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Exponential stabilization of a neutrally delayed viscoelastic Timoshenko beam

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Abstract: A Timoshenko type beam subject to a viscoelastic damping in the rotational displacement component is considered. Taking into account a neutral type delay, we prove a fast stability result despite the previously observed destabilizing effect due to delays in such systems. The proof relies on the introduction of nine different functionals with which we modify the energy of the system. These functionals are carefully selected and adapted to cope with both the viscoelasticity and the neutral delay.

Key words: Exponential decay, modified energy, multiplier technique, neutral delay, stability, viscoelasticity

1. Introduction

The scientific community is witnessing a considerable growth of interest in problems involving time delays. This is due mainly to the widespread appearances of such phenomena. Time delays are peculiar to the dependence of the rate of change on the past history of the system. This is the case, for instance, whenever there is a displacement of material or transmission of information. The class of differential equations treating delays is known as functional differential equations (FDEs) [5, 10, 14]. Models that necessitate the incorporation of the history of the (highest) derivative are commonly known as neutral delay differential equations (NDDEs).

NDDEs have been shown to be very useful in describing complicated phenomena in many fields including control theory, mechanical systems, chemical processes, oscillation theory, and biosciences [5, 10, 14, 33, 34].

It was established that differential equations are sensitive to the presence of delays. Many researchers have demonstrated that even initially stable systems may be destabilized when taking into account delays [1, 3, 4, 21]. This has forced scientists to find appropriate ways to fix this matter. The literature has been enriched by many results in this respect. Nevertheless, this class of NDDEs remains not well explored so far. We note here that delays may play a positive role in many cases. It has been well established that, in contrast to the sensitivity issue raised above, large neutral delays may stabilize systems. As a matter of fact, for better achievements, engineers have been adding neutral delays premeditatedly in the models.

We consider the neutrally retarded viscoelastic Timoshenko system

$$\begin{cases} \varphi_{tt} = (\varphi_x + \psi)_x, \\ \left[\psi_t + \int_0^t k(t-s)\psi_t(s)ds\right]' = \psi_{xx} - \int_0^t g(t-s)\,\psi_{xx}(s)ds - (\varphi_x + \psi), \end{cases} \tag{1}$$

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for $t > 0$, $0 < x < 1$ with initial and boundary conditions

$$\begin{cases} \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \ t \geq 0, \\ \varphi(x,0) = \varphi_0(x), \ \varphi_t(x,0) = \varphi_1(x), \ 0 < x < 1, \\ \psi(x,0) = \psi_0(x), \ \psi_t(x,0) = \psi_1(x), \ 0 < x < 1, \end{cases} \tag{2}$$

where $\varphi_0(x)$, $\varphi_1(x)$, $\psi_0(x)$, and $\psi_1(x)$ are given initial data. Here φ is the transversal displacement of the beam from its equilibrium and ψ is the rotational displacement of the beam.

As its name indicates, the model is based on the standard Timoshenko beam model [27, 33]. One of its components (the rotational displacement) is viscoelastic (described by the convolution involving the relaxation function g). The convolution term involving the kernel k describes the neutral delay.

This type of delay appears in the study of vibrating masses attached to an elastic bar and also (as the Euler equation) in some variational problems [11, 13, 16]. It appears also in the study of wave propagation in viscoelastic media [11, 16, 32]:

$$u_{tt} + K * u_{tt} = v^2 \nabla^2 u + \delta(t)\delta(x).$$

Moreover, it is used as a poroacoustic model in acoustic waves propagation:

$$\rho * u_{tt} = \nabla.[K * \nabla u]$$

(see [12, 13]).

In this work we shall prove that the neutral delay in (1) does not prevent the system from being stabilized by the viscoelastic term. In fact, we show that the system is exponentially stable under certain conditions on k. We refer the reader to other kinds of stabilization in [2, 15, 22, 23, 29–31].

The existence and uniqueness of a solution in $\left[H^2(0,1) \times H_0^1(0,1)\right]^2$ (weak solution in $\left[H_0^1(0,1) \times L^2(0,1)\right]^2$) may be proved by combining the results in [6, 7, 19, 34] and [8, 18, 24]. We shall assume therefore that the solution and the initial data are regular enough to justify our computation.

In the next section we present some preliminaries and introduce the different functionals that will be used in the sequel. Section 3 contains some useful lemmas that will help in proving our theorem. The last section is devoted to the statement and proof of our results.

2. Preliminaries

In this section we present our assumptions on both kernels and introduce the energy functional and some other functionals.

(K) The kernel k is a nonnegative continuously differentiable and summable function satisfying

$$k'(t) \leq -\eta\, k(t), \ \int_0^{+\infty} e^{\gamma s} |k'(s)| ds < \infty, \ t \geq 0,$$

for some positive constants η and γ.

The second condition is fulfilled if $k'(t) \geq -\tilde{\eta}\, k(t)$ with $\tilde{\eta} > \eta$ and $\gamma < \eta$.

(**G**) The relaxation function g is a nonnegative continuously differentiable and summable function satisfying $\bar{g} := \int_0^\infty g(t)\, dt < 1$ and there exists a constant $\xi > 0$ such that

$$g'(t) \leq -\xi\, g(t), \ t \geq 0.$$

For $t_* > 0$, we denote

$$g_* = \int_0^{t_*} g(s)ds,\ \bar{g} = \int_0^{\infty} g(s)ds,\ \bar{k} = \int_0^{\infty} k(s)ds,\ k_* = \int_0^{t_*} k(s)ds.$$

We caution the reader here that these assumptions are not the weakest ones possible. They are considered here only for simplicity. They may be weakened as in the case of Timoshenko systems without neutral delays (see [9, 17, 20, 25–28] for more general kernels).

We define the 'modified' energy (taking into account the viscoelasticity and the neutral delay) by

$$\begin{aligned} E(t) := \tfrac{1}{2}\{ &||\varphi_t||^2 + ||\psi_t||^2 + \left(1 - \textstyle\int_0^t g(s)ds\right) ||\psi_x||^2 + ||\varphi_x + \psi||^2 \\ &+ (g \,\square\, \psi_x) + \textstyle\int_0^t k(t-s)||\psi_t(s)||^2 ds\},\ t \geq 0, \end{aligned} \tag{3}$$

where $||.||$ denotes the L^2-norm and

$$(h \,\square v)(t) = \int_0^t h(t-s)||v(t) - v(s)||^2 ds,\ t \geq 0.$$

Proposition 1 *The modified energy $E(t)$ is nonincreasing and uniformly bounded. More precisely, we have*

$$E'(t) = \frac{1}{2}(k' \square \psi_t) + \frac{1}{2}(g' \square \psi_x) - \frac{k(t)}{2}||\psi_t||^2 - \frac{g(t)}{2}\,||\psi_x||^2 \leq 0,\, t \geq 0.$$

To prove the proposition we need to establish a useful identity.

Lemma 1 *We have the following identity:*

$$\begin{aligned} \textstyle\int_0^1 v(s) \int_0^t f(t-s)v_t(s)dsdx = &-\tfrac{1}{2}(f' \square v)(t) + \tfrac{1}{2}\tfrac{d}{dt}\textstyle\int_0^t f(t-s)||v(s)||^2 ds \\ &+ \tfrac{f(t)}{2}||v||^2 - f(t)\textstyle\int_0^1 v(t)v(0)dx,\ t \geq 0, \end{aligned}$$

for all $v \in C^1([0,\infty); L^2(0,1))$ and $f \in C^1[0,\infty)$.

Proof The identity is a direct consequence of

$$(f' \square v)(t) = f(t)||v(t) - v(0)||^2 - 2\int_0^t f(s)\int_0^1 v_t(t-s)(v(t) - v(t-s))dxds,\ t \geq 0$$

and

$$\begin{aligned} \tfrac{d}{dt}\textstyle\int_0^t f(t-s)||v(s)||^2 ds &= \tfrac{d}{dt}\textstyle\int_0^t f(s)\,\|v(t-s)\|\,|^2 ds \\ &= f(t)\,||v(0)||^2 + 2\textstyle\int_0^1\int_0^t f(s)v_t(t-s)v(t-s)dsdx,\ t \geq 0. \end{aligned}$$

□

Proof (of the Proposition)

A straightforward differentiation of $E(t)$, along solutions of (1)–(2), yields

$$\begin{aligned} E'(t) = &-k(t)\textstyle\int_0^1 \psi_t(0)\psi_t dx - \int_0^1 \psi_t \int_0^t k(t-s)\psi_{tt}(s)dsdx - \tfrac{g(t)}{2}\,||\psi_x||^2 \\ &+ \tfrac{1}{2}(g' \square \psi_x) + \tfrac{1}{2}\tfrac{d}{dt}\textstyle\int_0^t k(t-s)||\psi_t(s)||^2 ds,\ t \geq 0. \end{aligned}$$

Then, applying Lemma 1, we find the relation in the statement of the proposition. □

Next, we proceed with the introduction of several functionals and estimate their derivatives. These functionals are carefully selected and adapted to both the viscoelastic damping and the neutral delay.

$$\Lambda_1(t) := \int_0^1 \psi \left[\psi_t + \int_0^t k(t-s)\psi_t(s)ds\right] dx,\ \Lambda_2(t) := -\int_0^1 \varphi_t \varphi dx,$$

$$\begin{aligned}\Lambda_3(t) := & \int_0^1 (\varphi_x + \psi) \left[\psi_t + \int_0^t k(t-s)\psi_t(s)ds\right] dx \\ & + \int_0^1 \varphi_t \left[\psi_x - \int_0^t g(t-s)\,\psi_x(s)ds\right] dx,\end{aligned}$$

$$\Lambda_4(t) := e^{-\gamma t} \int_0^t e^{\gamma s} \tilde{K}(t-s) \left\|\psi_x(s)\right\|^2 ds,\ \tilde{K}(t) := \int_t^{+\infty} e^{\gamma s} |k'(s)| ds,$$

for some $\gamma > 0$,

$$\Lambda_5(t) := \int_0^1 p(x) \left[\psi_x - \int_0^t g(t-s)\,\psi_x(s)ds\right] \left[\psi_t + \int_0^t k(t-s)\psi_t(s)ds\right] dx,$$

and

$$\Lambda_6(t) := \int_0^1 p(x)\varphi_t\varphi_x dx,\ p(x) = -4x + 2,\ 0 \le x \le 1,\ t \ge 0.$$

In addition to these functionals, we consider

$$\Lambda_7(t) := \int_0^1 \varphi_t \chi dx,\ t \ge 0,$$

where χ is the solution of

$$\begin{cases} -\chi_{xx} = \left[p(x)\left(\psi_x - \int_0^t g(t-s)\,\psi_x(s)ds\right)\right]_x,\ 0 < x < 1, \\ \chi(0) = \chi(1) = 0, \end{cases}$$

$$\Lambda_8(t) := \int_0^1 \varphi_t q dx,\ t \ge 0,$$

where q is the solution of

$$\begin{cases} -q_{xx} = \psi_x,\ 0 < x < 1, \\ q(0) = q(1) = 0, \end{cases}$$

and

$$\Lambda_9(t) := -\int_0^1 \left[\psi_t + \int_0^t k(t-s)\psi(s)ds\right] \left[\int_0^t g(t-s)(\psi(t) - \psi(s))ds\right] dx,\ t \ge 0.$$

The usefulness of these functionals will be clear in the next section.

3. Some lemmas

In this section, we prepare several lemmas containing useful estimations of the derivatives of the functionals introduced above in the second section.

Lemma 2 *The derivative of* $\Lambda_1(t)$*, along solutions of (1)–(2), is estimated as follows:*

$$\Lambda_1'(t) \leq -\left(1-\bar{g}-\delta_0\right)\left\|\psi_x\right\|^2+\left(1+\bar{k}+\delta_0\right)\left\|\psi_t\right\|^2-\int_0^1 \psi(\varphi_x+\psi)dx+\tfrac{\bar{k}}{4\delta_0}(k\Box\psi_t)+\tfrac{\bar{g}}{4\delta_0}(g\Box\psi_x),\ t\geq 0,\ \delta_0>0.$$

Proof Using the system (1)–(2) we find

$$\begin{aligned}\Lambda_1'(t) &= \int_0^1 \psi_t\left[\left(1+\int_0^t k(s)ds\right)\psi_t+\int_0^t k(t-s)\left(\psi_t(s)-\psi_t(t)\right)ds\right]dx\\ &\quad+\int_0^1 \psi[\psi_{xx}-\int_0^t g(t-s)\psi_{xx}(s)ds-(\varphi_x+\psi)]\,dx\\ &=\left(1+\int_0^t k(s)ds\right)\left\|\psi_t\right\|^2+\int_0^1\psi_t\int_0^t k(t-s)\left(\psi_t(s)-\psi_t(t)\right)dsdx\\ &\quad-\left\|\psi_x\right\|^2+\int_0^1\psi_x\int_0^t g(t-s)\,\psi_x(s)dsdx-\int_0^1\psi(\varphi_x+\psi)dx,\ t\geq 0.\end{aligned}$$

The estimations (using the Young inequality)

$$\int_0^1 \psi_t\int_0^t k(t-s)\ \left(\psi_t(s)-\psi_t(t)\right)dsdx \leq \frac{\bar{k}}{4\delta_0}(k\Box\psi_t)+\delta_0\left\|\psi_t\right\|^2,\ \delta_0>0$$

and

$$\begin{aligned}\int_0^1\psi_x\int_0^t g(t-s)\,\psi_x(s)dsdx &= \int_0^1\psi_x\int_0^t g(t-s)\left[\,\psi_x(s)-\psi_x(t)\right]dsdx+\left(\int_0^t g(s)ds\right)\left\|\psi_x\right\|^2\\ &\leq \delta_0\left\|\psi_x\right\|^2+\tfrac{\bar{g}}{4\delta_0}(g\Box\psi_x)+\left(\int_0^t g(s)ds\right)\left\|\psi_x\right\|^2,\ \delta_0>0\end{aligned}$$

conclude. □

Lemma 3 *The derivative of* $\Lambda_2(t)$ *is equal to*

$$\Lambda_2'(t)=-\left\|\varphi_t\right\|^2+\left\|\varphi_x+\psi\right\|^2-\int_0^1\psi[\varphi_x+\psi]dx,\ t\geq 0.$$

Proof Clearly

$$\Lambda_2'(t)=-\left\|\varphi_t\right\|^2-\int_0^1\varphi[\varphi_x+\psi]_x\,dx=-\left\|\varphi_t\right\|^2+\int_0^1\varphi_x[\varphi_x+\psi]dx,\ t\geq 0$$

and therefore

$$\Lambda_2'(t)=-\left\|\varphi_t\right\|^2+\left\|\varphi_x+\psi\right\|^2-\int_0^1\psi[\varphi_x+\psi]dx,\ t\geq 0.$$

□

Lemma 4 *The derivative of the functional* $\Lambda_3(t)$ *is evaluated by*

$$\begin{aligned}\Lambda_3'(t) &\leq \tfrac{15}{4}\left[\left(\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)^2(1)+\left(\psi_x-\int_0^t g(t-s)\psi_x(s)ds\right)^2(0)\right]\\ &\quad+\tfrac{1}{15}\left[\varphi_x^2(1)+\varphi_x^2(0)\right]-\left\|\varphi_x+\psi\right\|^2+(1+\delta_0+\bar{k})\left\|\psi_t\right\|^2+\tfrac{\bar{k}}{4\delta_0}(k\Box\psi_t)\\ &\quad+\left[\tfrac{1}{4}+\tfrac{k(t)}{2}+\delta_0+g(t)\right]\left\|\varphi_t\right\|^2+\tfrac{1}{4}\left(8k^2(0)+g(t)\right)\left\|\psi_x\right\|^2\\ &\quad+\tfrac{k(t)}{2}\left\|\psi_{0x}\right\|^2+2k(0)\int_0^t\left|k'(t-s)\right|\left\|\psi_x(s)\right\|^2ds+\tfrac{g(0)}{4\delta_0}(\left|g'\right|\Box\psi_x)\,,\ t\geq 0.\end{aligned}$$

Proof It is easy to see that from the equations in (1) that

$$\Lambda_3'(t) = \int_0^1 (\varphi_x+\psi)[\psi_{xx} - \int_0^t g(t-s)\psi_{xx}(s)ds - (\varphi_x+\psi)]dx + \int_0^1 (\varphi_x+\psi)_t \\ \times \left[\psi_t + \int_0^t k(t-s)\psi_t(s)ds\right]dx + \int_0^1 \varphi_{tt}\left[\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right]dx \\ + \int_0^1 \varphi_t\left[\psi_{xt} - g(0)\psi_x(t) - \int_0^t g'(t-s)\psi_x(s)ds\right]dx,\ t\geq 0.$$

Therefore,

$$\Lambda_3'(t) = \left[\varphi_x\left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)\right]_0^1 - \int_0^1 \psi_x(\varphi_x+\psi)_x dx \\ + \int_0^1 (\varphi_x+\psi)_x \int_0^t g(t-s)\psi_x(s)dsdx - \|\varphi_x+\psi\|^2 + \int_0^1 \psi_t(\varphi_x+\psi)_t dx \\ + \int_0^1 (\varphi_x+\psi)_t \int_0^t k(t-s)\psi_t(s)dsdx + \int_0^1 \left[\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right](\varphi_x+\psi)_x dx \\ + \int_0^1 \varphi_t\left[\psi_{xt} - g(0)\psi_x - \int_0^t g'(t-s)\psi_x(s)ds\right]dx \tag{4}$$

and

$$\int_0^1 \varphi_{xt}\int_0^t k(t-s)\psi_t(s)dsdx = \int_0^1 \varphi_{xt}\left[k(t-s)\psi(s)|_0^t + \int_0^t k'(t-s)\psi(s)dsdx\right] \\ = \int_0^1 \varphi_{xt}\left[k(0)\psi(t) - k(t)\psi(0)\right]dx - \int_0^1 \varphi_t \int_0^t k'(t-s)\psi_x(s)dsdx \\ = -\int_0^1 \varphi_t\left[k(0)\psi_x(t) - k(t)\psi_x(0)\right]dx - \int_0^1 \varphi_t\int_0^t k'(t-s)\psi_x(s)dsdx \\ \leq \frac{1}{4}\|\varphi_t\|^2 + 2k^2(0)\|\psi_x\|^2 + \frac{k(t)}{2}(\|\varphi_t\|^2 + \|\psi_x(0)\|^2) \\ +2k(0)\int_0^t |k'(t-s)|\,\|\psi_x(s)\|^2 ds,\ t\geq 0. \tag{5}$$

On the other hand,

$$\int_0^1 \varphi_t\left[-g(0)\psi_x - \int_0^t g'(t-s)\psi_x(s)ds\right]dx = \int_0^1 \varphi_t\left\{-g(0)\psi_x + g(0)\psi_x - g(t)\psi_x \right. \\ \left. - \int_0^t g'(t-s)\left[\psi_x(s) - \psi_x(t)\right]ds\right\}dx \\ \leq (g(t)+\delta_0)\|\varphi_t\|^2 + \frac{g(t)}{4}\|\psi_x\|^2 + \frac{g(0)}{4\delta_0}(|g'|\,\Box\psi_x), \tag{6}$$

$$\left[\varphi_x\left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)\right]_0^1 \leq \frac{1}{15}\left[\varphi_x^2(1) + \varphi_x^2(0)\right] \\ + \frac{15}{4}\left[\left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2(1) + \left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2(0)\right], \tag{7}$$

and

$$\int_0^1 \psi_t\int_0^t k(t-s)\psi_t(s)dsdx = \int_0^1 \psi_t\int_0^t k(t-s)\left[\psi_t(s) - \psi_t(t)\right]dsdx + \left(\int_0^t k(s)ds\right)\|\psi_t\|^2 \\ \leq \left(\delta_0 + \int_0^t k(s)ds\right)\|\psi_t\|^2 + \frac{\bar{k}}{4\delta_0}(k\Box\psi_t),\ t\geq 0. \tag{8}$$

Taking (5)–(8) into account in (4), we obtain

$$\begin{aligned}\Lambda_3'(t) \leq \tfrac{15}{4} & \left[\left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2(1) + \left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2(0)\right] \\ & + \tfrac{1}{15}\left[\varphi_x^2(1) + \varphi_x^2(0)\right] - \|\varphi_x + \psi\|^2 + (1+\delta_0+\bar{k})\|\psi_t\|^2 + \tfrac{\bar{k}}{4\delta_0}(k\Box\psi_t) \\ & + \left[\tfrac{k(t)}{2} + \delta_0 + g(t) + \tfrac{1}{4}\right]\|\varphi_t\|^2 + \left(2k^2(0) + \tfrac{g(t)}{4}\right)\|\psi_x\|^2 + \tfrac{k(t)}{2}\|\psi_{0x}\|^2 \\ & + 2k(0)\int_0^t |k'(t-s)|\,\|\psi_x(s)\|^2 ds + \tfrac{g(0)}{4\delta_0}(|g'|\Box\psi_x)\end{aligned}$$

for $t \geq 0$ and $\delta_0 > 0$. □

Lemma 5 *The following identity holds:*

$$\Lambda_4'(t) = -\gamma\Lambda_4(t) + \tilde{K}(0)\|\psi_x\|^2 - \int_0^t |k'(t-s)|\,\|\psi_x(s)\|^2 ds,\ t \geq 0.$$

Lemma 6 *The derivative of* Λ_5*, along solutions of (1)–(2), satisfies*

$$\begin{aligned}\Lambda_5'(t) \leq & -\left[\left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2(1) + \left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2(0)\right] \\ & - \int_0^1 p(x)(\varphi_x+\psi)\left[\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right]dx + 2k(t)\delta_0\|\psi_{0x}\|^2 \\ & + \left[4\left(\tfrac{1}{2}+\delta_0\right) + 6k(0)\delta_0 + 2(1+\bar{k})\delta_0 g(t) + g(t)\right]\|\psi_x\|^2 + 2\bar{g}\left(1+\tfrac{1}{2\delta_0}\right)(g\Box\psi_x) \\ & + 2\left[1 + \tfrac{3(k(0)+k(t)+1)+\bar{k}g(t)}{4\delta_0} + \delta_0\left(1+\bar{k}\right)^2\right]\|\psi_t\|^2 + \tfrac{g(0)}{\delta_0}(|g'|\Box\psi_x) \\ & + 6\delta_0 k(0)\int_0^t |k'|(t-s)\|\psi_x(s)\|^2 ds + \bar{k}\left(g(t)+2\delta_0\right)(k\Box\psi_t),\ t\geq 0,\ \delta_0 > 0.\end{aligned}$$

Proof From the second equation in (1) we see that

$$\begin{aligned}\Lambda_5'(t) = & \int_0^1 p(x)\left[\psi_{xx} - \int_0^t g(t-s)\psi_{xx}(s)ds - (\varphi_x+\psi)\right] \\ & \times\left[\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right]dx \\ & + \int_0^1 p(x)\left[\psi_t + \int_0^t k(t-s)\psi_t(s)ds\right]\left[\psi_{xt} - g(0)\psi_x - \int_0^t g'(t-s)\psi_x(s)ds\right]dx \\ = & \tfrac{1}{2}\int_0^1 p(x)\tfrac{d}{dx}\left[\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right]^2 dx - \int_0^1 p(x)(\varphi_x+\psi) \\ & \times\left[\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right]dx + \tfrac{1}{2}\int_0^1 p(x)\tfrac{d\psi_t^2}{dx}dx \\ & + \int_0^1 p(x)\psi_{xt}\int_0^t k(t-s)\psi_t(s)dsdx \\ & - \int_0^1 p(x)\left[\left(1+\int_0^t k(s)ds\right)\psi_t + \int_0^t k(t-s)\left(\psi_t(s)-\psi_t(t)\right)ds\right] \\ & \times\left[g(t)\psi_x + \int_0^t g'(t-s)\left[\psi_x(s)-\psi_x(t)\right]ds\right]dx,\ t\geq 0.\end{aligned}$$

By integration by parts, we may write for $t \geq 0$

$$
\begin{aligned}
\Lambda_5'(t) \leq & \tfrac{1}{2}\left[p(x)\left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2\right]_0^1 + 4\left(\tfrac{1}{2}+\delta_0\right)\|\psi_x\|^2 \\
& +2\bar{g}\left(1+\tfrac{1}{2\delta_0}\right)(g\Box\psi_x) - \int_0^1 p(x)(\varphi_x+\psi)\left[\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right]dx \\
& +\tfrac{1}{2}\left[p(x)\psi_t^2\right]_0^1 - \tfrac{1}{2}\int_0^1 p'(x)\psi_t^2 dx + \int_0^1 p(x)\psi_{xt}\int_0^t k(t-s)\psi_t(s)dsdx \\
& -g(t)\left(1+\int_0^t k(s)ds\right)\int_0^1 p(x)\psi_t\psi_x dx \\
& -g(t)\int_0^1 p(x)\psi_x\int_0^t k(t-s)\left(\psi_t(s)-\psi_t(t)\right)dsdx \\
& -\left(1+\int_0^t k(s)ds\right)\int_0^1 p(x)\psi_t\left[\int_0^t g'(t-s)\left[\psi_x(s)-\psi_x(t)\right]ds\right]dx \\
& -\int_0^1 p(x)\left[\int_0^t k(t-s)\left(\psi_t(s)-\psi_t(t)\right)ds\right]\left[\int_0^t g'(t-s)\left[\psi_x(s)-\psi_x(t)\right]ds\right]dx,
\end{aligned}
$$

and applying the Young inequality, we get

$$
\begin{aligned}
\Lambda_5'(t) \leq & -\left[\left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2(1) + \left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2(0)\right] \\
& -\int_0^1 p(x)(\varphi_x+\psi)\left[\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right]dx + 4\left(\tfrac{1}{2}+\delta_0\right)\|\psi_x\|^2 \\
& +2\bar{g}\left(1+\tfrac{1}{2\delta_0}\right)(g\Box\psi_x) + 2\|\psi_t\|^2 + \int_0^1 p(x)\psi_{xt}\int_0^t k(t-s)\psi_t(s)dsdx \\
& +\tfrac{g(t)}{2\delta_0}\left(1+\int_0^t k(s)ds\right)\|\psi_t\|^2 + 2\delta_0 g(t)\left(1+\int_0^t k(s)ds\right)\|\psi_x\|^2 + g(t)\|\psi_x\|^2 \\
& +\bar{k}g(t)(k\Box\psi_t) + 2\delta_0\left(1+\bar{k}\right)^2\|\psi_t\|^2 + \tfrac{g(0)}{2\delta_0}(|g'|\Box\psi_x) \\
& +2\delta_0\bar{k}(k\Box\psi_t) + \tfrac{g(0)}{2\delta_0}(|g'|\Box\psi_x),\ t\geq 0.
\end{aligned}
$$

The sixth term in the right-hand side of the previous relation is handled in the following way:

$$
\begin{aligned}
\int_0^1 p(x)\psi_{xt}\int_0^t k(t-s)\psi_t(s)dsdx &= \int_0^1 p(x)\psi_{xt}\left[k(t-s)\psi(s)|_0^t + \int_0^t k'(t-s)\psi(s)ds\right]dx \\
&= \int_0^1 p(x)\psi_{xt}\left[k(0)\psi(t) - k(t)\psi(0) + \int_0^t k'(t-s)\psi(s)ds\right]dx \\
&= -k(0)\int_0^1 p(x)\psi_t\psi_x dx - k(0)\int_0^1 p'(x)\psi_t\psi dx + k(t)\int_0^1 p(x)\psi_t\psi_x(0)dx \\
&\quad +k(t)\int_0^1 p'(x)\psi_t\psi(0)dx - \int_0^1 p(x)\psi_t\int_0^t k'(t-s)\psi_x(s)dsdx \\
&\quad -\int_0^1 p'(x)\psi_t\int_0^t k'(t-s)\psi(s)dsdx,\ t\geq 0.
\end{aligned}
$$

Passing to the estimations, we find

$$
\begin{aligned}
\int_0^1 p(x)\psi_{xt}\int_0^t k(t-s)\psi_t(s)dsdx \leq & \tfrac{3(k(0)+k(t)+1)}{2\delta_0}\|\psi_t\|^2 + 6k(0)\delta_0\|\psi_x\|^2 + 2k(t)\delta_0\|\psi_{0x}\|^2 \\
& +6\delta_0 k(0)\int_0^t |k'|(t-s)\|\psi_x(s)\|^2 ds,\ t\geq 0,\ \delta_0>0.
\end{aligned}
$$

Hence,

$$\begin{aligned}\Lambda_5'(t) \leq & -\left[\left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2(1) + \left(\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right)^2(0)\right] \\ & - \int_0^1 p(x)(\varphi_x+\psi)\left[\psi_x - \int_0^t g(t-s)\psi_x(s)ds\right]dx + 2k(t)\delta_0 \left\|\psi_{0x}\right\|^2 \\ & + \left[4\left(\tfrac{1}{2}+\delta_0\right) + 6k(0)\delta_0 + 2\left(1+\bar{k}\right)\delta_0 g(t) + g(t)\right]\left\|\psi_x\right\|^2 + 2\bar{g}\left(1+\tfrac{1}{2\delta_0}\right)(g\Box\psi_x) \\ & +2\left[1 + \tfrac{3(k(0)+k(t)+1)+\left(1+\bar{k}\right)g(t)}{4\delta_0} + \delta_0\left(1+\bar{k}\right)^2\right]\left\|\psi_t\right\|^2 + \tfrac{g(0)}{\delta_0}(\left|g'\right|\Box\psi_x) \\ & +6\delta_0 k(0)\int_0^t \left|k'\right|(t-s)\left\|\psi_x(s)\right\|^2 ds + \bar{k}\left(g(t)+2\delta_0\right)(k\Box\psi_t),\ t \geq 0.\end{aligned}$$

□

Lemma 7 *It holds, for the derivative of* $\Lambda_6(t)$*, that*

$$\begin{aligned}\Lambda_6'(t) \leq & \ 3\left\|\varphi_x+\psi\right\|^2 - [\varphi_x^2(1)+\varphi_x^2(0)] + \left\|\psi_x\right\|^2 \\ & -4\int_0^1(\varphi_x+\psi)\psi dx + 2\left\|\varphi_t\right\|^2,\ t\geq 0.\end{aligned}$$

Proof A simple differentiation gives

$$\begin{aligned}\Lambda_6'(t) &= \int_0^1 p(x)(\varphi_x+\psi)_x\varphi_x\,dx + \int_0^1 p(x)\varphi_t\varphi_{xt}dx \\ &= \int_0^1 p(x)(\varphi_x+\psi)_x\varphi_x dx + \tfrac{1}{2}\int_0^1 p(x)\tfrac{d\varphi_t^2}{dx}dx \\ &= \int_0^1 p(x)(\varphi_x+\psi)_x\varphi_x dx + \tfrac{1}{2}[p(x)\varphi_t^2]_0^1 - \tfrac{1}{2}\int_0^1 p'(x)\varphi_t^2 dx \\ &= \int_0^1 p(x)(\varphi_x+\psi)_x(\varphi_x+\psi)dx - \int_0^1 p(x)(\varphi_x+\psi)_x\psi dx \\ &\quad -[\varphi_t^2(1)+\varphi_t^2(0)] + 2\left\|\varphi_t\right\|^2 \\ &= \tfrac{1}{2}\int_0^1 p(x)\tfrac{d(\varphi_x+\psi)^2}{dx}dx + \int_0^1 p(x)(\varphi_x+\psi)\psi_x dx \\ &\quad -4\int_0^1(\varphi_x+\psi)\psi dx + 2\left\|\varphi_t\right\|^2,\ t\geq 0.\end{aligned}$$

Then, using integration by parts and the Young inequality, we arrive at

$$\begin{aligned}\Lambda_6'(t) \leq & -[\varphi_x^2(1)+\varphi_x^2(0)] + 2\left\|\varphi_x+\psi\right\|^2 + \left\|\psi_x\right\|^2 + \left\|\varphi_x+\psi\right\|^2 \\ & +2\left\|\varphi_t\right\|^2 - 4\int_0^1(\varphi_x+\psi)\psi dx\end{aligned}$$

or

$$\begin{aligned}\Lambda_6'(t) \leq & \ 3\left\|\varphi_x+\psi\right\|^2 - [\varphi_x^2(1)+\varphi_x^2(0)] + \left\|\psi_x\right\|^2 \\ & -4\int_0^1(\varphi_x+\psi)\psi dx + 2\left\|\varphi_t\right\|^2,\ t\geq 0.\end{aligned}$$

□

Lemma 8 *We evaluate the derivative of* $\Lambda_7(t)$ *as follows:*

$$\begin{aligned}\Lambda_7'(t) \leq & \ \tfrac{5}{\delta_0}\left\|\psi_t\right\|^2 + 3\delta_0\left\|\varphi_t\right\|^2 + \tfrac{4g(0)}{\delta_0}(\left|g'\right|\Box\psi_x) + \tfrac{\bar{g}}{\delta_0}(g\Box\psi_x) \\ & +2g(t)\left(\left\|\varphi_t\right\|^2 + \left\|\psi_x\right\|^2\right) + 10\left\|\psi_x\right\|^2 + \left(\tfrac{1}{10}+\delta_0\right)\left\|\varphi_x+\psi\right\|^2 \\ & + \int_0^1 p(x)(\varphi_x+\psi)\left(\psi_x - \int_0^t g(t-s)\,\psi_x(s)ds\right)dx,\ t\geq 0,\ \delta_0>0.\end{aligned}$$

Proof Differentiating $\Lambda_7(t)$, and taking into account (1), we get for $t \geq 0$

$$\begin{aligned} \Lambda_7'(t) &= \int_0^1 [\varphi_x + \psi]_x \chi dx + \int_0^1 \varphi_t \chi_t dx = -\int_0^1 (\varphi_x + \psi)\chi_x dx + \int_0^1 \varphi_t \chi_t dx \\ &= \int_0^1 \varphi_t \chi_t dx + \int_0^1 p(x)(\varphi_x + \psi)\left(\psi_x - \int_0^t g(t-s)\,\psi_x(s)ds\right)dx \\ &\quad - \int_0^1 (\varphi_x + \psi)\int_0^1 p(z)\left(\psi_x - \int_0^t g(t-s)\,\psi_x(s)ds\right)dzdx. \end{aligned} \tag{9}$$

For the first term in (9), we have

$$\int_0^1 \varphi_t \chi_t dx = \int_0^1 \left(\frac{d}{dx}\int_0^x \varphi_t dy\right)\chi_t dx = \left[\chi_t \int_0^x \varphi_t dx\right]_0^1 - \int_0^1 \chi_{tx}\int_0^x \varphi_t dy dx$$

and

$$\begin{aligned} -\chi_{tx} &= p(x)\left(\psi_{xt} - g(0)\,\psi_x - \int_0^t g'(t-s)\,\psi_x(s)ds\right) \\ &\quad - \int_0^1 p(z)\left(\psi_{xt} - g(0)\,\psi_x - \int_0^t g'(t-s)\,\psi_x(s)ds\right)dz,\ t \geq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^1 \varphi_t \chi_t dx &= \int_0^1 p(x)\left(\int_0^x \varphi_t dy\right)\left(\psi_{xt} - g(0)\,\psi_x - \int_0^t g'(t-s)\,\psi_x(s)ds\right)dx \\ &\quad - \int_0^1 \left(\int_0^x \varphi_t dy\right)\int_0^1 p(z)\left(\psi_{xt} - g(0)\,\psi_x - \int_0^t g'(t-s)\,\psi_x(s)ds\right)dzdx \\ &= \int_0^1 \left(\int_0^x \varphi_t dy\right)\left\{\frac{d}{dx}[p(x)\psi_t] + 4\psi_t - p(x)\left[g(0)\psi_x + \int_0^t g'(t-s)\,\psi_x(s)ds\right]\right\}dx \\ &\quad - \int_0^1 \left(\int_0^x \varphi_t dy\right)\left(\int_0^1 p(z)\psi_{xt}dz\right)dx \\ &\quad + \int_0^1 \left(\int_0^x \varphi_t dy\right)\int_0^1 p(z)\left(g(0)\,\psi_x + \int_0^t g'(t-s)\,\psi_x(s)ds\right)dzdx,\ t \geq 0 \end{aligned}$$

or

$$\begin{aligned} \int_0^1 \varphi_t \chi_t dx &= -\int_0^1 p(x)\psi_t\varphi_t dx + 4\int_0^1 \left(\int_0^x \varphi_t dy\right)\psi_t dx \\ &\quad -g(0)\int_0^1 \left(\int_0^x \varphi_t dy\right)p(x)\psi_x dx - \int_0^1 p(x)\left(\int_0^x \varphi_t dy\right)\int_0^t g'(t-s)\,\psi_x(s)dsdx \\ &\quad -4\int_0^1 \left(\int_0^x \varphi_t dy\right)\left(\int_0^1 \psi_t dy\right)dx + g(0)\int_0^1 \left(\int_0^x \varphi_t dy\right)\int_0^1 p(z)\psi_x dx \\ &\quad + \int_0^1 \left(\int_0^x \varphi_t dy\right)\int_0^1 p(z)\int_0^t g'(t-s)\,\psi_x(s)dsdx,\ t \geq 0. \end{aligned}$$

This identity may also be written as

$$\begin{aligned} \int_0^1 \varphi_t \chi_t dx &= -\int_0^1 p(x)\psi_t\varphi_t dx + 4\int_0^1 \left(\int_0^x \varphi_t dy\right)\psi_t dx + \int_0^1 \left(\int_0^x \varphi_t dy\right)p(x) \\ &\quad \times \int_0^t g'(t-s)\,[\psi_x(t) - \psi_x(s)]\,dsdx - g(t)\int_0^1 \left(\int_0^x \varphi_t dy\right)p(x)\psi_x dx \\ &\quad -4\int_0^1 \left(\int_0^x \varphi_t dy\right)\left(\int_0^1 \psi_t dy\right)dx + g(t)\int_0^1 \left(\int_0^x \varphi_t dy\right)\int_0^1 p(z)\psi_x dx \\ &\quad + \int_0^1 \left(\int_0^x \varphi_t dy\right)\int_0^1 p(z)\int_0^t g'(t-s)\,[\psi_x(s) - \psi_x(t)]\,dsdzdx,\ t \geq 0 \end{aligned}$$

and estimated as follows:

$$\begin{aligned} \int_0^1 \varphi_t \chi_t dx &\leq \tfrac{5}{\delta_0}\|\psi_t\|^2 + 3\delta_0\|\varphi_t\|^2 + \tfrac{4g(0)}{\delta_0}(|g'|\,\Box\psi_x) \\ &\quad +2g(t)\left(\|\varphi_t\|^2 + \|\psi_x\|^2\right),\ t \geq 0,\ \delta_0 > 0. \end{aligned} \tag{10}$$

Substituting estimation (10) in (9) leads to

$$\begin{aligned}\Lambda_7'(t) &= \int_0^1[\varphi_x+\psi]_x\chi dx+\int_0^1\varphi_t\chi_t dx = -\int_0^1(\varphi_x+\psi)\chi_x dx+\int_0^1\varphi_t\chi_t dx\\ &\leq \tfrac{5}{\delta_0}\left\|\psi_t\right\|^2+3\delta_0\left\|\varphi_t\right\|^2+\tfrac{4g(0)}{\delta_0}(|g'|\,\Box\psi_x)+2g(t)\left(\left\|\varphi_t\right\|^2+\left\|\psi_x\right\|^2\right)\\ &+\int_0^1 p(x)(\varphi_x+\psi)\left(\psi_x-\int_0^t g(t-s)\,\psi_x(s)ds\right)dx-\int_0^1(\varphi_x+\psi)\\ &\times\int_0^1[p(z)\left[\left(1-\int_0^t g(s)ds\right)\psi_x+\int_0^t g(t-s)\,\left[\psi_x(t)-\psi_x(s)\right]ds\right]dzdx.\end{aligned}$$

The rest of the terms are evaluated in the following manner:

$$\begin{aligned}\Lambda_7'(t) \leq{}& \tfrac{5}{\delta_0}\left\|\psi_t\right\|^2+3\delta_0\left\|\varphi_t\right\|^2+\tfrac{4g(0)}{\delta_0}(|g'|\,\Box\psi_x)+2g(t)\left(\left\|\varphi_t\right\|^2+\left\|\psi_x\right\|^2\right)\\ &+\tfrac{\bar{g}}{\delta_0}(g\Box\psi_x)+10\left\|\psi_x\right\|^2+\left(\tfrac{1}{10}+\delta_0\right)\left\|\varphi_x+\psi\right\|^2\\ &+\int_0^1 p(x)(\varphi_x+\psi)\left(\psi_x-\int_0^t g(t-s)\,\psi_x(s)ds\right)dx,\ t\geq 0,\ \delta_0>0.\end{aligned}$$

□

Lemma 9 *The derivative of* Λ_8, *along solutions of (1)–(2), satisfies*

$$\Lambda_8'(t)\leq\int_0^1(\varphi_x+\psi)\psi dx+\delta_0\left\|\varphi_t\right\|^2+\frac{1}{\delta_0}\left\|\psi_t\right\|^2,\ t\geq 0,\ \delta_0>0.$$

Proof A differentiation of $\Lambda_8(t)$ gives

$$\Lambda_8'(t)=\int_0^1\varphi_{tt}q dx+\int_0^1\varphi_t q_t dx=\int_0^1(\varphi_x+\psi)_x q dx+\int_0^1\varphi_t q_t dx,\ t\geq 0.$$

Next, we have

$$\int_0^1\varphi_t q_t dx=\int_0^1\left(\frac{d}{dx}\int_0^x\varphi_t dy\right)q_t dx=\left[q_t\int_0^x\varphi_t dx\right]_0^1-\int_0^1 q_{tx}\int_0^x\varphi_t dydx$$

and

$$\int_0^1\varphi_t q_t dx=-\int_0^1 q_{tx}\int_0^x\varphi_t dydx=\int_0^1\left(\int_0^x\varphi_t dy\right)\left(\psi_t-\int_0^1\psi_t dz\right)dx.$$

Therefore,

$$\begin{aligned}\Lambda_8'(t)&=-\int_0^1(\varphi_x+\psi)q_x dx+\int_0^1\left(\int_0^x\varphi_t dy\right)\left(\psi_t-\int_0^1\psi_t dz\right)dx\\ &=\int_0^1(\varphi_x+\psi)\left(\psi-\int_0^1\psi dz\right)dx+\int_0^1\left(\int_0^x\varphi_t dy\right)\left(\psi_t-\int_0^1\psi_t dz\right)dx,\ t\geq 0\end{aligned}$$

and

$$\Lambda_8'(t)\leq\int_0^1(\varphi_x+\psi)\psi dx+\delta_0\left\|\varphi_t\right\|^2+\frac{1}{\delta_0}\left\|\psi_t\right\|^2,\ t\geq 0,\ \delta_0>0.$$

□

Lemma 10 *The derivative of $\Lambda_9(t)$ satisfies*

$$\Lambda_9'(t) \leq \delta_1 \left\|\psi_x\right\|^2 + \bar{g}\left(\frac{1}{2\delta_1}+1\right)(g\Box\psi_x) + \delta_1\left\|\varphi_x+\psi\right\|^2 + \frac{\left(2+\bar{k}\right)g(0)}{4\delta_1}(\left|g'\right|\Box\psi_x)$$
$$+\left[\left(1+\int_0^t k(s)ds\right)\left(\delta_1 - \int_0^t g(s)ds\right) + \delta_1\bar{g}\right]\left\|\psi_t\right\|^2 + \bar{k}\left(\delta_1 + \frac{\bar{g}}{4\delta_1}\right)(k\Box\psi_t)$$

for $t \geq 0$ and $\delta_1 > 0$.

Proof We have

$$\begin{aligned}\Lambda_9'(t) = &-\int_0^1\left[\psi_{xx} - \int_0^t g(t-s)\,\psi_{xx}(s)ds - (\varphi_x+\psi)\right]\\ &\times\left[\int_0^t g(t-s)(\psi(t)-\psi(s))ds\right]dx\\ &-\int_0^1\left[\left(1+\int_0^t k(s)ds\right)\psi_t + \int_0^t k(t-s)\left(\psi_t(s)-\psi_t(t)\right)ds\right]\\ &\times\left[\int_0^t g'(t-s)(\psi(t)-\psi(s))ds + \psi_t(t)\int_0^t g(s)ds\right]dx\end{aligned}$$

or

$$\begin{aligned}\Lambda_9'(t) = &\left(1-\int_0^t g(s)ds\right)\int_0^1\psi_x\left[\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right]dx\\ &+\int_0^1\left[\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))ds\right]^2dx\\ &+\int_0^1(\varphi_x+\psi)\left[\int_0^t g(t-s)(\psi(t)-\psi(s))ds\right]dx - \left(1+\int_0^t k(s)ds\right)\\ &\times\int_0^1\psi_t\left[\int_0^t g'(t-s)(\psi(t)-\psi(s))ds\right]dx - \left(1+\int_0^t k(s)ds\right)\left(\int_0^t g(s)ds\right)\left\|\psi_t\right\|^2\\ &-\int_0^1\left(\int_0^t k(t-s)\left(\psi_t(s)-\psi_t(t)\right)ds\right)\int_0^t g'(t-s)(\psi(t)-\psi(s))dsdx\\ &-\left(\int_0^t g(s)ds\right)\int_0^1\psi_t\int_0^t k(t-s)\left(\psi_t(s)-\psi_t(t)\right)dsdx,\ t\geq 0.\end{aligned}$$

This may be estimated as follows:

$$\begin{aligned}\Lambda_9'(t) \leq\ &\delta_1\left(1-\int_0^t g(s)ds\right)\left\|\psi_x\right\|^2 + \left(1-\int_0^t g(s)ds\right)\frac{\bar{g}}{4\delta_1}(g\Box\psi_x) + \bar{g}(g\Box\psi_x)\\ &+\delta_1\left\|\varphi_x+\psi\right\|^2 + \frac{\bar{g}}{4\delta_1}(g\Box\psi_x) + \delta_1\left(1+\int_0^t k(s)ds\right)\left\|\psi_t\right\|^2\\ &+\frac{g(0)}{4\delta_1}\left(1+\int_0^t k(s)ds\right)(\left|g'\right|\Box\psi_x) - \left(1+\int_0^t k(s)ds\right)\left(\int_0^t g(s)ds\right)\left\|\psi_t\right\|^2\\ &+\delta_1\bar{k}(k\Box\psi_t) + \frac{g(0)}{4\delta_1}(\left|g'\right|\Box\psi_x) + \delta_1\left(\int_0^t g(s)ds\right)\left\|\psi_t\right\|^2\\ &+\frac{\bar{k}}{4\delta_1}\left(\int_0^t g(s)ds\right)(k\Box\psi_t),\ \delta_0>0,\end{aligned}$$

or, for $t \geq 0$,

$$\Lambda_9'(t) \leq \delta_0 \left\|\psi_x\right\|^2 + \bar{g}\left(\frac{1}{2\delta_0}+1\right)(g\Box\psi_x) + \delta_0\left\|\varphi_x+\psi\right\|^2 + \frac{\left(2+\bar{k}\right)g(0)}{4\delta_0}(\left|g'\right|\Box\psi_x)$$
$$+\left[\left(1+\int_0^t k(s)ds\right)\left(\delta_0 - \int_0^t g(s)ds\right) + \delta_0\bar{g}\right]\left\|\psi_t\right\|^2 + \bar{k}\left(\delta_0 + \frac{\bar{g}}{4\delta_0}\right)(k\Box\psi_t).$$

□

4. Stability of the system

In this section, with the help of the previous lemmas, we prove that the energy is exponentially decaying to zero. We define the functional

$$L(t) := E(t) + \sum_{i=1}^{9} \lambda_i \Lambda_i(t),\ t \geq 0,$$

where $\lambda_i,\ i = 1, ..., 9$ are positive constants to be determined later. It is easy to see that $L(t)$ is equivalent to $E(t) + \lambda_4 \Lambda_4(t)$; that is,

$$A_1 E(t) \leq L(t) \leq A_2 E(t) + \lambda_4 \Lambda_4(t),\ t \geq 0, \tag{11}$$

for some positive constants A_1 and A_2.

Theorem 1 *Under the above assumptions (**G**) and (**K**) on the kernels, we have*

$$E(t) \leq M e^{-Ct},\ t \geq 0, \tag{12}$$

for some positive constants M and C.

Proof Thanks to Proposition 1 (on the derivative of $E(t)$) and Lemmas 2 to 10 (on the derivatives of the functional Λ_1 to Λ_9), we find

$$\begin{aligned} L'(t) \leq\ & B_1(k\Box\psi_t) + B_2(|g'|\,\Box\psi_x) + B_3 \|\psi_x\|^2 + B_4 \|\psi_t\|^2 + B_5 \|\varphi_t\|^2 \\ & + B_6 \|\varphi_x + \psi\|^2 - \tfrac{\xi}{4}(g\,\Box\,\psi_x) + B_7 \Lambda_4(t) + B_8 k(t) \|\psi_{0x}\|^2 + B_9 \textstyle\int_0^1 \psi(\varphi_x + \psi) dx \\ & + B_{10} \textstyle\int_0^t |k'|\,(t-s) \|\psi_x(s)\|^2 ds + B_{11} \left[\varphi_x^2(1) + \varphi_x^2(0)\right] \\ & + B_{12} \left[\left(\psi_x - \textstyle\int_0^t g(t-s)\psi_x(s) ds\right)^2 (1) + \left(\psi_x - \textstyle\int_0^t g(t-s)\psi_x(s) ds\right)^2 (0) \right] \\ & + B_{13} \textstyle\int_0^1 p(x)(\varphi_x + \psi) \left[\psi_x - \textstyle\int_0^t g(t-s)\psi_x(s) ds\right] dx \end{aligned} \tag{13}$$

where

$$B_1 = -\frac{\eta}{2} + \bar{k} \left[\frac{\lambda_1 + \lambda_3}{4\delta_0} + \lambda_5 \left(g(t) + 2\delta_0\right) + \lambda_9 \left(\delta_1 + \frac{\bar{g}}{4\delta_1}\right) \right],$$

$$B_2 = g(0) \left[\frac{\lambda_3 + 4\left(\lambda_5 + 4\lambda_7\right)}{4\delta_0} + \frac{2 + \bar{k}}{4\delta_1} \lambda_9 \right] + \frac{\bar{g}}{\xi} \left[\frac{\lambda_1}{4\delta_0} + 2\lambda_5 \left(1 + \frac{1}{2\delta_0}\right) + \frac{\lambda_7}{\delta_0} + \lambda_9 \left(\frac{1}{2\delta_1} + 1\right) \right] - \frac{1}{4},$$

$$\begin{aligned} B_3 =\ & -\lambda_1 \left(1 - \bar{g} - \delta_0\right) - \tfrac{g(t)}{2} + \tfrac{\lambda_3}{4} \left(8k^2(0) + g(t)\right) + \lambda_4 \tilde{K}(0) \\ & + 2\lambda_7 \left(5 + g(t)\right) + \lambda_5 \left[4\left(\tfrac{1}{2} + \delta_0\right) + 6k(0)\delta_0 + \left[2(1+\bar{k})\delta_0 + 1\right] g(t)\right] + \lambda_6 + \lambda_9 \delta_1, \end{aligned}$$

$$\begin{aligned} B_4 =\ & \left(\lambda_1 + \lambda_3\right)\left(1 + \delta_0 + \bar{k}\right) + 2\lambda_5 \left[1 + \tfrac{3(k(0) + k(t) + 1) + (1+\bar{k})g(t)}{4\delta_0} + \delta_0 \left(1 + \bar{k}\right)^2\right] \\ & + \tfrac{5\lambda_7 + \lambda_8}{\delta_0} + \lambda_9 \left[\left(1 + \textstyle\int_0^t k(s) ds\right)\left(\delta_1 - g^*\right) + \delta_1 \bar{g}\right] - \tfrac{k(t)}{2}, \end{aligned}$$

$$B_5 = -\lambda_2 + \lambda_3 \left[\frac{k(t)}{2} + \delta_0 + g(t) + \frac{1}{4}\right] + 2\lambda_6 + \left(3\delta_0 + 2g(t)\right)\lambda_7 + \lambda_8 \delta_0,$$

$$B_6 = \lambda_2 - \lambda_3 + 3\lambda_6 + \lambda_7 \left(\frac{1}{10} + \delta_0\right) + \lambda_9 \delta_1,\ B_7 = -\lambda_4 \gamma,\ B_8 = 2\delta_0 \lambda_5 + \frac{\lambda_3}{2},$$

$$B_9 = \lambda_8 - \lambda_1 - \lambda_2 - 4\lambda_6,\ B_{10} = 2k(0)\left(\lambda_3 + 3\delta_0\lambda_5\right) - \lambda_4,$$

$$B_{11} = \frac{\lambda_3}{15} - \lambda_6,\ B_{12} = \frac{15\lambda_3}{4} - \lambda_5,\ B_{13} = \lambda_7 - \lambda_5.$$

Let us take $\lambda_7 = \lambda_5 = \frac{15\lambda_3}{4},\ \lambda_6 = \frac{\lambda_3}{15},\ \lambda_2 = \frac{2}{5}\lambda_3,\ \lambda_8 = \lambda_1 + \lambda_2 + 4\lambda_6$ and use the assumption on g to find

$$\begin{aligned} L'(t) \le\ & B_1(k\Box\psi_t) + B_2(|g'|\,\Box\psi_x) + B_3\left\|\psi_x\right\|^2 + B_4\left\|\psi_t\right\|^2 + B_5\left\|\varphi_t\right\|^2 \\ & -\tfrac{\xi}{4}(g\,\Box\,\psi_x) + B_6\left\|\varphi_x + \psi\right\|^2 + B_7\Lambda_4(t) + B_8 k(t)\left\|\psi_{0x}\right\|^2 \\ & + B_{10}\textstyle\int_0^t |k'|\,(t-s)\left\|\psi_x(s)\right\|^2 ds. \end{aligned} \tag{14}$$

Let us focus first on $B_i,\ i = 3, ..., 6, 10.$ We need these coefficients to be negative, i.e.

$$\begin{cases} \frac{\lambda_3}{4}\left(8k^2(0) + g(t)\right) + \lambda_4\tilde{K}(0) + 2\lambda_7(5 + g(t)) + \lambda_6 + \lambda_9\delta_1 \\ +\lambda_5\left[4\left(\frac{1}{2} + \delta_0\right) + 6k(0)\delta_0 + \left[2(1+\bar{k})\delta_0 + 1\right]g(t)\right] < \lambda_1\left(1 - \bar{g} - \delta_0\right) + \frac{g(t)}{2}, \\ (\lambda_1 + \lambda_3)\left(1 + \delta_0 + \bar{k}\right) + 2\lambda_5\left[1 + \frac{3(k(0)+k(t)+1)+(1+\bar{k})g(t)}{4\delta_0} + \delta_0\left(1 + \bar{k}\right)^2\right] \\ +\frac{5\lambda_7 + \lambda_8}{\delta_0} + \lambda_9\left[(1 + k_*)\left(\delta_1 - g^*\right) + \delta_1\bar{g}\right] < \frac{k(t)}{2}, \\ \lambda_3\left[\frac{k(t)}{2} + \delta_0 + g(t) + \frac{1}{4}\right] + 2\lambda_6 + \left(3\delta_0 + 2g(t)\right)\lambda_7 + \lambda_8\delta_0 < \lambda_2, \\ \lambda_2 + 3\lambda_6 + \lambda_7\left(\frac{1}{10} + \delta_0\right) + \lambda_9\delta_1 < \lambda_3, \\ 2k(0)\left(\lambda_3 + 3\delta_0\lambda_5\right) < \lambda_4. \end{cases} \tag{15}$$

We ignore $\delta_0,\ g(t),\ k(t)$, and the second condition for a moment. Using the above choices, we obtain (the fourth condition is trivially satisfied)

$$\begin{cases} \lambda_3\left(2k^2(0) + 46\right) + \lambda_4\tilde{K}(0) < \lambda_1\left(1 - \bar{g}\right), \\ 2k(0)\lambda_3 < \lambda_4. \end{cases} \tag{16}$$

This is possible (and therefore λ_4 exists) if

$$\lambda_3\left(2k^2(0) + 2k(0)\tilde{K}(0) + 46\right) < \lambda_1\left(1 - \bar{g}\right). \tag{17}$$

First, we select the following:

- λ_1 large enough so that the condition in (17) is satisfied, then
- λ_4 so that both conditions in (16) are valid.
- Select δ_0 so small and t large so that the 1st, 3rd, 4th, and 5th conditions in (15) (without the terms in δ_1) hold.
- Next, we pass to choosing λ_9 large enough so that the second condition in (15) holds.
- Then select δ_1 so small that the 1st, 2nd, and 4th conditions (this time including the terms in δ_1) are satisfied.
- Finally, λ_3 is selected so small that the first 2 coefficients in (14) are negative.

Therefore, we are left with

$$L'(t) \leq -C_1 E(t) - C_2 \Lambda_4(t) + C_3 k(t),\ t \geq t_*$$

with $C_3 := \left(2\delta_0\lambda_5 + \frac{\lambda_3}{2}\right) \|\psi_{0x}\|^2$, and by the equivalence (11)

$$L'(t) \leq -C_4 L(t) + C_3 k(t),\ t \geq t_*$$

for positive constants $C_i,\ i = 1, ..., 4.$

Hence, for smaller $C_4,\ C_4 < \eta,$ if necessary, we see that

$$E(t) \leq M e^{-C_5 t},\ t \geq t_*$$

for some positive constants M and C_5, and thereafter this estimation holds for all $t \geq 0$ (with a different constant M). □

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