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Inclusion properties of Lucas polynomials for bi-univalent functions introduced through the $\mathfrak{q}$-analogue of the Noor integral operator

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Abstract: In this paper, by using the $(\mathbf{P},\mathbf{Q})$-Lucas polynomials and the $\mathfrak{q}$-analogue of the Noor integral operator, we aim to build a bridge between the theory of geometric functions and that of special functions.

Key words: $(\mathbf{P},\mathbf{Q})$-Lucas polynomials, coefficient bounds, bi-univalent functions

1. Introduction, preliminaries, and known results

In modern science there is a huge interest in the theory and application of the Fibonacci polynomials, the Lucas polynomials, the Chebyshev polynomials, the Pell polynomials, the Pell–Lucas polynomials, the Lucas–Lehmer polynomials, the families of orthogonal polynomials, and other special polynomials as well as their generalizations. These polynomials play a major role in a variety of disciplines in the mathematical, physical, statistical, and engineering sciences (see, for example, [4, 8, 11, 14, 18, 19] and references therein).

Definition 1 *(See [9]) Let* $\mathbf{P}(\varkappa)$ *and* $\mathbf{Q}(\varkappa)$ *represent polynomials with real coefficients. The* $(\mathbf{P},\mathbf{Q})$*-Lucas polynomials* $L_{\mathbf{P},\mathbf{Q},j}(\varkappa)$ *are introduced by the recurrence relevance*

$$L_{\mathbf{P},\mathbf{Q},j}(\varkappa)=\mathbf{P}(\varkappa)L_{\mathbf{P},\mathbf{Q},j-1}(\varkappa)+\mathbf{Q}(\varkappa)L_{\mathbf{P},\mathbf{Q},j-2}(\varkappa)\quad(j\geq 2),$$

with the first few terms of the $(\mathbf{P},\mathbf{Q})$*-Lucas polynomials as follows:*

$$\begin{aligned}
L_{\mathbf{P},\mathbf{Q},0}(\varkappa)&=2,\\
L_{\mathbf{P},\mathbf{Q},1}(\varkappa)&=\mathbf{P}(\varkappa),\\
L_{\mathbf{P},\mathbf{Q},2}(\varkappa)&=\mathbf{P}^2(\varkappa)+2\mathbf{Q}(\varkappa),\\
L_{\mathbf{P},\mathbf{Q},3}(\varkappa)&=\mathbf{P}^3(\varkappa)+3\mathbf{P}(\varkappa)\mathbf{Q}(\varkappa),\\
&\vdots
\end{aligned}\tag{1}$$

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Table. Specific cases of $L_{\mathbf{P},\mathbf{Q}.,j}(\varkappa)$

$\mathbf{P}(\varkappa)$	$\mathbf{Q}(\varkappa)$	$L_{\mathbf{P},\mathbf{Q},j}(\varkappa)$
$\varkappa$	1	Lucas polynomials $L_n(\varkappa)$
$2\varkappa$	1	Pell–Lucas polynomials $D_n(\varkappa)$
1	$2\varkappa$	Jacobsthal–Lucas polynomials $j_n(\varkappa)$
$3\varkappa$	-2	Fermat–Lucas polynomials $f_n(\varkappa)$
$2\varkappa$	-1	Chebyshev polynomials $T_n(\varkappa)$

Theorem 2 *(See [9]) Let* $\Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(z)$ *indicate the generating function of the* $(\mathbf{P},\mathbf{Q})$*-Lucas polynomial sequence* $L_{\mathbf{P},\mathbf{Q},j}(\varkappa)$*. Then*

$$\Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(z)=\sum_{j\geq 0}^{\infty}L_{\mathbf{P},\mathbf{Q},j}(\varkappa)z^{j}=\frac{2-\mathbf{P}(\varkappa)z}{1-\mathbf{P}(\varkappa)z-\mathbf{Q}(\varkappa)z^{2}}.$$

Let $\mathcal{A}$ indicate the class of functions $\mathfrak{f}$ normalized by

$$\mathfrak{f}(z)=z+a_2z^2+a_3z^3+\cdots, \tag{2}$$

which are analytic in the open unit disk $\Delta=\{z\in\mathbb{C}:|z|<1\}$. The class of this kind of functions is represented by S.

With a view to recalling the rule of subordination for analytic functions, let $\mathfrak{f}$, $\mathfrak{g}$ be analytic in Δ. A function $\mathfrak{f}$ is *subordinate* to $\mathfrak{g}$, indicated as $\mathfrak{f}\prec\mathfrak{g}$, if there exists a function τ, analytic in Δ, such that

$$\tau(0)=0,\ \ |\tau(z)|<1$$

and

$$\mathfrak{f}(z)=\mathfrak{g}(\tau(z))\qquad(z\in\Delta).$$

According to the Koebe one-quarter theorem, every univalent function $\mathfrak{f}\in\mathcal{A}$ has an inverse $\mathfrak{f}^{-1}$ satisfying

$$\mathfrak{f}^{-1}(\mathfrak{f}(z))=z$$

and

$$\mathfrak{f}\left(\mathfrak{f}^{-1}(w)\right)=w\ \ \left(|w|<\frac{1}{4}\right),$$

where

$$\mathfrak{g}(\mathfrak{w})=\mathfrak{f}^{-1}(w)=w\ -a_2w^2+\left(2a_2^2-a_3\right)w^3-\left(5a_2^3-5a_2a_3+a_4\right)w^4+\cdots. \tag{3}$$

A function $\mathfrak{f}\in\mathcal{A}$ is called *bi-univalent* in Δ if both $\mathfrak{f}$ and $\mathfrak{f}^{-1}$ are univalent in Δ. Let denote the class of bi-univalent functions in Δ given by (2). For several interesting examples in the class , see [17] (see also [1–3, 6, 10, 12, 15]).

It may be of interest to recall that Srivastava used the basic (or $\mathfrak{q}$-)hypergeometric functions in a book chapter (see, for details, [16]). Thus, the theory of univalent functions was characterized by the concept of $\mathfrak{q}$-calculus. For simplicity, we provide some fundamental descriptions of $\mathfrak{q}$-calculus that are used in this paper. Next, we recall some identities of fractional $\mathfrak{q}$-calculus operators of complex valued function $\mathfrak{f}$.

For $0<\mathfrak{q}<1$, the $\mathfrak{q}$-derivative of a function $\mathfrak{f}\in\mathcal{A}$ is defined as follows:

$$\partial_{\mathfrak{q}}\mathfrak{f}(z)=\frac{\mathfrak{f}(z)-\mathfrak{f}(\mathfrak{q}z)}{z(1-\mathfrak{q})},\qquad (z\in\Delta).$$

Obviously, for $j\in\mathbb{N}:=\{1,2,\ldots\}$ and $z\in\Delta$,

$$\partial_{\mathfrak{q}}\left(\sum_{j\geq 1}^{\infty}a_jz^j\right)=\sum_{j\geq 1}^{\infty}[j,\mathfrak{q}]\,a_jz^{j-1},$$

where

$$[j,\mathfrak{q}]=\frac{1-\mathfrak{q}^j}{1-\mathfrak{q}},\qquad [0,\mathfrak{q}]=0.$$

Moreover, it is worth mentioning that

$$[j,\mathfrak{q}]!=\begin{cases}1, & j=0\\ [1,\mathfrak{q}]\,[2,\mathfrak{q}]\,[3,\mathfrak{q}]\ldots[j,\mathfrak{q}], & j\in\mathbb{N}\end{cases}.$$

Also, the $\mathfrak{q}$-generalized Pochhammer symbol for $\wp>0$ is given by

$$[\wp,\mathfrak{q}]_j=\begin{cases}1, & j=0\\ [\wp,\mathfrak{q}]\,[\wp+1,\mathfrak{q}]\ldots[\wp+j-1,\mathfrak{q}], & j\in\mathbb{N}\end{cases}.$$

Very recently, Arif et al. [5] defined the function $F^{-1}_{\mathfrak{q},\mu+1}(z)$ by

$$F^{-1}_{\mathfrak{q},\mu+1}(z)*F_{\mathfrak{q},\mu+1}(z)=z\partial_{\mathfrak{q}}\mathfrak{f}(z)\quad(\mu>-1),$$

where the function $F_{\mathfrak{q},\mu+1}(z)$ is given by

$$F_{\mathfrak{q},\mu+1}(z)=z+\sum_{j=2}^{\infty}\frac{[\mu+1,\mathfrak{q}]_{j-1}}{[j-1,\mathfrak{q}]!}z^j,\qquad (z\in\Delta).\tag{4}$$

Because the series defined in (4) is convergent absolutely in Δ, by making use of the description of the $\mathfrak{q}$-derivative through convolution, we now define the integral operator $\zeta^{\mu}_{\mathfrak{q}}:\Delta\to\Delta$ by

$$\zeta^{\mu}_{\mathfrak{q}}\mathfrak{f}(z)=F^{-1}_{\mathfrak{q},\mu+1}(z)*\mathfrak{f}(z)=z+\sum_{j=2}^{\infty}\Theta_{j-1}a_jz^j,\qquad (z\in\Delta),\tag{5}$$

where

$$\Theta_{j-1}=\frac{[j,\mathfrak{q}]!}{[\mu+1,\mathfrak{q}]_{j-1}}.$$

We note that $\zeta^0_{\mathfrak{q}}\mathfrak{f}(z)=z\partial_{\mathfrak{q}}\mathfrak{f}(z)$, $\zeta^1_{\mathfrak{q}}\mathfrak{f}(z)=\mathfrak{f}(z)$, and

$$\lim_{\mathfrak{q}\to 1^-}\zeta^{\mu}_{\mathfrak{q}}\mathfrak{f}(z)=z+\sum_{j=2}^{\infty}\frac{j!}{(\mu+1)_{j-1}}a_jz^j.$$

This shows that, by taking $\mathfrak{q} \to 1^{-}$, the operator defined in (5) reduces to the familiar Noor integral operator introduced in [13].

We want to assert evidently that, in the paper, by using the $L_{\mathbf{P},\mathbf{Q},j}(\varkappa)$ functions, our methodology constructs a bridge between the theory of geometric functions and that of special functions. Thus, we aim to introduce a new class of bi-univalent functions introduced through the $(\mathbf{P},\mathbf{Q})$-Lucas polynomials. Furthermore, we derive coefficient inequalities and obtain the Fekete–Szegö problem for this new function class.

Definition 3 *A function* $\mathfrak{f} \in$ *is said to be in the class*

$$U^{\mu}(\mathfrak{q};\varkappa) \quad (\mu > -1,\ 0 < \mathfrak{q} < 1,\ z,w \in \Delta)$$

if the following subordinations are fulfilled:

$$\frac{z\partial_{\mathfrak{q}}\left(\zeta_{\mathfrak{q}}^{\mu}\mathfrak{f}(z)\right)}{\zeta_{\mathfrak{q}}^{\mu}\mathfrak{f}(z)} \prec \Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(z) - 1$$

and

$$\frac{z\partial_{\mathfrak{q}}\left(\zeta_{\mathfrak{q}}^{\mu}\mathfrak{g}(w)\right)}{\zeta_{\mathfrak{q}}^{\mu}\mathfrak{g}(w)} \prec \Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(w) - 1,$$

where the function $\mathfrak{g}$ *is given by (3).*

Remark 4 *Upon setting* $\mathfrak{q} \to 1^{-}$ *in Definition 3, it is readily seen that a function* $\mathfrak{f} \in$ *is in the class*

$$U^{\mu}(\varkappa) \quad (\mu > -1,\ z,w \in \Delta)$$

if the following conditions are fulfilled:

$$\frac{z\left(\zeta^{\mu}\mathfrak{f}(z)\right)'}{\zeta^{\mu}\mathfrak{f}(z)} \prec \Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(z) - 1,$$

$$\frac{z\left(\zeta^{\mu}\mathfrak{g}(w)\right)'}{\zeta^{\mu}\mathfrak{g}(w)} \prec \Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(w) - 1,$$

where the function $\mathfrak{g}$ *is given by (3).*

Remark 5 *Upon setting* $\mu = 1$ *in Definition 3, it is readily seen that a function* $\mathfrak{f} \in$ *is in the class*

$$S(\varkappa) \quad (z,w \in \Delta)$$

if the following conditions are fulfilled:

$$\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)} \prec \Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(z) - 1$$

and

$$\frac{w\mathfrak{g}'(w)}{\mathfrak{g}(w)} \prec \Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(w) - 1,$$

where the function $\mathfrak{g}$ *is given by (3).*

Remark 6 *Upon setting* $\mu = 0$ *in Definition 3, it is readily seen that a function* $\mathfrak{f} \in$ *is in the class*

$$C_\Sigma(\varkappa) \quad (z, w \in \Delta)$$

if the following conditions are fulfilled:

$$\left(1 + \frac{z\mathfrak{f}''(z)}{\mathfrak{f}(z)}\right) \prec \Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(z) - 1$$

and

$$\left(1 + \frac{w\mathfrak{g}''(w)}{\mathfrak{g}(w)}\right) \prec \Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(w) - 1,$$

where the function $\mathfrak{g}$ *is given by (3).*

2. Inequalities for the Taylor–Maclaurin coefficients

Theorem 7 *Let the function* $\mathfrak{f}$ *given by (2) be in the class* $U^\mu(\mathfrak{q};\varkappa)$*. Then*

$$|a_2| \leq \frac{|\mathbf{P}(\varkappa)|\sqrt{|\mathbf{P}(\varkappa)|}}{\sqrt{|(\mathfrak{q}+1)(\Theta_2 - \Theta_1^2)\mathbf{P}^2(\varkappa) - 2\mathfrak{q}\Theta_1^2\mathbf{Q}(\varkappa)|\,\mathfrak{q}}}$$

and

$$|a_3| \leq \frac{\mathbf{P}^2(\varkappa)}{\mathfrak{q}^2\Theta_1^2} + \frac{|\mathbf{P}(\varkappa)|}{\mathfrak{q}(\mathfrak{q}+1)\Theta_2}.$$

Proof Let $\mathfrak{f} \in U^\mu(\mathfrak{q};\varkappa)$. From Definition 3, for some analytic functions Φ, Ψ such that

$$\Phi(0) = 0,\ |\Phi(z)| = \left|m_1 z + m_2 z^2 + m_3 z^3 + \cdots\right| < 1 \quad (z \in \Delta),$$

$$\Psi(0) = 0,\ |\Psi(w)| = \left|n_1 w + n_2 w^2 + n_3 w^3 + \cdots\right| < 1 \quad (w \in \Delta),$$

we can write

$$\frac{z\partial_\mathfrak{q}\left(\zeta_\mathfrak{q}^\mu \mathfrak{f}(z)\right)}{\zeta_\mathfrak{q}^\mu \mathfrak{f}(z)} = \Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(\Phi(z)) - 1$$

and

$$\frac{z\partial_\mathfrak{q}\left(\zeta_\mathfrak{q}^\mu \mathfrak{g}(w)\right)}{\zeta_\mathfrak{q}^\mu \mathfrak{g}(w)} = \Upsilon_{\{L_{\mathbf{P},\mathbf{Q},j}(\varkappa)\}}(\Psi(w)) - 1,$$

or equivalently

$$\frac{z\partial_\mathfrak{q}\left(\zeta_\mathfrak{q}^\mu \mathfrak{f}(z)\right)}{\zeta_\mathfrak{q}^\mu \mathfrak{f}(z)} = -1 + L_{\mathbf{P},\mathbf{Q},0}(\varkappa) + L_{\mathbf{P},\mathbf{Q},1}(\varkappa)\Phi(z) + L_{\mathbf{P},\mathbf{Q},2}(\varkappa)\Phi^2(z) + \cdots \tag{6}$$

and

$$\frac{z\partial_\mathfrak{q}\left(\zeta_\mathfrak{q}^\mu \mathfrak{g}(w)\right)}{\zeta_\mathfrak{q}^\mu \mathfrak{g}(w)} = -1 + L_{\mathbf{P},\mathbf{Q},0}(\varkappa) + L_{\mathbf{P},\mathbf{Q},1}(\varkappa)\Psi(w) + L_{\mathbf{P},\mathbf{Q},2}(\varkappa)\Psi^2(w) + \cdots. \tag{7}$$

From equalities (6) and (7), we obtain that

$$\frac{z\partial_{\mathfrak{q}}\left(\zeta_{\mathfrak{q}}^{\mu}\mathfrak{f}(z)\right)}{\zeta_{\mathfrak{q}}^{\mu}\mathfrak{f}(z)} = 1 + L_{\mathbf{P},\mathbf{Q},1}(\varkappa)m_1 z + \left[L_{\mathbf{P},\mathbf{Q},1}(\varkappa)m_2 + L_{\mathbf{P},\mathbf{Q},2}(\varkappa)m_1^2\right] z^2 + \cdots \tag{8}$$

and

$$\frac{z\partial_{\mathfrak{q}}\left(\zeta_{\mathfrak{q}}^{\mu}\mathfrak{g}(w)\right)}{\zeta_{\mathfrak{q}}^{\mu}\mathfrak{g}(w)} = 1 + L_{\mathbf{P},\mathbf{Q},1}(\varkappa)n_1 w + \left[L_{\mathbf{P},\mathbf{Q},1}(\varkappa)n_2 + L_{\mathbf{P},\mathbf{Q},2}(\varkappa)n_1^2\right] w^2 + \cdots . \tag{9}$$

Additionally, it is well known that

$$|m_k| \leq 1 \quad \text{and} \quad |n_k| \leq 1 \quad (k \in \mathbb{N}).$$

By comparing the corresponding coefficients in (8) and (9), we have

$$\mathfrak{q}\Theta_1 a_2 = L_{\mathbf{P},\mathbf{Q},1}(\varkappa)m_1, \tag{10}$$

$$\mathfrak{q}(\mathfrak{q}+1)\Theta_2 a_3 - \mathfrak{q}\Theta_1^2 a_2^2 = L_{\mathbf{P},\mathbf{Q},1}(\varkappa)m_2 + L_{\mathbf{P},\mathbf{Q},2}(\varkappa)m_1^2, \tag{11}$$

$$-\mathfrak{q}\Theta_1 a_2 = L_{\mathbf{P},\mathbf{Q},1}(\varkappa)n_1, \tag{12}$$

and

$$\mathfrak{q}(\mathfrak{q}+1)\Theta_2(2a_2^2 - a_3) - \mathfrak{q}\Theta_1^2 a_2^2 = L_{\mathbf{P},\mathbf{Q},1}(\varkappa)n_2 + L_{\mathbf{P},\mathbf{Q},2}(\varkappa)n_1^2. \tag{13}$$

From equations (10) and (12), we can easily see that

$$m_1 = -n_1, \tag{14}$$

$$2\mathfrak{q}^2\Theta_1^2 a_2^2 = L_{\mathbf{P},\mathbf{Q},1}^2(\varkappa)\left(m_1^2 + n_1^2\right). \tag{15}$$

If we add (11) to (13), we get

$$2\mathfrak{q}\left[(\mathfrak{q}+1)\Theta_2 - \Theta_1^2\right] a_2^2 = L_{\mathbf{P},\mathbf{Q},1}(\varkappa)\left(m_2 + n_2\right) + L_{\mathbf{P},\mathbf{Q},2}(\varkappa)\left(m_1^2 + n_1^2\right). \tag{16}$$

By using (15) in equality (16), we have

$$2\mathfrak{q}\left\{\left[(\mathfrak{q}+1)\Theta_2 - \Theta_1^2\right] L_{\mathbf{P},\mathbf{Q},1}^2(\varkappa) - \mathfrak{q}\Theta_1^2 L_{\mathbf{P},\mathbf{Q},2}(\varkappa)\right\} a_2^2 = L_{\mathbf{P},\mathbf{Q},1}^3(\varkappa)\left(m_2 + n_2\right). \tag{17}$$

We obtain the following inequality from (17), by using equation (1) and taking the modulus of a_2:

$$|a_2| \leq \frac{|\mathbf{P}(\varkappa)|\sqrt{|\mathbf{P}(\varkappa)|}}{\sqrt{|(\mathfrak{q}+1)(\Theta_2 - \Theta_1^2)\mathbf{P}^2(\varkappa) - 2\mathfrak{q}\Theta_1^2\mathbf{Q}(\varkappa)|\,\mathfrak{q}}}.$$

Moreover, if we subtract (13) from (11), we obtain

$$2\mathfrak{q}(\mathfrak{q}+1)\Theta_2(a_3 - a_2^2) = L_{\mathbf{P},\mathbf{Q},1}(\varkappa)\left(m_2 - n_2\right) + L_{\mathbf{P},\mathbf{Q},2}(\varkappa)\left(m_1^2 - n_1^2\right). \tag{18}$$

Then, in view of (14) and (15), (18) becomes

$$a_3 = \frac{L^2_{\mathbf{P},\mathbf{Q},1}(\varkappa)}{2\mathfrak{q}^2\Theta_1^2}\left(m_1^2+n_1^2\right)+\frac{L_{\mathbf{P},\mathbf{Q},1}(\varkappa)}{2\mathfrak{q}(\mathfrak{q}+1)\Theta_2}\left(m_2-n_2\right).$$

Then, with the help of (1), we finally deduce

$$|a_3| \leq \frac{\mathbf{P}^2(\varkappa)}{\mathfrak{q}^2\Theta_1^2}+\frac{|\mathbf{P}(\varkappa)|}{\mathfrak{q}(\mathfrak{q}+1)\Theta_2}.$$

□

Corollary 8 *If* $\mathfrak{f} \in U^{\mu}(\varkappa)$, *then*

$$|a_2| \leq \frac{|\mathbf{P}(\varkappa)|\sqrt{|\mathbf{P}(\varkappa)|}}{\sqrt{2\left|(\Theta_2-\Theta_1^2)\mathbf{P}^2(\varkappa)-\Theta_1^2\mathbf{Q}(\varkappa)\right|}}$$

and

$$|a_3| \leq \frac{\mathbf{P}^2(\varkappa)}{\Theta_1^2}+\frac{|\mathbf{P}(\varkappa)|}{2\Theta_2}.$$

Corollary 9 *If* $\mathfrak{f} \in S(\varkappa)$, *then*

$$|a_2| \leq |\mathbf{P}(\varkappa)|\sqrt{\left|\frac{\mathbf{P}(\varkappa)}{2\mathbf{Q}(\varkappa)}\right|}$$

and

$$|a_3| \leq \mathbf{P}^2(\varkappa)+\frac{|\mathbf{P}(\varkappa)|}{2}.$$

Corollary 10 *If* $\mathfrak{f} \in C(\varkappa)$, *then*

$$|a_2| \leq \frac{|\mathbf{P}(\varkappa)|\sqrt{|\mathbf{P}(\varkappa)|}}{\sqrt{2\left|\mathbf{P}^2(\varkappa)+4\mathbf{Q}(\varkappa)\right|}}$$

and

$$|a_3| \leq \frac{\mathbf{P}^2(\varkappa)}{4}+\frac{|\mathbf{P}(\varkappa)|}{6}.$$

3. Fekete–Szegö problem

The classical Fekete–Szegö inequality for the coefficients of $\mathfrak{f}$ selected from S is

$$\left|a_3-\vartheta a_2^2\right| \leq 1+2\exp(-2\vartheta/(1-\vartheta)) \quad \text{for } \vartheta \in [0,1).$$

As $\vartheta \to 1^-$, we have the elementary inequality $\left|a_3-a_2^2\right| \leq 1$. Moreover, the coefficient functional

$$F_{\vartheta}(\mathfrak{f}) = a_3-\vartheta a_2^2$$

for the normalized analytic functions $\mathfrak{f}$ in the open unit disk Δ revives a major role in geometric function theory. The problem of maximizing the absolute value of the functional $\digamma_{\vartheta}(\mathfrak{f})$ presents the Fekete–Szegö problem (see, for details, [7]).

Next, in this section, we aim to provide Fekete–Szegö inequalities for functions in the class $U^{\mu}(\mathfrak{q};\varkappa)$. These inequalities are given in the following theorem.

Theorem 11 *For $\vartheta\in\mathbb{R}$, let the function f given by (2) be in the class $U^{\mu}(\mathfrak{q};\varkappa)$. Then*

$$\left|a_3-\vartheta a_2^2\right|\leq\begin{cases}\frac{|\mathbf{P}(\varkappa)|}{\mathfrak{q}(\mathfrak{q}+1)\Theta_2}; & |\vartheta-1|\leq\left|1-\left(1+\frac{2\mathfrak{q}\mathbf{Q}(\varkappa)}{(\mathfrak{q}+1)\mathbf{P}^2(\varkappa)}\right)\frac{\Theta_1^2}{\Theta_2}\right|\\ \frac{|1-\vartheta|\left|\mathbf{P}^3(\varkappa)\right|}{\mathfrak{q}\left|(\mathfrak{q}+1)(\Theta_2-\Theta_1^2)\mathbf{P}^2(\varkappa)-2\mathfrak{q}\Theta_1^2\mathbf{Q}(\varkappa)\right|}; & |\vartheta-1|\geq\left|1-\left(1+\frac{2\mathfrak{q}\mathbf{Q}(\varkappa)}{(\mathfrak{q}+1)\mathbf{P}^2(\varkappa)}\right)\frac{\Theta_1^2}{\Theta_2}\right|\end{cases}.$$

Proof From (17) and (18)

$$\begin{aligned}a_3-\vartheta a_2^2 &= \frac{L_{\mathbf{P},\mathbf{Q},1}^3(\varkappa)\,(1-\vartheta)\,(m_2+n_2)}{2\mathfrak{q}\left\{\left[(\mathfrak{q}+1)\Theta_2-\Theta_1^2\right]L_{\mathbf{P},\mathbf{Q},1}^2(\varkappa)-\mathfrak{q}\Theta_1^2L_{\mathbf{P},\mathbf{Q},2}(\varkappa)\right\}}\\ &+\frac{L_{\mathbf{P},\mathbf{Q},1}(\varkappa)\,(m_2-n_2)}{2\mathfrak{q}(\mathfrak{q}+1)\Theta_2}\\ &= L_{\mathbf{P},\mathbf{Q},1}(\varkappa)\left[\left(\hbar\,(\vartheta;\varkappa)+\frac{1}{2\mathfrak{q}(\mathfrak{q}+1)\Theta_2}\right)m_2+\left(\hbar\,(\vartheta;\varkappa)-\frac{1}{2\mathfrak{q}(\mathfrak{q}+1)\Theta_2}\right)n_2\right],\end{aligned}$$

where

$$\hbar\,(\vartheta;\varkappa)=\frac{L_{\mathbf{P},\mathbf{Q},1}^2(\varkappa)\,(1-\vartheta)}{2\mathfrak{q}\left\{\left[(\mathfrak{q}+1)\Theta_2-\Theta_1^2\right]L_{\mathbf{P},\mathbf{Q},1}^2(\varkappa)-\mathfrak{q}\Theta_1^2L_{\mathbf{P},\mathbf{Q},2}(\varkappa)\right\}}.$$

In view of (1), we conclude that

$$\left|a_3-\vartheta a_2^2\right|\leq\begin{cases}\dfrac{|\mathbf{P}(\varkappa)|}{\mathfrak{q}(\mathfrak{q}+1)\Theta_2}, & 0\leq|\hbar\,(\vartheta;\varkappa)|\leq\dfrac{1}{2\mathfrak{q}(\mathfrak{q}+1)\Theta_2}\\ 2\,|\mathbf{P}(\varkappa)|\,|\hbar\,(\vartheta;\varkappa)|\,, & |\hbar\,(\vartheta;\varkappa)|\geq\dfrac{1}{2\mathfrak{q}(\mathfrak{q}+1)\Theta_2}\end{cases}.$$

□

Corollary 12 *For $\vartheta\in\mathbb{R}$, let $\mathfrak{f}$ given by (2) be in the class $U^{\mu}(\varkappa)$. Then*

$$\left|a_3-\vartheta a_2^2\right|\leq\begin{cases}\dfrac{|\mathbf{P}(\varkappa)|}{2\Theta_2}; & |\vartheta-1|\leq\left|1-\left(1+\frac{\mathbf{Q}(\varkappa)}{\mathbf{P}^2(\varkappa)}\right)\frac{\Theta_1^2}{\Theta_2}\right|\\ \dfrac{|1-\vartheta|\left|\mathbf{P}^3(\varkappa)\right|}{2\left|(\Theta_2-\Theta_1^2)\mathbf{P}^2(\varkappa)-\Theta_1^2\mathbf{Q}(\varkappa)\right|}; & |\vartheta-1|\geq\left|1-\left(1+\frac{\mathbf{Q}(\varkappa)}{\mathbf{P}^2(\varkappa)}\right)\frac{\Theta_1^2}{\Theta_2}\right|\end{cases}.$$

Corollary 13 *If* $\mathfrak{f} \in S(\varkappa)$, *then*

$$\left|a_3-\vartheta a_2^2\right| \leq \begin{cases} \dfrac{|\mathbf{P}(\varkappa)|}{2}, & |\vartheta-1| \leq \dfrac{|\mathbf{Q}(\varkappa)|}{\mathbf{P}^2(\varkappa)} \\ \\ \dfrac{|1-\vartheta|\left|\mathbf{P}^3(\varkappa)\right|}{2\left|\mathbf{Q}(\varkappa)\right|}, & |\vartheta-1| \geq \dfrac{|\mathbf{Q}(\varkappa)|}{\mathbf{P}^2(\varkappa)} \end{cases}.$$

Corollary 14 *If* $f \in C(\varkappa)$, *then*

$$\left|a_3-\vartheta a_2^2\right| \leq \begin{cases} \dfrac{|\mathbf{P}(\varkappa)|}{6}, & |\vartheta-1| \leq \dfrac{\left|\mathbf{P}^2(\varkappa)+4\mathbf{Q}(\varkappa)\right|}{3\mathbf{P}^2(\varkappa)} \\ \\ \dfrac{|1-\vartheta|\left|\mathbf{P}^3(\varkappa)\right|}{2\left|\mathbf{P}^2(\varkappa)+4\mathbf{Q}(\varkappa)\right|}, & |\vartheta-1| \geq \dfrac{\left|\mathbf{P}^2(\varkappa)+4\mathbf{Q}(\varkappa)\right|}{3\mathbf{P}^2(\varkappa)} \end{cases}.$$

If we choose $\vartheta=1$ in Theorem 11, we get the next corollary.

Corollary 15 *If* $\mathfrak{f} \in U^{\mu}(\mathfrak{q};\varkappa)$, *then*

$$\left|a_3-a_2^2\right| \leq \frac{|\mathbf{P}(\varkappa)|}{\mathfrak{q}(\mathfrak{q}+1)\Theta_2}.$$

Corollary 16 *If* $\mathfrak{f} \in U^{\mu}(\varkappa)$, *then*

$$\left|a_3-a_2^2\right| \leq \frac{|\mathbf{P}(\varkappa)|}{2\Theta_2}.$$

Corollary 17 *If* $\mathfrak{f} \in S(\varkappa)$, *then*

$$\left|a_3-a_2^2\right| \leq \frac{|\mathbf{P}(\varkappa)|}{2}.$$

Corollary 18 *If* $\mathfrak{f} \in C(\varkappa)$, *then*

$$\left|a_3-a_2^2\right| \leq \frac{|\mathbf{P}(\varkappa)|}{6}.$$

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