = R^{-1}sRu = \Phi(s)(u) = 0.$$

As R^{-1} is one to one, we have $s = 0$. Since H is ru-complete, $Orth(H)$ is ru-complete [15]. As Φ is an algebra homomorphism from ru-complete f-algebra $Orth(H)$ into f-algebra $Orth(G)$, Φ is a lattice homomorphism from the Theorem 5.1 in [4]. □

Theorem 29 *Let H be an ru-complete Riesz space and has rich center and G be a Riesz space has bounded band preserving property, and $R : G \to H$ be a bijective band operator. Then R^{-1} is an order bounded disjointness preserving operator.*

Proof From the Proposition 1.2 in [5], it is enough to show that if $|z| \leq |w|$ in H, then $\left|R^{-1}z\right| \leq \left|R^{-1}w\right|$. Let $x = R^{-1}z$ and $y = R^{-1}w$. Since H has rich center there exists $s \in Z(H)$ such that $s(w) = z$ and $|s| \leq I$. We have

$$s(w) = s(Ry) = Rx \text{ and } R^{-1}s(Ry) = x.$$

On the other hand, since Φ is a Riesz homomorphism, $\left|R^{-1}sR\right| = |\Phi(s)| \leq \Phi(I) = I$. Thus, we obtain $\left|R^{-1}z\right| = |x| = \left|R^{-1}s(Ry)\right| \leq |y| = \left|R^{-1}w\right|$. □

Corollary 30 *Let H be an ru-complete Riesz space and has rich center and G be a Riesz space has bounded band preserving property, and $R : G \to H$ be a bijective band operator. Then R is order bounded operator.*

Proof From the Theorem 29, R^{-1} is an order bounded disjointness preserving operator. Hence, we obtain that $(R^{-1})^{-1} = R$ is order bounded from the Theorem 4.12 in [1]. □

Example 31 *There exists a Dedekind complete Riesz space without the bounded band preserving property.*

Proof We will use G and R defined in the Example 24. Thus, we know that G is a Dedekind complete Riesz space, R is a bijective disjointness preserving operator and R^{-1} is not disjointness preserving operator. Assume now that G has bounded band preserving property. Since G has projection property and R is onto, R is a band operator. If we take $H = G$ in the Theorem 29, we have that R^{-1} is an order bounded disjointness preserving operator. This is a contradiction. □

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