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On S-prime submodules

Esra ŞENGELEN SEVİM[1], **Tarık ARABACI**[1,*], **Ünsal TEKİR**[2], **Suat KOÇ**[2]
[1]Department of Mathematics, Faculty of Science, İstanbul Bilgi University, İstanbul, Turkey
[2]Department of Mathematics, Faculty of Science, Marmara University, Ziverbey, Göztepe, İstanbul, Turkey

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Abstract: In this study, we introduce the concepts of S-prime submodules and S-torsion-free modules, which are generalizations of prime submodules and torsion-free modules. Suppose $S \subseteq R$ is a multiplicatively closed subset of a commutative ring R, and let M be a unital R-module. A submodule P of M with $(P :_R M) \cap S = \emptyset$ is called an S-prime submodule if there is an $s \in S$ such that $am \in P$ implies $sa \in (P :_R M)$ or $sm \in P$. Also, an R-module M is called S-torsion-free if $ann(M) \cap S = \emptyset$ and there exists $s \in S$ such that $am = 0$ implies $sa = 0$ or $sm = 0$ for each $a \in R$ and $m \in M$. In addition to giving many properties of S-prime submodules, we characterize certain prime submodules in terms of S-prime submodules. Furthermore, using these concepts, we characterize some classical modules such as simple modules, S-Noetherian modules, and torsion-free modules.

1. Introduction

In every part of this study, we focus on only commutative rings with $1 \neq 0$ and nonzero unital modules. Let R always denote such a ring and M denote such an R-module. The notion of prime submodule has a significant place in the theory of modules, and it is used to characterize certain classes of modules. For years, there have been many studies and generalizations on this issue. See, for example, [6], [8], [14], and [18]. The aim of this article is to introduce S-prime submodules and S-torsion-free modules and to characterize certain prime submodules, S-Noetherian modules, simple modules, and torsion-free modules in terms of these concepts.

For the sake of completeness, we begin with some definitions and notations that will be followed in this paper. Let P, K be two submodules of an R-module M and J an ideal of R. Then the residual P by K and J is defined as follows:

$$(P :_R K) = \{a \in R : aK \subseteq P\},$$
$$(P :_M J) = \{m \in M : Jm \subseteq P\}.$$

Particularly, we use $ann(M)$ instead of $(0 :_R M)$, and we use $(P :_M s)$ instead of $(P :_M Rs)$, where Rs is the principal ideal generated by an element $s \in R$. Also, for any $s \in R$ and $m \in M$, we use $ann_M(s)$ to denote $(0 :_M Rs)$ and also we use $ann_R(m)$ to denote $(0 :_R Rm)$, where Rm is a cyclic submodule generated by $m \in M$. The sets of prime ideals and maximal ideals are denoted by $Spec(R)$ and $Max(R)$, respectively. A ring R is called quasilocal if $|Max(R)| = 1$. Recall from [11] that a prime submodule is a proper submodule

*Correspondence: tarik.arabaci@bilgi.edu.tr
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P of M having the property that $am \in P$ implies $a \in (P :_R M)$ or $m \in P$ for each $a \in R$ and $m \in M$. In that case, $(P :_R M) \in Spec(R)$. An R-module M is called a multiplication module if $P = (P :_R M)M$ for every submodule P of M [9]. If the only submodules of M are 0 and M, then we call M a simple module [15].

Consider a nonempty subset S of R. We call S a multiplicatively closed subset (briefly, m.c.s.) of R if (i) $0 \notin S$, (ii) $1 \in S$, and (iii) $ss' \in S$ for all $s, s' \in S$ [17]. Note that $S_P = R - P$ is a m.c.s. of R for every $P \in Spec(R)$. Let S be a m.c.s. of R and P a submodule of M with $(P :_R M) \cap S = \emptyset$. Then the submodule P is called an S-prime submodule if there exists $s \in S$, and whenever $am \in P$, then $sa \in (P :_R M)$ or $sm \in P$ for each $a \in R, m \in M$. Particularly, an ideal I of R is called an S-prime ideal if I is an S-prime submodule of R-module R. Note that all prime submodules P whose residual by M is disjoint from S become an S-prime submodule since $1 \in S$. Also, if we take $S \subseteq u(R)$, where $u(R)$ denotes the set of units in R, the notions of S-prime submodules and prime submodules are equal. Here, we denote the sets of all prime submodules and all S-prime submodules by $Spec(_RM)$ and $Spec_S(_RM)$, respectively. In particular, we write $Spec_S(R)$ to express the set of all S-prime ideals of R. Among many results in Section 2, we investigate the properties of S-prime submodules similar to prime submodules. In particular, we investigate the behavior of S-prime submodules under localization, under homomorphism, in factor modules, and in Cartesian products of modules (see Proposition 2.2, Proposition 2.7, Corollary 2.8, and Theorem 2.14). We give some characterizations of S-prime submodules in multiplication modules (see Proposition 2.9, Corollary 2.10, and Theorem 2.11). We characterize in Theorem 2.19 certain prime submodules in terms of S-prime submodules. Using Theorem 2.19, we determine all prime submodules of a module M over a quasilocal ring R in terms of S-prime submodules (see Corollary 2.20).

Recall the following well-known definition: an R-module M is torsion-free if the torsion subset $T(M) = \{m \in M : ann_R(m) \neq 0\}$ is zero. Let M be an R-module and $S \subseteq R$ be a m.c.s. of R with $ann(M) \cap S = \emptyset$. We call M an S-torsion-free module with $am = 0$ implying either $sa = 0$ or $sm = 0$ for some fixed $s \in S$ and for each $a \in R, m \in M$. It can be easily seen that being a torsion-free module is a sufficient condition for being an S-torsion-free module. Also, we can see that the class of S-torsion-free modules properly contains the class of torsion-free modules (observe Example 2.3). It is known that a proper submodule P of M is a prime submodule if and only if M/P is a torsion-free $R/(P :_R M)$-module [12, Lemma 1.1]. We prove that in Proposition 2.24, a sufficient and necessary condition for P being an S-prime submodule is that the factor module M/P is a $\pi(S)$-torsion-free $R/(P :_R M)$-module, where $\pi : R \to R/(P :_R M)$ is the canonical homomorphism. Furthermore, we give a characterization of torsion-free modules by using S-torsion-free modules (see Theorem 2.25). Finally, we characterize S-Noetherian modules and simple modules in terms of S-prime submodules (see Proposition 2.22 and Theorem 2.26).

2. Characterization of S-prime submodules

Definition 2.1 *Let $S \subseteq R$ be a m.c.s. and P a submodule of M with $(P :_R M) \cap S = \emptyset$. Then P is said to be an S-prime submodule if there exists $s \in S$ and whenever $am \in P$ then either $sa \in (P :_R M)$ or $sm \in P$ for each $a \in R$ and $m \in M$.*

Let $S \subseteq R$ be a m.c.s. and M an R-module. The quotient module of M is thus denoted by $S^{-1}M$. Note that $S^{-1}M$ is both an R- and $S^{-1}R$-module. Here we just consider $S^{-1}M$ as an $S^{-1}R$-module. Recall that the saturation S^* of S is defined as $S^* = \{x \in R : \frac{x}{1}$ is a unit of $S^{-1}R\}$. It is obvious that S^* is a m.c.s. of R containing S [10].

Proposition 2.2 *Assume that $S \subseteq R$ is a m.c.s. and M is an R-module. Then:*

(i) If $P \in Spec(_RM)$ provided that $(P :_R M)$ and S are disjoint, then $P \in Spec_S(_RM)$. In fact, if $S \subseteq u(R)$ and $P \in Spec_S(_RM)$, then $P \in Spec(_RM)$.

(ii) If $S_1 \subseteq S_2$ are m.c.s. of R and $P \in Spec_{S_1}(_RM)$, then $P \in Spec_{S_2}(_RM)$ in case $(P :_R M) \cap S_2 = \emptyset$.

(iii) $P \in Spec_S(_RM)$ if and only if $P \in Spec_{S^*}(_RM)$.

(iv) If $P \in Spec_S(_RM)$, then $S^{-1}P$ is a prime submodule of $S^{-1}M$.

Proof (i), (ii): It is clear.

(iii): Assume that $P \in Spec_S(_RM)$. We need to prove that $(P :_R M)$ and S^* are disjoint. Suppose there exists $x \in (P :_R M) \cap S^*$. As $x \in S^*$, $\frac{x}{1}$ is a unit of $S^{-1}R$ and so $\frac{x}{1}\frac{a}{s} = 1$ for some $a \in R$ and $s \in S$. This yields that $us = uxa$ for some $u \in S$. Now put $us = s' \in S$. Then note that $s' = us = uxa \in (P :_R M) \cap S$, a contradiction. Thus, $(P :_R M) \cap S^* = \emptyset$. As $S \subseteq S^*$, by (ii), $P \in Spec_{S^*}(_RM)$. Conversely, assume that $P \in Spec_{S^*}(_RM)$. Let $rm \in P$. As $P \in Spec_{S^*}(_RM)$, there is an $x \in S^*$ so that either $xr \in (P :_R M)$ or $xm \in P$. As $\frac{x}{1}$ is a unit of $S^{-1}R$, there exist $u, s \in S$ and $a \in R$ such that $us = uxa$. Put $us = s' \in S$. Then note that $s'r = (us)r = uaxr \in (P :_R M)$ or $s'm = ua(xm) \in P$. Therefore, $P \in Spec_S(_RM)$.

(iv): Assume that $P \in Spec_S(_RM)$. Let $\frac{r}{s}\frac{m}{t} \in S^{-1}P$, where $\frac{r}{s} \in S^{-1}R$ and $\frac{m}{t} \in S^{-1}M$. Then $urm \in P$ for some $u \in S$. Since $P \in Spec_S(_RM)$, there is an $s' \in S$ so that $s'ur \in (P :_R M)$ or $s'm \in P$. This yields $\frac{r}{s} = \frac{s'ur}{s'us} \in S^{-1}(P :_R M) \subseteq (S^{-1}P :_{S^{-1}R} S^{-1}M)$ or $\frac{m}{t} = \frac{s'm}{s't} \in S^{-1}P$. Hence, $S^{-1}P$ is a prime submodule of $S^{-1}M$. □

The converses of Proposition 2.2(i) and (iv) are not true in general. See the following two examples.

Example 2.3 *Take the $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z}_2$ and the zero submodule $P = 0 \times 0$. First note that $(P :_{\mathbb{Z}} \mathbb{Z} \times \mathbb{Z}_2) = 0$ and $2(0, \overline{1}) = (0, \overline{0}) \in P$. Since $2 \notin (P :_{\mathbb{Z}} \mathbb{Z} \times \mathbb{Z}_2)$ and $(0, \overline{1}) \notin P$, P is not a prime submodule of $\mathbb{Z} \times \mathbb{Z}_2$. Now, take the m.c.s. $S = \mathbb{Z} - \{0\}$ of $\mathbb{Z}$ and put $s = 2$. Let $m(a, \overline{x}) = (ma, \overline{mx}) \in P$. Then $ma = 0$ and $m\overline{x} = 0$. If $m = 0$, there is nothing to show, so assume that $a = 0$. Then it is clear that $s(a, \overline{x}) \in P$. Therefore, $P \in Spec_S(_{\mathbb{Z}}\mathbb{Z} \times \mathbb{Z}_2)$.*

Example 2.4 *Consider the $\mathbb{Z}$-module $\mathbb{Q} \times \mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers. Take the submodule $N = \mathbb{Z} \times 0$ and the m.c.s. $S = \mathbb{Z} - \{0\}$ of $\mathbb{Z}$. Then it is clear that $(N :_{\mathbb{Z}} \mathbb{Q} \times \mathbb{Q}) = 0$. Let s be an arbitrary element of S. Choose a prime number p with $\gcd(p, s) = 1$. Then note that $p(\frac{1}{p}, 0) = (1, 0) \in N$. Since $sp \notin (N :_{\mathbb{Z}} \mathbb{Q} \times \mathbb{Q})$ and $s(\frac{1}{p}, 0) = (\frac{s}{p}, 0) \notin N$, it follows that N is not an S-prime submodule. Since $S^{-1}\mathbb{Z} = \mathbb{Q}$ is a field, $S^{-1}(\mathbb{Q} \times \mathbb{Q})$ is a vector space so that the proper submodule $S^{-1}N$ is a prime submodule of $S^{-1}(\mathbb{Q} \times \mathbb{Q})$.*

Lemma 2.5 *Suppose P is a submodule of M and S is a m.c.s. of R satisfying $(P :_R M) \cap S = \emptyset$. The following are equivalent:*

(i) $P \in Spec_S(_RM)$.

(ii) There is an $s \in S$, and $JN \subseteq P$ implies $sJ \subseteq (P :_R M)$ or $sN \subseteq P$ for each ideal J of R and submodule N of M.

Proof $(i) \Rightarrow (ii)$: Let $P \in Spec_S(_RM)$. Suppose that $JN \subseteq P$ for some ideal J of R and submodule N of M. As $P \in Spec_S(_RM)$, there is an $s \in S$ so that $rm \in P$ implies $sr \in (P :_R M)$ or $sm \in P$ for each $r \in R$

and $m \in M$. Assume that $sN \nsubseteq P$. Then there is an $n \in N$ with $sn \notin P$. Then note that for each $a \in J$, we have $an \in P$. Since $P \in Spec_S({}_RM)$, we can conclude that $sa \in (P :_R M)$ and so $sJ \subseteq (P :_R M)$.

$(ii) \Rightarrow (i)$: Take $a \in R$ and $m \in M$ with $am \in P$. Now, put $J = Ra$ and $N = Rm$. Then we can conclude that $JN = Ram \subseteq P$. By assumption, there is an $s \in S$ so that $sJ = Ras \subseteq (P :_R M)$ or $sN = Rsm \subseteq P$ and so either $sa \in (P :_R M)$ or $sm \in P$. Therefore, $P \in Spec_S({}_RM)$. □

Corollary 2.6 *Suppose S is a m.c.s. of R and take any ideal P with $P \cap S = \emptyset$. Then the following are equivalent:*

(i) $P \in Spec_S(R)$.

(ii) There is an $s \in S$, for each ideal I, J of R with $IJ \subseteq P$, so either $sI \subseteq P$ or $sJ \subseteq P$.

Proposition 2.7 *Suppose $f : M \to M'$ is an R-homomorphism. Then:*

(i) If $P' \in Spec_S({}_RM')$ provided that $(f^{-1}(P') :_R M) \cap S = \emptyset$, then $f^{-1}(P') \in Spec_S({}_RM)$.

(ii) If f is an epimorphism and $P \in Spec_S({}_RM)$ with $Ker(f) \subseteq P$, then $f(P) \in Spec_S({}_RM')$.

Proof (i): Let $am \in f^{-1}(P')$ for some $a \in R, m \in M$. This yields that $f(am) = af(m) \in P'$. Since $P' \in Spec_S({}_RM')$, there is an $s \in S$ so that $sa \in (P' :_R M')$ or $sf(m) = f(sm) \in P'$. Now we will show that $(P' :_R M') \subseteq (f^{-1}(P') :_R M)$. Take $x \in (P' :_R M')$. Then we have $xM' \subseteq P'$. Since $f(M) \subseteq M'$, we conclude that $f(xM) = xf(M) \subseteq xM' \subseteq P'$. This implies that $xM \subseteq xM + Ker(f) = f^{-1}(f(xM)) \subseteq f^{-1}(P')$ and thus $x \in (f^{-1}(P') :_R M)$. As $(P' :_R M') \subseteq (f^{-1}(P') :_R M)$, we can conclude either $sa \in (f^{-1}(P') :_R M)$ or $sm \in f^{-1}(P')$. Hence, $f^{-1}(P') \in Spec_S({}_RM)$.

(ii): First note that $(f(P) :_R M') \cap S = \emptyset$. Otherwise there would be an $s \in (f(P) :_R M') \cap S$. Since $s \in (f(P) :_R M')$, $sM' \subseteq f(P)$, but then $f(sM) = sf(M) \subseteq sM' \subseteq f(P)$. By taking their inverse images under f, we have $sM \subseteq sM + Ker(f) \subseteq P + Ker(f) = P$. That means $sM \subseteq P$. Then s must be in $(P :_R M)$, which contradicts $P \in Spec_S({}_RM)$. Now take $a \in R, m' \in M$ with $am' \in f(P)$. As f is an epimorphism, there is an $m \in M$ such that $m' = f(m)$. Then $am' = af(m) = f(am) \in f(P)$. Since $Ker(f)$ is a subset of P, we get $am \in P$. As $P \in Spec_S({}_RM)$, there is an $s \in S$ so that $sa \in (P :_R M)$ or $sm \in P$. Since $(P :_R M) \subseteq (f(P) :_R M')$, we have $sa \in (f(P) :_R M')$ or $f(sm) = sf(m) = sm' \in f(P)$. Accordingly, $f(P) \in Spec_S({}_RM')$. □

Corollary 2.8 *Let $S \subseteq R$ be a m.c.s. of R and take a submodule L of M.*

(i) If $P' \in Spec_S({}_RM)$ with $(P' :_R L) \cap S = \emptyset$, then $L \cap P' \in Spec_S({}_RL)$.

(ii) Suppose that P is a submodule of M with $L \subseteq P$. Then $P \in Spec_S({}_RM)$ if and only if $P/L \in Spec_S({}_R(M/L))$.

Proof (i): Consider the injection $i : L \to M$ defined by $i(m) = m$ for all $m \in L$. Then note that $i^{-1}(P') = L \cap P'$. Now we will show that $(i^{-1}(P') :_R L) \cap S = \emptyset$. Assume that $s \in (i^{-1}(P') :_R L) \cap S$. Then we have $sL \subseteq i^{-1}(P') = L \cap P' \subseteq P'$ and thus $s \in (P' :_R L) \cap S$, a contradiction. The rest follows from Proposition 2.7 (i).

(ii) Assume that $P \in Spec_S({}_RM)$. Then consider the canonical homomorphism $\pi : M \to M/L$ defined by $\pi(m) = m + L$ for all $m \in M$. By Proposition 2.7 (ii), $P/L \in Spec_S({}_R(M/L))$. Conversely, assume that

$P/L \in Spec_S(_R(M/L))$. Let $rm \in P$ for some $r \in R, m \in M$. This yields that $r(m+L) = rm + L \in P/L$. As $P/L \in Spec_S(_R(M/L))$, there is an $s \in S$ so that $sr \in (P/L :_R M/L) = (P :_R M)$ or $s(m+L) = sm + L \in P/L$. Therefore, we have $sr \in (P :_R M)$ or $sm \in P$. Hence, $P \in Spec_S(_RM)$. □

Proposition 2.9 *Let M be an R-module and S a m.c.s. of R. The following statements hold:*

(i) If $P \in Spec_S(_RM)$, then $(P :_R M) \in Spec_S(R)$.

(ii) If M is a multiplication module and $(P :_R M) \in Spec_S(R)$, then $P \in Spec_S(_RM)$.

Proof (i) Let $xy \in (P :_R M)$ for some $x, y \in R$. As $P \in Spec_S(_RM)$, there is an $s \in S$ so that $rm \in P$ implies $sr \in (P :_R M)$ or $sm \in P$. Then note that $xym \in P$ for all $m \in M$. If $sx \in (P :_R M)$, there is nothing to prove. Suppose that $sx \notin (P :_R M)$. As $P \in Spec_S(_RM)$, $sym \in P$ for all $m \in M$ so that $sy \in (P :_R M)$. Therefore, $(P :_R M) \in Spec_S(R)$.

(ii) Assume that M is a multiplication module and $(P :_R M) \in Spec_S(R)$. Let J be an ideal of R and N a submodule of M with $JN \subseteq P$. Then we can conclude that $J(N :_R M) \subseteq (JN :_R M) \subseteq (P :_R M)$. As $(P :_R M) \in Spec_S(R)$, by Corollary 2.6, there is an $s \in S$ so that $sJ \subseteq (P :_R M)$ or $s(N :_R M) \subseteq (P :_R M)$. Thus, we can conclude that $sJ \subseteq (P :_R M)$ or $sN = s(N :_R M)M \subseteq (P :_R M)M = P$. By Lemma 2.5, $P \in Spec_S(_RM)$. □

Assume that M is a multiplication module and K, L are two submodules of M. The product of K and L is defined as $KL = (K :_R M)(L :_R M)M$ [2]. As an immediate consequence of the previous proposition and Lemma 2.5, we have the following explicit result.

Corollary 2.10 *Suppose that M is a multiplication module and P is a submodule of M with $(P :_R M) \cap S = \emptyset$, where S is a m.c.s. of R. Then the following are equivalent:*

(i) $P \in Spec_S(_RM)$.

(ii) There is an $s \in S$, for every submodule L, N of M with $LN \subseteq P$, and then $sL \subseteq P$ or $sN \subseteq P$.

Theorem 2.11 *Let M be a finitely generated multiplication module and P a submodule of M provided that $(P :_R M) \cap S = \emptyset$, where S is a m.c.s. of R. Then the following are equivalent:*

(i) $P \in Spec_S(_RM)$.

(ii) $(P :_R M) \in Spec_S(R)$.

(iii) $P = IM$ for some $I \in Spec_S(R)$ with $ann(M) \subseteq I$.

Proof $(i) \Leftrightarrow (ii)$: It is clear from Proposition 2.9.

$(ii) \Rightarrow (iii)$: It is obvious.

$(iii) \Rightarrow (i)$: Suppose that $P = IM$ for some $I \in Spec_S(R)$ with $ann(M) \subseteq I$. Assume that $JN \subseteq P$ for some ideal J of R and some submodule N of M. This yields $J(N :_R M)M \subseteq IM$. As M is a finitely generated multiplication module, by [16, Theorem 9 Corollary], $J(N :_R M) \subseteq I + ann(M) = I$. Since $I \in Spec_S(R)$, by Corollary 2.6, there is an $s \in S$ so that $sJ \subseteq I \subseteq (P :_R M)$ or $s(N :_R M) \subseteq I \subseteq (P :_R M)$ and hence $sJ \subseteq (P :_R M)$ or $sN \subseteq P$. □

Proposition 2.12 *Let M be a multiplication module and $P \in Spec_S(_RM)$. Suppose that $N \cap L \subseteq P$ for some submodules N, L of M. Then $sN \subseteq P$ or $sL \subseteq P$ for some $s \in S$.*

Proof As $P \in Spec_S({}_RM)$, there is an $s \in S$ so that $rm \in P$ implies $sr \in (P :_R M)$ or $sm \in P$ for each $r \in R$ and $m \in M$. Let $sL \nsubseteq P$. Then $sm \notin P$ for some $m \in L$. Take an element $a \in (N :_R M)$. This yields $am \in (N :_R M)L \subseteq L \cap N \subseteq P$. As $P \in Spec_S({}_RM)$ and $sm \notin P$, we can conclude that $sa \in (P :_R M)$ so that $s(N :_R M) \subseteq (P :_R M)$. As M is a multiplication module, $sN = s(N :_R M)M \subseteq (P :_R M)M = P$. □

Lemma 2.13 *Let $R = R_1 \times R_2$ and $S = S_1 \times S_2$, where S_i is a m.c.s. of R_i. Suppose $P = P_1 \times P_2$ is an ideal of R. So the following are equivalent:*

(i) $P \in Spec_S(R)$.

(ii) $P_1 \in Spec_{S_1}(R_1)$ and $P_2 \cap S_2 \neq \emptyset$ or $P_1 \cap S_1 \neq \emptyset$ and $P_2 \in Spec_{S_2}(R_2)$.

Proof $(i) \Rightarrow (ii)$: Suppose $P \in Spec_S(R)$. Since $(1,0)(0,1) = (0,0) \in P$, there exists $s = (s_1, s_2) \in S$ so that $s(1,0) = (s_1, 0) \in P$ or $s(0,1) = (0, s_2) \in P$ and thus $P_1 \cap S_1 \neq \emptyset$ or $P_2 \cap S_2 \neq \emptyset$. We may assume that $P_1 \cap S_1 \neq \emptyset$. As $P \cap S = \emptyset$, we have $P_2 \cap S_2 = \emptyset$. Let $xy \in P_2$ for some $x, y \in R_2$. Since $(0,x)(0,y) \in P$ and $P \in Spec_S(R)$, we get either $s(0,x) = (0, s_2x) \in P$ or $s(0,y) = (0, s_2y) \in P$ and this yields $s_2x \in P_2$ or $s_2y \in P_2$. Therefore, $P_2 \in Spec_{S_2}(R_2)$. In the other case, one can easily show that $P_1 \in Spec_{S_1}(R_1)$.

$(ii) \Rightarrow (i)$: Assume that $P_1 \cap S_1 \neq \emptyset$ and $P_2 \in Spec_{S_2}(R_2)$. Then there exists $s_1 \in P_1 \cap S_1$. Let $(a,b)(c,d) = (ac, bd) \in P$ for some $a, c \in R_1$ and $b, d \in R_2$. This yields that $bd \in P_2$ and thus there exists $s_2 \in S_2$ so that $s_2b \in P_2$ or $s_2d \in P_2$. Put $s = (s_1, s_2) \in S$. Then note that $s(a,b) = (s_1a, s_2b) \in P$ or $s(c,d) = (s_1c, s_2d) \in P$. Therefore, $P \in Spec_S(R)$. In other case, one can similarly prove that $P \in Spec_S(R)$. □

Theorem 2.14 *Suppose that $M = M_1 \times M_2$ is an $R = R_1 \times R_2$ module and $S = S_1 \times S_2$ is a m.c.s. of R, where M_i is an R_i-module and S_i is a m.c.s. of R_i for each $i = 1, 2$. Assume $P = P_1 \times P_2$ is a submodule of M. The following are equivalent:*

(i) $P \in Spec_S({}_RM)$.

(ii) $P_1 \in Spec_{S_1}({}_{R_1}M_1)$ and $(P_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$ or $(P_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$ and $P_2 \in Spec_{S_2}({}_{R_2}M_2)$.

Proof $(i) \Rightarrow (ii)$: Assume that $P \in Spec_S({}_RM)$. Then by Proposition 2.9, $(P :_R M) = (P_1 :_{R_1} M_1) \times (P_2 :_{R_2} M_2) \in Spec_S(R)$ and so by Lemma 2.13, either $(P_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$ or $(P_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$. We may assume that $(P_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$. Now we will show that $P_2 \in Spec_{S_2}({}_{R_2}M_2)$. Let $rm \in P_2$ for some $r \in R_2, m \in M_2$. Then $(1,r)(0,m) = (0, rm) \in P$. As $P \in Spec_S({}_RM)$, there is an $s = (s_1, s_2) \in S$ so that $s(1,r) = (s_1, s_2r) \in (P :_R M)$ or $s(0,m) = (0, s_2m) \in P$. This implies that $s_2r \in (P_2 :_{R_2} M_2)$ or $s_2m \in P_2$. Therefore, $P_2 \in Spec_{S_2}({}_{R_2}M_2)$. In the other case, it can be similarly shown that $P_1 \in Spec_{S_1}({}_{R_1}M_1)$.

$(ii) \Rightarrow (i)$: Assume that $(P_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$ and $P_2 \in Spec_{S_2}({}_{R_2}M_2)$. Then there exists $s_1 \in (P_1 :_{R_1} M_1) \cap S_1$. Let $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2) \in P$ for some $r_i \in R_i, m_i \in M_i$, where $i = 1, 2$. Then $r_2m_2 \in P_2$. As $P_2 \in Spec_{S_2}({}_{R_2}M_2)$, there is an $s_2 \in S_2$ so that $s_2r_2 \in (P_2 :_{R_2} M_2)$ or $s_2m_2 \in P_2$. Now put $s = (s_1, s_2) \in S$. Then note that $s(r_1, r_2) = (s_1r_1, s_2r_2) \in (P :_R M)$ or $s(m_1, m_2) = (s_1m_1, s_2m_2) \in P_1 \times P_2 = P$. Therefore, $P \in Spec_S({}_RM)$. Similarly one can show that if $P_1 \in Spec_{S_1}({}_{R_1}M_1)$ and $(P_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$, then $P \in Spec_S({}_RM)$. □

Theorem 2.15 *Let $M = M_1 \times M_2 \times \ldots \times M_n$ be an $R = R_1 \times R_2 \times \ldots \times R_n$ module and $S = S_1 \times S_2 \times \ldots \times S_n$, where M_i is an R_i-module and S_i is a m.c.s. of R_i for each $i = 1, 2, \ldots, n$. Assume $P = P_1 \times P_2 \times \ldots \times P_n$ is a submodule of M. The following statements are equivalent:*

(i) $P \in Spec_S({}_RM)$.

(ii) $P_i \in Spec_{S_i}({}_{R_i}M_i)$ *for some* $i \in \{1,2,\ldots,n\}$ *and* $(P_j :_{R_j} M_j) \cap S_j \neq \emptyset$ *for all* $j \in \{1,2,\ldots,n\} - \{i\}$.

Proof We apply induction on n. For $n = 1$, the result is true. If $n = 2$, then $(i) \Leftrightarrow (ii)$ follows from Theorem 2.14. Assume that (i) and (ii) are equal when $k < n$. Now, we shall prove $(i) \Leftrightarrow (ii)$ when $k = n$. Let $P = P_1 \times P_2 \times \ldots \times P_n$. Put $P' = P_1 \times P_2 \times \ldots \times P_{n-1}$ and $S' = S_1 \times S_2 \times \ldots \times S_{n-1}$. Then by Theorem 2.14, the necessary and sufficient condition for $P = P' \times P_n \in Spec_S({}_RM)$ is that $P' \in Spec_{S'}({}_{R'}M')$ and $(P_n :_{R_n} M_n) \cap S_n \neq \emptyset$ or $(P' :_{R'} M') \cap S' \neq \emptyset$ and $P_n \in Spec_{S_n}({}_{R_n}M_n)$, where $M' = M_1 \times M_2 \times \ldots \times M_{n-1}$ and $R' = R_1 \times R_2 \times \ldots \times R_{n-1}$. The rest follows from the induction hypothesis. $\square$

Lemma 2.16 *Suppose that* $P \in Spec_S({}_RM)$. *The following statements hold for some* $s \in S$:

(i) $(P :_M s') \subseteq (P :_M s)$ *for all* $s' \in S$.

(ii) $((P :_R M) :_R s') \subseteq ((P :_R M) :_R s)$ *for all* $s' \in S$.

Proof (i): Suppose that $P \in Spec_S({}_RM)$. Then there is an $s \in S$ so that $rm \in P$ implies $sr \in (P :_R M)$ or $sm \in P$. Take an element $m' \in (P :_M s')$, where $s' \in S$. Then $s'm' \in P$. Since $P \in Spec_S({}_RM)$, either $ss' \in (P :_R M)$ or $sm' \in P$. As $(P :_R M) \cap S = \emptyset$, we get $sm' \in P$, namely $m' \in (P :_M s)$.

(ii): Follows from (i). $\square$

Proposition 2.17 *Suppose that* M *is a finitely generated* R*-module,* $S \subseteq R$ *is a m.c.s. of* R, *and* P *is a submodule of* M *satisfying* $(P :_R M) \cap S = \emptyset$. *The following are equivalent:*

(i) $P \in Spec_S({}_RM)$.

(ii) $S^{-1}P$ *is a prime submodule of* $S^{-1}M$ *and there is an* $s \in S$ *satisfying* $(P :_M s') \subseteq (P :_M s)$ *for all* $s' \in S$.

Proof $(i) \Rightarrow (ii)$: It is clear from Proposition 2.2 and Lemma 2.16.

$(ii) \Rightarrow (i)$: Take $a \in R, m \in M$ with $am \in P$. Then $\frac{a}{1}\frac{m}{1} \in S^{-1}P$. Since $S^{-1}P$ is a prime submodule of $S^{-1}M$ and M is finitely generated, we can conclude that $\frac{a}{1} \in (S^{-1}P :_{S^{-1}R} S^{-1}M) = S^{-1}(P :_R M)$ or $\frac{m}{1} \in S^{-1}P$. Then $ua \in (P :_R M)$ or $u'm \in P$ for some $u, u' \in S$. By assumption, there is an $s \in S$ so that $(P :_M s') \subseteq (P :_M s)$ for all $s' \in S$. If $ua \in (P :_R M)$, then $aM \subseteq (P :_M u) \subseteq (P :_M s)$ and thus $sa \in (P :_R M)$. If $u'm \in P$, a similar argument shows that $sm \in P$. Therefore, $P \in Spec_S({}_RM)$. $\square$

Theorem 2.18 *Suppose that* P *is a submodule of* M *provided* $(P :_R M) \cap S = \emptyset$. *Then* $P \in Spec_S({}_RM)$ *if and only if* $(P :_M s) \in Spec({}_RM)$ *for some* $s \in S$.

Proof Assume $(P :_M s) \in Spec({}_RM)$ for some $s \in S$. Let $am \in P$ for some $a \in R, m \in M$. As $am \in (P :_M s)$ and $(P :_M s) \in Spec({}_RM)$, we get $a \in ((P :_M s) :_R M)$ or $m \in (P :_M s)$. This yields that $as \in (P :_R M)$ or $sm \in P$. Conversely, assume that $P \in Spec_S({}_RM)$. Then there is an $s \in S$ so that $am \in P$ implies $sa \in (P :_R M)$ or $sm \in P$. Now we prove that $(P :_M s) \in Spec({}_RM)$. Take $r \in R, m \in M$ with $rm \in (P :_M s)$. Then $(sr)m \in P$. As $P \in Spec_S({}_RM)$, we get $s^2r \in (P :_R M)$ or $sm \in P$. If $sm \in P$, then there is nothing to show. Assume that $sm \notin P$. Then $s^2r \in (P :_R M)$ and so

$r \in ((P :_R M) :_R s^2) = ((P :_R M) :_R s))$ by Lemma 2.16. Thus, we can conclude that $r \in ((P :_M s) :_R M)$ and hence $(P :_M s) \in Spec({}_RM)$. □

Consider an R-module M with a submodule P. It is known that if $P \in Spec({}_RM)$, then $(P :_R M) \in Spec(R)$, but the converse is not true in general. Now we characterize certain prime submodules in terms of S-prime submodules.

Theorem 2.19 *Suppose that P is a submodule of M provided $(P :_R M) \subseteq Jac(R)$, where $Jac(R)$ is the Jacobson radical of R. The following statements are equivalent:*

(i) $P \in Spec({}_RM)$.

(ii) $(P :_R M) \in Spec(R)$ *and* $P \in Spec_{R-\mathfrak{M}}({}_RM)$ *for each* $\mathfrak{M} \in Max(R)$.

Proof $(i) \Rightarrow (ii)$: Assume that $P \in Spec({}_RM)$. Since $(P :_R M) \subseteq Jac(R)$, $(P :_R M) \subseteq \mathfrak{M}$ for each $\mathfrak{M} \in Max(R)$ and hence $(P :_R M) \cap (R - \mathfrak{M}) = \emptyset$. The rest follows from Proposition 2.2(i).

$(ii) \Rightarrow (i)$: Suppose $(P :_R M) \in Spec(R)$ and $P \in Spec_{R-\mathfrak{M}}({}_RM)$ for each $\mathfrak{M} \in Max(R)$. Let $am \in P$ with $a \notin (P :_R M)$ for some $a \in R, m \in M$. Let $\mathfrak{M} \in Max(R)$. As $P \in Spec_{R-\mathfrak{M}}({}_RM)$, we can guarantee the existence of an $s_{\mathfrak{M}} \notin \mathfrak{M}$ so that $as_{\mathfrak{M}} \in (P :_R M)$ or $s_{\mathfrak{M}}m \in P$. As $(P :_R M) \in Spec(R)$ and $s_{\mathfrak{M}} \notin (P :_R M)$, we have $as_{\mathfrak{M}} \notin (P :_R M)$ and so $s_{\mathfrak{M}}m \in P$. Now consider the set $\Omega = \{s_{\mathfrak{M}} : \exists\, \mathfrak{M} \in Max(R),\ s_{\mathfrak{M}} \notin \mathfrak{M}$ and $s_{\mathfrak{M}}m \in P\}$. Then note that $(\Omega) = R$. To see this, take any maximal ideal $\mathfrak{M}'$ containing Ω. Then the definition of Ω requires that there exists $s_{\mathfrak{M}'} \in \Omega$ and $s_{\mathfrak{M}'} \notin \mathfrak{M}'$. As $\Omega \subseteq \mathfrak{M}'$, we have $s_{\mathfrak{M}'} \in \Omega \subseteq \mathfrak{M}'$, a contradiction. Thus, $(\Omega) = R$, and this yields $1 = r_1 s_{\mathfrak{M}_1} + r_2 s_{\mathfrak{M}_2} + \ldots + r_n s_{\mathfrak{M}_n}$ for some $r_i \in R$ and $s_{\mathfrak{M}_i} \notin \mathfrak{M}_i$ with $s_{\mathfrak{M}_i}m \in P$, where $\mathfrak{M}_i \in Max(R)$ for each $i = 1, 2, \ldots, n$. This yields that $m = r_1 s_{\mathfrak{M}_1}m + r_2 s_{\mathfrak{M}_2}m + \ldots + r_n s_{\mathfrak{M}_n}m \in P$. Therefore, $P \in Spec({}_RM)$. □

Now we determine all prime submodules of a module over a quasilocal ring in terms of S-prime submodules.

Corollary 2.20 *Suppose M is a module over a quasilocal ring $(R, \mathfrak{M})$. The following are equivalent:*

(i) $P \in Spec({}_RM)$.

(ii) $(P :_R M) \in Spec(R)$ *and* $P \in Spec_{R-\mathfrak{M}}({}_RM)$.

Proof It is clear from Theorem 2.19. □

Suppose that M is an R-module. The idealization $R(+)M = \{(a, m) : a \in R, m \in M\}$ of M is a commutative ring whose addition is componentwise and whose multiplication is defined as $(a, m)(b, m') = (ab, am' + bm)$ for each $a, b \in R$; $m, m' \in M$ [13]. If S is a m.c.s. of R and P is a submodule of M, then $S(+)P = \{(s, p) : s \in S, p \in P\}$ is a m.c.s. of $R(+)M$ [4].

Proposition 2.21 *Suppose that S is a m.c.s. of R and P is an ideal of R provided $P \cap S = \emptyset$. The following are equivalent:*

(i) $P \in Spec_S(R)$.

(ii) $P(+)M \in Spec_{S(+)0}(R(+)M)$.

(iii) $P(+)M \in Spec_{S(+)M}(R(+)M)$.

Proof $(i) \Rightarrow (ii)$: Suppose $P \in Spec_S(R)$. Let $(x, m)(y, m') = (xy, xm' + ym) \in P(+)M$ for some $x, y \in R$; $m, m' \in M$. Then we get $xy \in P$. As $P \in Spec_S(R)$, we can find $s \in S$ so that $sx \in P$ or $sy \in P$.

Now put $s' = (s,0) \in S(+)0$. Then we have $s'(x,m) = (sx,sm) \in P(+)M$ or $s'(y,m') = (sy,sm') \in P(+)M$. Therefore, $P(+)M \in Spec_{S(+)0}(R(+)M)$.

$(ii) \Rightarrow (iii):$ It is clear from Proposition 2.2.

$(iii) \Rightarrow (i):$ Suppose $P(+)M \in Spec_{S(+)M}(R(+)M)$. Let $xy \in P$ for some $x,y \in R$. Then $(x,0)(y,0) \in P(+)M$. Since $P(+)M \in Spec_{S(+)M}(R(+)M)$, there is an $s = (s_1,m_1) \in S(+)M$ so that $s(x,0) = (s_1x, xm_1) \in P(+)M$ or $s(y,0) = (s_1y, ym_1) \in P(+)M$ and hence we get $s_1x \in P$ or $s_1y \in P$. Therefore, $P \in Spec_S(R)$. □

Let M be an R-module and S a m.c.s. of R. The concept of S-Noetherian modules, a generalization of Noetherian modules, was first studied by Anderson and Dumitrescu in [3]. Recently it has drawn attention and there have been many studies on this issue. See, for example, [1] and [7]. Suppose P is a submodule of M. We can call P an S-finite submodule if there is an $s \in S$ and a finitely generated submodule K of M so that $sP \subseteq K \subseteq P$. If all submodules of M are S-finite, M is called an S-Noetherian module. Note that if $S \subseteq u(R)$, then the concepts of S-Noetherian modules and Noetherian modules coincide. Also, if R is an S-Noetherian R-module, then we call R an S-Noetherian ring. The authors in [3] extended many properties of Noetherian rings and modules to S-Noetherian rings and S-Noetherian modules. In particular, they proved the S-version of Cohen's theorem: let M be an S-finite R-module. Then M is an S-Noetherian R-module if and only if PM is S-finite for every prime ideal P of R (disjoint from S). As every prime ideal P disjoint from S is an S-prime ideal, we give the following explicit result.

Proposition 2.22 *Assume M is an S-finite R-module, where S is a m.c.s. of R. Then M is an S-Noetherian module if and only if PM is an S-finite submodule for every $P \in Spec_S(R)$.*

Proof It is clear from Proposition 2.2 and [3, Proposition 4]. □

Definition 2.23 *Suppose M is an R-module and $S \subseteq R$ is a m.c.s. of R and none of the elements of S is an annihilator of M. M is said to be an S-torsion-free module in the case that we can find $s \in S$ and whenever $rm = 0$, so $sr = 0$ or $sm = 0$ for each $r \in R$ and $m \in M$.*

Consider an ideal J of R. The canonical homomorphism that maps the ring R to the factor ring R/J is denoted by $\pi(a) = a + J$ for each $a \in R$.

Proposition 2.24 *Let M be an R-module. Assume that $S \subseteq R$ is a m.c.s. of R and P is a submodule of M. Then $P \in Spec_S({}_RM)$ if and only if the factor module M/P is a $\pi(S)$-torsion-free $R/(P :_R M)$-module, where $\pi : R \to R/(P :_R M)$ is the canonical homomorphism.*

Proof Suppose $P \in Spec_S({}_RM)$. Let $\overline{a}\overline{m} = 0_{M/P}$, where $\overline{a} = a + (P :_R M)$ and $\overline{m} = m + P$ for some $a \in R, m \in M$. This yields that $am \in P$. As $P \in Spec_S({}_RM)$, there is an $s \in S$ so that $sa \in (P :_R M)$ or $sm \in P$. Then we can conclude that $\pi(s)\overline{a} = 0_{R/(P:_RM)}$ or $\pi(s)\overline{m} = 0_{M/P}$. Therefore, M/P is a $\pi(S)$-torsion-free $R/(P :_R M)$-module. For the other direction, we will suppose that M/P is a $\pi(S)$-torsion-free $R/(P :_R M)$-module. Take $a \in R$ and $m \in M$ with $am \in P$. Put $\overline{a} = a + (P :_R M)$ and $\overline{m} = m + P$. Then note that $\overline{a}\overline{m} = 0_{M/P}$. As M/P is a $\pi(S)$-torsion-free $R/(P :_R M)$-module, we can find $s \in S$, which satisfies $\pi(s)\overline{a} = 0_{R/(P:_RM)}$ or $\pi(s)\overline{m} = 0_{M/P}$. This yields that $sa \in (P :_R M)$ or $sm \in P$. Accordingly, $P \in Spec_S({}_RM)$. □

It is known that if M is a torsion-free module, then R is an integral domain and M is a faithful module. However, sometimes the reverse of this statement may not be true. Consider the $\mathbb{Z}$-module $\prod_{i=1}^{\infty}\mathbb{Z}_{p^i}$, where p is a prime number. Then it is easy to see that $\mathbb{Z}$-module $\prod_{i=1}^{\infty}\mathbb{Z}_{p^i}$ is a faithful module. Take $a = p$ and $m = (\overline{1}, \overline{0}, \overline{0}, \ldots)$. It is clear that $am = 0$ but $a \neq 0$ and $m \neq 0$, and so $\mathbb{Z}$-module $\prod_{i=1}^{\infty}\mathbb{Z}_{p^i}$ is not a torsion-free module. Now, for the next step, we characterize torsion-free modules in terms of S-torsion-free modules.

Theorem 2.25 *Let M be a module over an integral domain R. The following are equivalent:*

(i) M is a torsion-free module.

(ii) M is an $(R-P)$-torsion-free for each $P \in Spec(R)$.

(iii) M is an $(R-\mathfrak{M})$-torsion-free for each $\mathfrak{M} \in Max(R)$.

Proof $(i) \Rightarrow (ii)$: It is clear.

$(ii) \Rightarrow (iii)$: It is obvious.

$(iii) \Rightarrow (i)$: Suppose M is $(R-\mathfrak{M})$-torsion-free for each $\mathfrak{M} \in Max(R)$. Let $am = 0$ for some $a \in R, m \in M$. Assume that $a \neq 0$. Take $\mathfrak{M} \in Max(R)$. As M is $(R-\mathfrak{M})$-torsion-free, there exists $s_m \notin \mathfrak{M}$ so that $s_m a = 0$ or $s_m m = 0$. As R is an integral domain, $s_m a \neq 0$ and thus $s_m m = 0$. Now, put $\Omega = \{s_m : \exists \mathfrak{M} \in Max(R),\ s_m \notin \mathfrak{M} \text{ and } s_m m = 0\}$. A similar argument in the proof of Theorem 2.19 shows that $(\Omega) = R$. Then we have $(s_{m_1}) + (s_{m_2}) + \ldots + (s_{m_n}) = R$ for some $s_{m_i} \in \Omega$. This yields that $Rm = \sum_{i=1}^{n}(s_{m_i})m = 0$ and hence $m = 0$. This means M is a torsion-free module. □

Let M be an R-module. The set of zero divisors $Z(M)$ on M is defined as $Z(M) = \{a \in R : ann_M(a) \neq 0\}$. Now we characterize simple modules in terms of S-prime submodules.

Theorem 2.26 *Suppose M is a finitely generated multiplication module and S is a m.c.s. of R provided $ann(M) \cap S = \emptyset$. The following are equivalent:*

(i) Each proper submodule is an S-prime.

(ii) M is a simple module.

Proof $(i) \Rightarrow (ii)$: Suppose every proper submodule is an S-prime. First we will show that $Z(M) = ann(M)$. Let $a \in Z(M)$. Then there is a $0 \neq m' \in M$ with $am' = 0$. Since the zero submodule is S-prime and $am' = 0$, there is an $s \in S$ so that $sa \in ann(M)$ or $sm' = 0$. If $sm' = 0$, we have $s \in ann_R(m')$. Now put $P' = ann_R(m')M$ and note that $S \cap (P' :_R M) \neq \emptyset$. Thus, we have $P' = ann_R(m')M = M$. By [5, Corollary 2.5], $1 - x \in ann(M) \subseteq ann_R(m')$ for some $x \in ann_R(m')$. This yields that $ann_R(m') = R$ and so $m' = 0$, which is a contradiction. We have $sa \in ann(M)$. Then we can conclude that $s \in (ann_M(a) :_R M)$ and hence by assumption $ann_M(a) = M$. Thus, we get $a \in ann(M)$. Therefore, $Z(M) = ann(M)$. Let $a \notin Z(M)$. Now we will show that $aM = M$. If $a^2M = M$, then $aM = M$. Suppose $a^2M \neq M$. Since a^2M is an S-prime submodule and $(a)(aM) \subseteq a^2M$, by Lemma 2.5, there is an $s \in S$ so that $saM \subseteq a^2M$. Then for all $m \in M$, $sam = a^2m'$ for some $m' \in M$. Since $a \notin Z(M)$, we get $sm - am' \in ann_M(a) = 0$ and hence $sm = am'$. This yields that $sM \subseteq aM$ and thus $s \in (aM :_R M)$. By assumption, we have

$aM = M$. Now take a submodule P of M. If $(P :_R M) = ann(M)$, then $P = ann(M)M = 0$. Take an element $a \in (P :_R M) - ann(M)$. As $Z(M) = ann(M)$, $a \notin Z(M)$ and so $aM = M$. Then we get $M = aM \subseteq (P :_R M)M = P$. Therefore, M is a simple module.

$(ii) \Rightarrow (i)$: Note that the zero submodule is a prime submodule in a simple module. Since $ann(M) \cap S = \emptyset$, by Proposition 2.2, the zero submodule is S-prime. □

Corollary 2.27 *For any ring R and a m.c.s. S of R, R is a field if and only if each proper ideal is S-prime.*

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