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Properties of a generalized class of analytic functions with coefficient inequality

Ben WONGSAIJAI, Nattakorn SUKANTAMALA*
Center of Excellence in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, Thailand

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Abstract: Let $(\beta_n)_{n\geq 2}$ be a sequence of nonnegative real numbers and δ be a positive real number. We introduce the subclass $\mathcal{A}(\beta_n,\delta)$ of analytic functions, with the property that the Taylor coefficients of the function f satisfies $\sum_{n\geq 2}^{\infty}\beta_n|a_n|\leq\delta$, where $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$. The class $\mathcal{A}(\beta_n,\delta)$ contains nonunivalent functions for some choices of $(\beta_n)_{n\geq 2}$. In this paper, we provide some general properties of functions belonging to the class $\mathcal{A}(\beta_n,\delta)$, such as the radii of univalence, distortion theorem, and invariant property. Furthermore, we derive the best approximation of an analytic function in such class by using the semiinfinite quadratic programming. Applying our results, we recover some known results on subclasses related to coefficient inequality. Some applications to starlike and convex functions of order α are also mentioned.

Key words: Analytic function, starlike function of order α, convex function of order α , coefficient inequality , quadratic programming, Karush–Kuhn–Tucker conditions

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ be the class of analytic functions in the open unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ and $\mathcal{A}$ be the class of analytic functions normalized by the conditions that $f(0)=0$ and $f'(0)=1$; that is, $f\in\mathcal{A}$ can be written in the form

$$f(z)=z+\sum_{k=2}^{\infty}a_nz^n,\quad z\in\mathbb{D}. \tag{1}$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions that are univalent on $\mathbb{D}$. Let $\mathcal{T}$ be a subset of $\mathcal{A}$ containing negative coefficient functions; that is, $a_n\leq 0$ for $n\geq 2$.

A considerable number of publications have focused on finding sufficient conditions via coefficient inequality for understanding the construction and its properties of subclasses of univalent functions. In [13], Goodman derived a sufficient condition for starlike functions via coefficient inequality. In [27], Silverman obtained some results on coefficient inequality for univalent functions with negative coefficients that are starlike and convex of order α. Darwish [8] established the quasi-Hadamard product results of starlike and convex functions of order α type β. Some coefficient estimates and related properties of the class of bounded starlike functions of complex order were obtained by Attiya [6]. Recently, Dzjok et al. [10] applied the extreme points theory to obtain coefficients estimate and other properties for some generalizations of starlike harmonic functions. To

*Correspondence: ben.wongsaijai@gmail.com

date, the coefficient inequality is one of the essential tools to study subclasses of univalent functions, especially subclasses involving differential-integral operators (see [3, 5, 6, 9, 15, 18, 20, 28, 30–33]).

In addition, many sufficient conditions for subclasses of analytic functions, which perhaps contain nonunivalent functions, can be derived in the form of coefficient inequality, especially subclasses of analytic functions defined by the q-derivative operator. For example, Purohit and Raina [23] introduced a subclass of analytic functions defined by the q-analogue operator of fractional calculus and derived some results including the coefficient inequality. In [35], the authors obtained some relations between various types of q-starlike functions of order α by using the coefficient inequality. Recently, Mahmood and Sokol [19] defined a new class of analytic functions by using the Ruscheweyh q-differential operator, while Govindaraj and Sivasubramanian [14] applied the concept of q-calculus to define the new class of analytic functions that are closely related to the domains bounded by conic sections. Some convolution properties for the classes of bounded q-starlike and q-convex univalent functions of complex order were investigated by Aouf and Seoudy [4]. For some interesting properties and recent related works on subclasses of analytic functions associated with the q-difference operator, we refer to [1, 2, 4, 14, 19, 24, 26, 29, 34–36].

In [11], Frasin introduced the class $H_\phi(c_k, \delta)$ of $\mathcal{S}$ consisting of functions f satisfying the coefficient inequality $\sum_{k=2}^{\infty} c_k|a_k| < \delta$, where the function $\phi(z) = z + \sum_{k=2}^{\infty} c_k z^k$ is fixed in $\mathcal{S}$. Later, Frasin and Aouf [12] followed the idea in [11] by introducing the classes $\mathcal{M}_\psi^0(c_n, \delta)$, $\mathcal{N}_\psi^0(c_n, \delta)$, and $\mathcal{B}_\psi^k(c_n, \delta)$ and studied some results related to the quasi-Hadamard product of functions in these classes. However, these classes can not be used to describe properties of the nonunivalent functions such as the q-extension starlike function and convex function and related subclasses. From the literature, it is therefore natural to define the following subclass of $\mathcal{A}$, which contains nonunivalent functions:

$$\mathcal{A}(\beta_n, \delta) = \left\{ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S} : \sum_{n=2}^{\infty} \beta_n |a_n| \leq \delta \right\},$$

where $(\beta_n)_{n\geq 2}$ is a fixed nonnegative real sequence and δ is a fixed positive real number. In the Table, some particular cases of $\mathcal{A}(\beta_n, \delta)$ are mentioned.

For a nonunivalent analytic function, it is interesting to study the problem of finding the best approximation to show how far an analytic function is from a subclass of univalent functions. In [21], Pascu and Pascu first paid attention to the problem of finding the best approximation of an analytic in the subclass of starlike functions by extending the Karush–Kuhn–Tucker conditions to semiinfinite quadratic programming. Later, they solved this kind of problem on the subclass of convex functions (see [22]). Recently, Kargar et al. [17] solved the best approximation of analytic functions by a locally univalent normalized analytic function. To date, the study of the best approximation has not been widely studied.

This paper has been arranged as follows. In Section 2, we provide some general properties of functions belonging to the class $\mathcal{A}(\beta_n, \delta)$, such as the radii of univalence, distortion theorem, and invariant property. In Section 3, we adopt the idea in [17, 21, 22] to solve the semiinfinite quadratic programming to obtain the best approximation in the class $\mathcal{A}(\beta_n, \delta)$. In Section 4, we apply our results to classes of starlike and convex function order α to obtain their general properties. Some examples of polynomial functions are presented. We finish the paper with observations and concluding remarks.

Table. Some typical classes included in $\mathcal{A}(\beta_n, \delta)$.

		β_n	δ	Refs.
1.	$\mathcal{T}^*(\alpha)$	$n-\alpha$	$1-\alpha$	[27]
2.	$\mathcal{C}^*(\alpha)$	$n(n-\alpha)$	$1-\alpha$	[27]
3.	$\mathcal{P}^*(\alpha)$	n	$1-\alpha$	[25]
4.	$\mathcal{P}^*(\alpha,\beta)$	$n(1+\beta)$	$2\beta(1-\alpha)$	[15]
5.	$\mathcal{G}(\alpha)$	$n[2(n-1)-\beta]$	α	[20]
6.	$\mathcal{R}[\alpha]$	$(n-\beta)\prod_{j=2}^{n}\frac{j-2\alpha}{(n-1)!}$	$1-\alpha$	[28]
7.	$\mathcal{C}_0^*(\alpha,\beta)$ $(a_1=1)$	$n[(1-\beta)n-\alpha\beta]$	$\beta(1-\alpha)$	[8]
8.	$\mathcal{S}_k^*(\alpha,\beta)$ $(a_1=1)$	$n^k[(1-\beta)n-\alpha\beta]$	$\beta(1-\alpha)$	[8]
9.	$\mathcal{S}_k(\alpha,\beta)$ $(a_1=1)$	$n^k[(n-1)+\beta(1+\alpha n)]$	$\beta(1+\alpha)$	[3]
10.	$H_k^b(A,B)$	$n^k((n-1)+\lvert(A-B)b-B(n-1)\rvert)$	$(A-B)\lvert b\rvert$	[6]
11.	$\mathcal{S}_s^r(\lambda,\kappa,\mu)$	$(2n(1+\kappa)-(\mu+\kappa)[1-(-1)^n])(\lambda[\varphi_n-1]+1)\psi_n$	$2(1-\mu)$	[9]
12.	$\mathcal{TS}^*(\alpha)$	$[n]_q-\alpha$	$1-\alpha$	[35]
13.	$\mathcal{K}_q(\alpha)$ $(g(z)=z)$	$[n]_q$	$1-\alpha$	[36]
14.	$\mathcal{S}_q(b,M)$	$[n]_q+\lvert b\rvert-1$	$\lvert b\rvert$	[4]
15.	$\mathcal{R}_{q,\lambda,l,1}^{\delta,m}(\alpha)$	$[n]_q\Psi_{q,\lambda,l,1}^{\delta,m}$	$1-\alpha$	[34]
16.	$\mathcal{J}_{q,\delta}^{\alpha}$	$(1+\beta)\frac{\Gamma_q(k+1)\Gamma_q(2-\alpha)}{\Gamma_q(2)\Gamma_q(k-\alpha+1)}$	$2\beta(1-\delta)$	[23]
17.	$k-\mathcal{UST}_q^{\lambda}(\gamma)$	$[q[n-1,q](k+1)+\lvert\gamma\rvert]\,\lvert\Psi_{n-1}\rvert$	$\lvert\gamma\rvert$	[19]
18.	$\mathcal{S}_q(k,\alpha,m)$	$[n]_q^m([n]_q(k+1)-k-\alpha)$	$1-\alpha$	[14]

2. Properties on the classes $\mathcal{A}(\beta_n, \delta)$

In this section, we provide some basic properties on $\mathcal{A}(\beta_n, \delta)$. We begin to derive the radii of univalence for the function $\mathcal{A}(\beta_n, \delta)$.

Theorem 1 *If $f \in \mathcal{A}(\beta_n, \delta)$, then f is univalent and starlike in $|z| < r_0$, where*

$$r_0 = \inf_{n\geq 2}\left[\frac{\beta_n}{\delta n}\right]^{\frac{1}{n-1}}. \tag{2}$$

Proof To obtain the result, we observe that if $\operatorname{Re}\{f'(z)\} > 0$ on $\mathbb{D}_{r_0}$, where $\mathbb{D}_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$, then f is univalent due to the following formula:

$$\operatorname{Re}\left\{\frac{f(z_1)-f(z_2)}{z_1-z_2}\right\} = \int_0^1 \operatorname{Re}\left\{f'(z_1+t(z_2-z_1))\right\}dt. \tag{3}$$

Therefore, we have

$$\operatorname{Re}\{f'(z)\} = \operatorname{Re}\left\{1-\sum_{n=2}^{\infty} n a_n z^{n-1}\right\} \geq 1-\sum_{n=2}^{\infty} n|a_n| r_0^{n-1}, \tag{4}$$

for all $|z| < r_0$. By the definition of $\mathcal{A}(\beta_n, \delta)$ and Eq. (4), the inequality $\operatorname{Re}\{f'(z)\} > 0$ holds on $\mathbb{D}_{r_0}$, where

$$r_0 = \inf_{n\geq 2}\left[\frac{\beta_n}{\delta n}\right]^{\frac{1}{n-1}}.$$

This completes the proof. □

Using Eq. (2), it is easy to see that r_0 is greater than unity when $\beta_n > \delta n$. We directly obtain the following result.

Corollary 2 *If $\beta_n > \delta n$, for all $n \geq 2$, then $\mathcal{A}(\beta_n, \delta) \subset \mathcal{S}$.*

Next, we derive the distortion inequalities for functions in the class $\mathcal{A}(\beta_n, \delta)$ that will be given by the following results.

Theorem 3 *If $\beta_0 = \inf_{n\geq 2} \beta_n > 0$, then*

$$|z| - \frac{\delta}{\beta_0}|z|^2 \leq |f(z)| \leq |z| + \frac{\delta}{\beta_0}|z|^2 \tag{5}$$

and

$$1 - \frac{2\delta}{\beta_0}|z| \leq |f'(z)| \leq 1 + \frac{2\delta}{\beta_0}|z|, \tag{6}$$

for all $f \in \mathcal{A}(\beta_n, \delta)$.

Proof Suppose that $\beta_0 = \inf_{n\geq 2} \beta_n > 0$. Letting $f \in \mathcal{A}(\beta_n, \delta)$, we see that

$$\beta_0 \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \beta_n |a_n| \leq \delta. \tag{7}$$

From Eq. (7), the consequence is that

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{\delta}{\beta_0}, \tag{8}$$

which yields

$$|z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \leq |f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} |a_n|. \tag{9}$$

The conjunction of inequalities Eqs. (8) and (9) implies the assertions (5) of Theorem 3. Hence, inequality (6) follows from

$$1 - |z| \sum_{n=2}^{\infty} n|a_n| \leq |f'(z)| \leq 1 + |z| \sum_{n=2}^{\infty} n|a_n|.$$

The proof is completed. □

Remark 4 *By letting $z \to 1^-$, Theorem 3 demonstrates that the disk $|z| < 1$ is mapped onto a domain that contains the disk*

$$|w| < 1 - \frac{\delta}{\beta_0},$$

under any analytic function $f \in \mathcal{A}(\beta_n, \delta)$.

In the following results, we focus on the invariant property of the class $\mathcal{A}(\beta_n,\delta)$. We shall consider some operators in terms of the standard convolution formula.

Definition 5 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *and* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ *be the function in* $\mathcal{A}$. *The convolution formula of* f *and* g *can be defined by* $f * g = z + \sum_{n=2}^{\infty} a_n b_n z^n$.

According to the definition of $\mathcal{A}(\beta_n,\delta)$, we easily obtain the following properties.

Theorem 6 *Let* $f \in \mathcal{A}(\beta_n,\delta)$ *and* $g(z) = z+\sum_{n=2}^{\infty} \mu_n z^n \in \mathcal{A}$. *Then for* $|\mu_n| \le 1$ $(n \ge 2)$ $(f*g)(z) \in \mathcal{A}(\beta_n,\delta)$.

Example 1 *(The Bernardi–Libera integral operator) The Bernardi–Libera integral operator* $L_\gamma : \mathcal{A} \to \mathcal{A}$ *is defined as follows:*

$$L_\gamma f(z) = \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1} f(t)dt, \tag{10}$$

which was studied by Bernardi in [7]. It is clear that

$$L_\gamma f(z) := (f*g)(z) = z + \sum_{n=2}^{\infty} \frac{\gamma+1}{\gamma+n} a_n z^n,$$

where $g = z + \sum_{k=2}^{\infty} \frac{\gamma+1}{\gamma+n} z^n$. *Combining the formula of* L_γ *and Theorem 6, we obtain the invariant properties under integral operator* L_γ *as*

$$L_\gamma\left[\mathcal{A}(\beta_n,\delta)\right] \subset \mathcal{A}(\beta_n,\delta).$$

3. Best approximation problem in the classes $\mathcal{A}(\beta_n,\delta)$

In this section, we investigate the problem of the best approximation of an analytic function by a function in the class $\mathcal{A}(\beta_n,\delta)$ and $f \in \mathcal{A}$. To define a measure of the nonunivalency of a function, we recall the definition introduced in [17, 21, 22]:

$$dist(f,\mathcal{A}(\beta_n,\delta)) = \inf_{g\in\mathcal{A}(\beta_n,\delta)} \left(\int_{\mathbb{D}} |f(x+iy) - g(x+iy)|^2 dxdy\right)^{1/2}, \tag{11}$$

for any sublclass $\mathcal{A}(\beta_n,\delta)$ of $\mathcal{S}$. In [21], it was also shown that $dist(f,\mathcal{S}) = 0$ if and only if $f \in \mathcal{S}$. In terms of the coefficients of the Taylor series of f, we obtain that

$$\int_{\mathbb{D}} |f(x+iy)|^2 dxdy = \pi \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1}, \tag{12}$$

where the calculation can be obtained by using Fubini's theorem and the orthogonality of the trigonometric functions. Then, by making use definition (11), the above identity leads us to consider the problem of finding

$$\inf_{(x_n)_{n\ge2}} \sum_{n=2}^{\infty} \frac{|x_n - a_n|^2}{n+1}, \tag{13}$$

subject to

$$\sum_{n=2}^{\infty} \beta_n |x_n| \leq \delta, \tag{14}$$

where the infimum is taken over all sequences $(x_n)_{n\geq 2}$ of complex numbers.

For the convenience, we provide some basic definitions and concept details of the Karush–Kuhn–Tucker conditions (see [16]) for quadratic programming, which are used in this paper. Given a minimization problem

$$f(x) = x^T Q x + cx \tag{15}$$

subject to

$$Ax \leq b \text{ and } x \geq 0, \tag{16}$$

where $x \in \mathbb{R}^n$, Q is an $n \times n$ symmetric matrix, A is an $m \times n$ matrix, and b and c^T are column vectors $m \times 1$ and $n \times 1$, respectively. It is well known that the problem has a unique global minimum when the objective function f is strictly convex. Consider the Lagrangian function L for the quadratic problem (15)–(16):

$$L = x^T Q x + cx + \mu(Ax - b). \tag{17}$$

The solution is given by the Karush–Kuhn–Tucker conditions:

$$\begin{aligned} \frac{\partial L}{\partial x_i} &\geq 0, \\ \frac{\partial L}{\partial \mu_j} &\leq 0, \\ x_i \frac{\partial L}{\partial x_i} &= 0, \\ \mu(Ax - b) &= 0, \\ x_i \geq 0, \quad \mu_j &\geq 0, \end{aligned}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. We also note that the Karush–Kuhn–Tucker conditions are the necessary conditions for the global minimum. The sufficient condition for a global minimum is given by the positive definite property of Q.

We are ready to prove the following theorem.

Theorem 7 *Let $f \in \mathcal{A}$ be a function of the form*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}.$$

Suppose that $\lim_{n\to\infty} \frac{|a_n|}{n\beta_n} = 0$ and $\sum_{n=2}^{\infty} n\beta_n^2 = +\infty$.

If $\sum_{n=2}^{\infty} \beta_n |a_n| \leq \delta$, then $dist(f, \mathcal{A}(\beta_n, \delta)) = 0$ when the minimum is attained for the function $g = f \in \mathcal{A}(\beta_n, \delta)$.

If $\sum_{n=2}^{\infty} \beta_n |a_n| > \delta$, *then*

$$dist(f, \mathcal{A}(\beta_n, \delta)) = \left(\pi \sum_{n \in \mathcal{I}^c} \frac{|a_n|^2}{n+1} + \pi \frac{\left(\sum_{m \in \mathcal{I}} (\beta_m |a_m|) - \delta \right)^2}{\sum_{m \in \mathcal{I}} (\beta_m^2 (m+1))} \right)^{1/2}, \tag{18}$$

where $\mathcal{I} = \{i_2, i_3, \ldots, i_N\}$*, and* $(i_n)_{n \geq 2}$ *is a permutation of the indices in* $\mathcal{P} = \{n \geq 2 : |a_n| > 0\}$ *such that* $\alpha_n = \frac{2|a_n|}{(n+1)\beta_n}$, $n = 2, \ldots, |\mathcal{P}| + 1$ *is a nonincreasing sequence. The infimum is attained for the function* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}(\beta_n, \delta)$*, where*

$$b_n = \begin{cases} \left(|a_n| - (n+1)\beta_n \frac{\sum_{n \in \mathcal{I}} (\beta_n |a_n|) - \delta}{\sum_{m \in \mathcal{I}} \beta_m^2 (m+1)} \right) e^{i \arg a_n} & \text{for} \quad n \in \mathcal{I}, \\ 0 & \text{for} \quad n \in \mathcal{I}^c. \end{cases} \tag{19}$$

Proof It is obvious if $\sum_{n=2}^{\infty} \beta_n |a_n| \leq \delta$. Assume that $\sum_{n=2}^{\infty} \beta_n |a_n| > \delta$. By using Eq. (12) and the triangle inequality, we get

$$\begin{aligned} dist(f, \mathcal{A}(\beta_n, \delta)) &= \inf_{g \in \mathcal{A}(\beta_n, \delta)} \left(\int_{\mathbb{D}} |f(x+iy) - g(x+iy)|^2 dx dy \right)^{1/2} \\ &= \left(\pi \inf \sum_{n=2}^{\infty} \frac{|x_n - a_n|^2}{n+1} \right)^{1/2} \\ &\geq \left(\pi \inf \sum_{n=2}^{\infty} \frac{(|x_n| - |a_n|)^2}{n+1} \right)^{1/2}. \end{aligned}$$

In fact, the equality holds in the triangle inequality if $\arg x_n = \arg a_n$. It follows that

$$dist(f, \mathcal{A}(\beta_n, \delta)) = \left(\pi \inf \sum_{n=2}^{\infty} \frac{|x_n - a_n|^2}{n+1} \right)^{1/2} = \left(\pi \inf \sum_{n=2}^{\infty} \frac{(|x_n| - |a_n|)^2}{n+1} \right)^{1/2}. \tag{20}$$

Therefore, it is natural to replace the complex sequence $(x_n)_{n \geq 2}$ by the nonnegative sequence satisfying $\sum_{n=2}^{\infty} \beta_n x_n \leq \delta$. This leads us to study the best approximation problem on the class $\mathcal{A}(\beta_n, \delta)$ with nonnegative coefficient. Explicitly, the quadratic problem (13)–(14) can be replaced by the problem of finding

$$\inf_{(x_n)_{n \geq 2}} \sum_{n=2}^{\infty} \frac{(x_n - a_n)^2}{n+1}, \tag{21}$$

subject to

$$\sum_{n=2}^{\infty} \beta_n x_n \leq \delta, \tag{22}$$

where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ and $x_n \geq 0$, for all $n \geq 2$.

Without loss of generality, we assume $|a_n| \neq 0$ for all $n \geq 2$. The proof can be followed by introducing the set of indices $\mathcal{P}^* = \{n : |a_n| > 0\}$. Here, the infimum problem becomes

$$\inf \sum_{n=2}^{\infty} \frac{(x_n - a_n)^2}{n+1} = \min \left\{ 0, \inf \sum_{n \in \mathcal{P}^*} \frac{(x_n - a_n)^2}{n+1} \right\}.$$

See more details in [17, 21, 22]. Thus, the objective function can be defined by $f(x) = \sum_{n=2}^{\infty} \frac{(x_n-a_n)^2}{n+1}$. which is a quadratic function, and the constrained inequality can be written as $Ax \leq \delta$, where $A = (\beta_2 \;\; \beta_3 \;\; \beta_4 \;\; \dots)$ and $x = (x_2 \;\; x_3 \;\; x_4 \;\; \dots)'$. The corresponding Lagrangian in the problem is given by

$$L = \sum_{n=2}^{\infty} \frac{(x_n - a_n)^2}{n+1} + \mu \left(\sum_{n=2}^{\infty} \beta_n x_n - \delta \right).$$

In this case, the Karush–Kuhn–Tucker conditions are

$$2\frac{x_n - a_n}{n+1} + \mu\beta_n \geq 0, \quad n \geq 2, \tag{23}$$

$$\sum_{n=2}^{\infty} \beta_n x_n - \delta \leq 0, \tag{24}$$

$$x_n \left(2\frac{x_n - a_n}{n+1} + \mu\beta_n \right) = 0, \quad n \geq 2, \tag{25}$$

$$\mu \left(\sum_{n=2}^{\infty} \beta_n x_n - \delta \right) = 0, \tag{26}$$

$$x_n \geq 0, (n \geq 2) \quad \mu \geq 0. \tag{27}$$

First of all, by using Eq. (26), we see that either $\mu = 0$ or $\sum_{n=2}^{\infty} \beta_n x_n = \delta$. Assuming $\mu = 0$, from Eq. (25), we have $x_n = 0$ or $x_n = a_n$, and since from Eq. (23) we obtain $x_n \geq a_n$, this implies that $x_n = a_n$ for all $n \geq 2$. Now we have $\sum_{n=2}^{\infty} \beta_n x_n = \sum_{n=2}^{\infty} \beta_n a_n > \delta$, which contradicts (24). Thus, we must have $\mu \neq 0$.

The Karush–Kuhn–Tucker conditions become in this case

$$2\frac{x_n - a_n}{n+1} + \mu\beta_n \geq 0, \quad n \geq 2, \tag{28}$$

$$\sum_{n=2}^{\infty} \beta_n x_n = \delta, \tag{29}$$

$$x_n \left(2\frac{x_n - a_n}{n+1} + \mu\beta_n \right) = 0, \quad n \geq 2, \tag{30}$$

$$x_n \geq 0 (n \geq 2) \quad \mu > 0. \tag{31}$$

The Eq. (30) shows that either $x_n = 0$ or $x_n = a_n - \frac{1}{2}\mu(n+1)\beta_n$. Let us introduce the set of indices $\mathcal{I}$ such as $x_n = a_n - \frac{1}{2}\mu(n+1)\beta_n$ for $n \in \mathcal{I}$, so $x_n = 0$ for $n \in \mathcal{I}^c$. Hence, by the Eq. (29), we see that

$$\delta = \sum_{n=2}^{\infty} \beta_n x_n = \sum_{n \in \mathcal{I}} \beta_n \left(a_n - \frac{1}{2}\mu(n+1)\beta_n \right) = \sum_{n \in \mathcal{I}} a_n \beta_n - \frac{\mu}{2} \sum_{n \in \mathcal{I}} (n+1)\beta_n^2,$$

which yields

$$\mu = 2\frac{\sum_{n \in \mathcal{I}} a_n \beta_n - \delta}{\sum_{n \in \mathcal{I}} (n+1)\beta_n^2} > 0. \tag{32}$$

Eq. (32) shows that $|\mathcal{I}|$ must be finite by the assumption $\sum_{n=2}^{\infty} n\beta_n^2 = +\infty$. Moreover, by using Eqs. (28) and (31), we have

$$\mu \geq \frac{2a_n}{(n+1)\beta_n} := \alpha_n, \quad n \in \mathcal{I}^c \tag{33}$$

and

$$\mu \leq \frac{2a_n}{(n+1)\beta_n} := \alpha_n, \quad n \in \mathcal{I}. \tag{34}$$

Next, we have to construct the set of indices $\mathcal{I}$ that satisfies Eqs. (33) and (34). By assumption $\lim_{n\to\infty} \alpha_n$, there exists a permutation $(i_n)_{n=2}^{\infty}$ of $\{2, 3, \dots\}$ such that $(\alpha_{i_n})_{n=2}^{\infty}$ is a nonincreasing sequence. We observe that

$$\sum_{n=2}^{\infty} \beta_{i_n} a_{i_n} = \sum_{n=2}^{\infty} \beta_n a_n.$$

Then there exists the smallest integer $n_0 \geq 2$ such that

$$\sum_{n=2}^{n_0} \beta_{i_n} a_{i_n} > \delta.$$

Using this fact, we now construct the method to find the indices set $\mathcal{I}$. Since $\beta_n > 0$ for $n \geq 2$ and $\sum_{n=2}^{n_0-1} (\beta_n a_n) < \delta$, we obtain

$$\begin{aligned}
\mu_{n_0} &= 2\frac{\sum_{n=2}^{n_0} (\beta_{i_n} a_{i_n}) - \delta}{\sum_{n=2}^{n_0} \beta_{i_n}^2 (i_n + 1)} \\
&= 2\frac{\beta_{i_{n_0}} a_{i_{n_0}} + \left(\sum_{n=2}^{n_0-1} (\beta_{i_n} a_{i_n}) - \delta\right)}{\beta_{i_{n_0}}^2 (i_{n_0} + 1) + \sum_{n=2}^{n_0-1} \beta_{i_n}^2 (i_n + 1)} \\
&\leq 2\frac{\beta_{i_{n_0}} a_{i_{n_0}}}{\beta_{i_{n_0}}^2 (i_{n_0} + 1) + \sum_{n=2}^{n_0-1} \beta_{i_n}^2 (i_n + 1)} \\
&\leq 2\frac{\beta_{i_{n_0}} a_{i_{n_0}}}{\beta_{i_{n_0}}^2 (i_{n_0} + 1)} := \alpha_{i_{n_0}}.
\end{aligned}$$

In the case of $n_0 = 2$, it is easy to check that $\mu_2 = 2\frac{\beta_{i_2} a_{i_2} - 1}{\beta_{i_2}^2 (i_2+1)} \leq 2\frac{\beta_{i_2} a_{i_2}}{\beta_{i_2}^2 (i_2+1)} := \alpha_{i_2}$, so we conclude that $\mu_{n_0} \leq \alpha_{i_{n_0}}$. Using the nonincreasing property of $(\alpha_{i_n})_{n=2}^{\infty}$, we have $\mu_{n_0} \leq \alpha_{i_{n_0}} \leq \alpha_{i_{n_0-1}} \leq \cdots \leq \alpha_{i_2}$. We need to find the smallest integer N satisfying such a property, i.e.

$$\alpha_{i_{N+1}} \leq \mu_N \leq \alpha_{i_N} \leq \alpha_{i_{N-1}} \leq \cdots \leq \alpha_{i_2}. \tag{35}$$

If $\alpha_{i_{n_0+1}} \leq \mu_{n_0}$, then we can choose $N = n_0$ and $\mathcal{I} = \{i_2, i_3, \dots, i_{n_0}\}$.

If $\alpha_{i_{n_0+1}} > \mu_{n_0}$, then we have

$$\mu_{n_0} = 2\frac{\sum_{n=2}^{n_0} (\beta_{i_n} a_{i_n}) - 1}{\sum_{n=2}^{n_0} \beta_{i_n}^2 (i_n + 1)} \leq \frac{2\left(\sum_{n=2}^{n_0} (\beta_{i_n} a_{i_n}) - 1\right) + (\beta_{i_{n_0+1}} a_{i_{n_0+1}})}{\sum_{n=2}^{n_0} (\beta_{i_n}^2 (i_n + 1)) + \beta_{i_{n_0+1}}^2 (i_{n_0+1} + 1)} := \mu_{n_0+1},$$

where we used the fact that if $\frac{a}{b} \leq \frac{c}{d}$ $(b, d > 0)$ then $\frac{a}{b} \leq \frac{a+c}{b+d}$. By the above observation, we also have $\mu_{n_0} \leq \mu_{n_0+1} \leq \alpha_{i_{n_0+1}}$. In this case, we distinguish the comparison between $\alpha_{i_{n_0+2}}$ and $\mu_{i_{n_0+1}}$ as in the following cases.

If $\alpha_{i_{n_0+2}} \leq \mu_{n_0+1}$, then we can choose $N = n_0 + 1$ and $\mathcal{I} = \{i_2, i_3, \ldots, i_{n_0+1}\}$.

If $\alpha_{i_{n_0+2}} > \mu_{n_0+1}$, by using the same argument, we then have

$$\mu_{n_0} \leq \mu_{n_0+1} \leq \mu_{n_0+2} \leq \alpha_{i_{n_0+2}}.$$

Proceeding inductively, we obtain that either there exists an integer $N = n_0 + k$ at some step for which the relation (35) holds, or

$$0 < \mu_{n_0} \leq \mu_{n_0+1} \leq \cdots \leq \mu_{n_0+k} \leq \alpha_{i_{n_0+k}}, \tag{36}$$

for all $k \geq 0$. However, inequality (36) cannot be true for all $k \geq 0$ due to $\lim\limits_{n\to\infty} \alpha_{i_n} = 0$, i.e. $0 < \mu_{n_0} \leq \lim\limits_{n\to\infty} \alpha_{i_n} = 0$. Hence, at some step, we can find $k \geq 0$ such that

$$\alpha_{i_{n_0+k+1}} \leq \mu_{n_0+k} \leq \alpha_{i_{n_0+k}} \leq \cdots \leq \alpha_{i_2}.$$

That is, $\mathcal{I} = \{i_2, i_3, \ldots, i_{n_0+k}\}$ satisfying Eqs. (33) and (34). Then we have

$$\begin{aligned}
\inf \sum_{n=2}^{\infty} \frac{(x_n - a_n)^2}{n+1} &= \sum_{n \in \mathcal{I}^c} \frac{(x_n - a_n)^2}{n+1} + \sum_{n \in \mathcal{I}} \frac{(x_n - a_n)^2}{n+1} \\
&= \sum_{n \in \mathcal{I}^c} \frac{a_n^2}{n+1} + \frac{\mu^2}{4} \sum_{n \in \mathcal{I}} (n+1)\beta_n^2 \\
&= \sum_{n \in \mathcal{I}^c} \frac{a_n^2}{n+1} + \frac{\left(\sum_{n \in \mathcal{I}} (\beta_n a_n) - \delta\right)^2}{\sum_{n \in \mathcal{I}} \beta_n^2 (n+1)}.
\end{aligned}$$

Next, we show that the Karush–Kuhn–Tucker conditions can be applied in this infinite-dimensional setting. We first observe that, for any fixed integer $m \geq 2$,

$$\inf \sum_{n=2}^{\infty} \frac{(x_n - a_n)^2}{n+1} \geq \inf \sum_{n=2}^{m} \frac{(x_n - a_n)^2}{n+1}, \tag{37}$$

where both infimums are taken over all nonnegative sequences $(x_n)_{n=2}^{\infty}$ satisfying $\sum_{n=2}^{\infty} \beta_n x_n \leq \delta$. Using the same argument, we can solve the Karush–Kuhn–Tucker conditions for finite-dimensional problem. Then, for $m \geq \max\{i_2, i_3, \ldots, i_N\}$, we obtain that the second infimum is attained for the sequence $(x_n)_{n=2}^{m}$ given by

$$x_n = \begin{cases} a_n - \frac{1}{2}\mu(n+1)\beta_n & \text{for} \quad n \in \mathcal{I}, \\ 0 & \text{for} \quad n \in \mathcal{I}^c \cap \{2, 3, \ldots, m\}. \end{cases} \tag{38}$$

We see that this argument can be applied for any arbitrary $m \geq \max\{i_2, i_3, \ldots, i_N\}$, so passing $m \to \infty$, we obtain that

$$\begin{aligned}
\inf \sum_{n=2}^{\infty} \frac{(x_n - a_n)^2}{n+1} &\geq \lim_{m\to\infty} \inf \sum_{n=2}^{m} \frac{(x_n - a_n)^2}{n+1} \\
&= \lim_{m\to\infty} \sum_{n\in\mathcal{I}^c\cap\{2,\ldots,m\}} \frac{a_n^2}{n+1} + \frac{\left(\sum_{n\in\mathcal{I}} \beta_n a_n - \delta\right)^2}{\sum_{n\in\mathcal{I}} \beta_n^2 (n+1)} \\
&= \sum_{n\in\mathcal{I}^c} \frac{a_n^2}{n+1} + \frac{\left(\sum_{n\in\mathcal{I}} \beta_n a_n - \delta\right)^2}{\sum_{n\in\mathcal{I}} \beta_n^2 (n+1)}.
\end{aligned}$$

This inequality guarantees that the Karush–Kuhn–Tucker conditions can be applied to the quadratic problem (21)–(22) in the case of an infinite-dimensional setting. In addition, we note that the infimum in Eq. (20) is attained for the sequence $(b_n)_{n\geq 2}$ given by

$$b_n = \begin{cases} \left(|a_n| - (n+1)\beta_n \frac{\sum_{m\in\mathcal{I}} (\beta_m |a_m|) - \delta}{\sum_{m\in\mathcal{I}} \beta_m^2 (m+1)}\right) e^{i \arg a_n} & \text{for} \quad n \in \mathcal{I}, \\ 0 & \text{for} \quad n \in \mathcal{I}^c, \end{cases}$$

where we use the fact that Eq. (20) holds in the triangle inequality if $\arg a_n = \arg b_n$ and apply this to Eq. (38). To complete this proof, we note that

$$\begin{aligned}
\sum_{n=2}^{\infty} \beta_n |b_n| &= \sum_{n\in\mathcal{I}} \left[\beta_n \left(|a_n| - (n+1)\beta_n \frac{\sum_{m\in\mathcal{I}} (\beta_m |a_m|) - \delta}{\sum_{m\in\mathcal{I}} \beta_m^2 (m+1)}\right)\right] \\
&= \sum_{n\in\mathcal{I}} (\beta_n |a_n|) - \left(\sum_{n\in\mathcal{I}} (\beta_n |a_n|) - \delta\right) = \delta.
\end{aligned}$$

That is, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}(\beta_n, \delta)$ and

$$dist(f, \mathcal{A}(\beta_n, \delta)) = \left(\pi \inf \sum_{n=2}^{\infty} \frac{(|a_n| - |b_n|)^2}{n+1}\right)^{1/2} = \left(\pi \sum_{n\in\mathcal{I}^c} \frac{|a_n|^2}{n+1} + \pi \frac{\left(\sum_{n\in\mathcal{I}} (\beta_n |a_n|) - \delta\right)^2}{\sum_{n\in\mathcal{I}} (\beta_n^2 |a_n|)}\right)^{1/2}.$$

The proof is complete. □

4. Applications on starlike and convex functions of order α

In this section, we provide some applications of our results to the classes of starlike and convex functions of order α. Here, we recall some sufficient coefficient inequality conditions for functions to be starlike and convex of order α. The following results are required.

Theorem 8 *[27] Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$.*

1. *If f satisfies the inequality*

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \leq 1 - \alpha, \tag{39}$$

then $f \in \mathcal{S}^*(\alpha)$. The converse is also true when $f \in \mathcal{T}$.

2. If f satisfies the inequality

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq 1-\alpha, \tag{40}$$

then $f \in \mathcal{K}(\alpha)$. The converse is also true when $f \in \mathcal{T}$.

We denote by $\mathcal{S}_\alpha^*$ and $\mathcal{K}_\alpha$ the subclass of $\mathcal{S}$ defined by Eqs. (39) and (40), respectively. We note that $\mathcal{S}_\alpha^* \subset \mathcal{S}^*(\alpha) \subset \mathcal{S}^*$ and $\mathcal{K}_\alpha \subset \mathcal{K}(\alpha) \subset \mathcal{K}$.

By applying Corollary 2, we obtain the fact that all functions in $\mathcal{S}_\alpha^*$ (and also $\mathcal{K}_\alpha$) are univalent; that is, $\mathcal{S}_\alpha^* \subset \mathcal{S}$ and $\mathcal{K}_\alpha \subset \mathcal{S}$. Theorem 3 also gives us the extension distortion inequalities for the functions in $\mathcal{S}_\alpha^*$ and $\mathcal{K}_\alpha$, which implies Theorem 4 and Theorem 6 in [27]. In addition, the invariant properties of classes $\mathcal{S}_\alpha^*$ and $\mathcal{K}_\alpha$ under some known operators can be easily derived by using Theorem 6.

Theorem 9 *The classes $\mathcal{S}_\alpha^*$ and $\mathcal{K}_\alpha$ are invariant under the Bernardi–Libera integral operator defined in Eq. (10). That is,*

$$L_\gamma\left[\mathcal{S}_\alpha^*\right] \subset \mathcal{S}_\alpha^*, \qquad \textit{and} \qquad L_\gamma\left[\mathcal{K}_\alpha\right] \subset \mathcal{K}_\alpha.$$

Next, we apply Theorem 7 to solve the best approximation problem in the class $\mathcal{S}_\alpha^*$ and $\mathcal{K}_\alpha$. The following theorem can be derived by using the same techniques of Theorem 3 in [22] and the characterization of $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, i.e. $f \in \mathcal{S}_\alpha^* \subset \mathcal{S}^*(\alpha) \iff \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha$ and $f \in \mathcal{K}_\alpha \subset \mathcal{K}(\alpha) \iff \operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha$.

Theorem 10 *For $f \in \mathcal{A}$, then:*

1. *$dist(f, \mathcal{S}_\alpha^*) = 0$ if and only if $f \in \mathcal{S}_\alpha^*$.*
2. *$dist(f, \mathcal{K}_\alpha) = 0$ if and only if $f \in \mathcal{K}_\alpha$.*

4.1. Starlike functions of order α

By Eq. (39), we have $\beta_n = \frac{n-\alpha}{1-\alpha}$. Applying Theorem 7, we obtain the best starlike order α approximation theorem.

Theorem 11 *Suppose that $f \in \mathcal{A}$ and $\lim_{n\to\infty} \frac{|a_n|}{n^2} = 0$.*

If $\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leq 1-\alpha$, then $dist(f, \mathcal{S}_\alpha^*) = 0$, where the infimum value of $dist(f, \mathcal{S}_\alpha^*)$ is attained for $g = f \in \mathcal{S}_\alpha^*$.

If $\sum_{n=2}^{\infty}(n-\alpha)|a_n| > 1-\alpha$, then

$$dist(f, \mathcal{S}_\alpha^*) = \left(\pi \sum_{n\in\mathcal{I}^c} \frac{|a_n|^2}{n+1} + \pi \frac{\left(\sum_{n\in\mathcal{I}}((n-\alpha)|a_n|) - (1-\alpha)\right)^2}{\sum_{n\in\mathcal{I}}((n-\alpha)^2(n+1))}\right)^{1/2},$$

where the infimum is attained for the function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$,

$$b_n = \begin{cases} \left(|a_n| - (n-\alpha)(n+1)\frac{\sum_{m\in\mathcal{I}}((m-\alpha)|a_m|)-(1-\alpha)}{\sum_{m\in\mathcal{I}}(m-\alpha)^2(m+1)}\right)e^{i\arg a_m}, & \textit{for} \quad n \in \mathcal{I}, \\ 0, & \textit{for} \quad n \in \mathcal{I}^c. \end{cases} \tag{41}$$

Remark 12 *By setting $\alpha = 0$ in Theorem 11, we obtain Theorem 5 in [21].*

Next, we provide an example to demonstrate our theoretical results.

Example 2 *Let $0 \le \alpha < 1$. Consider the function $f_{a,b} : \mathbb{D} \to \mathbb{C}$ defined by $f_{a,b}(z) = z + az^n + bz^m$, where $2 \le n < m$ and $a, b \in \mathbb{C}$ are fixed constants.*

From Theorem 8 and Theorem 11, we obtain the following.

If $(n-\alpha)|a| + (m-\alpha)|b| \le 1-\alpha$, then $f_{a,b} \in \mathcal{S}^*_\alpha$ and $dist(f_{a,b}, \mathcal{S}^*_\alpha) = 0$.

If $(n-\alpha)|a| + (m-\alpha)|b| > 1-\alpha$, then we distinguish following cases. If $|a| \ge \frac{(n-\alpha)(n+1)}{(m-\alpha)(m+1)}|b| + \frac{1-\alpha}{n-\alpha}$ we have $N = 2$ and $\mathcal{I} = \{i_2\} = \{n\}$, and if $|b| \ge \frac{(m-\alpha)(m+1)}{(n-\alpha)(n+1)}|a| + \frac{1-\alpha}{m-\alpha}$ we have $N = 2$ and $\mathcal{I} = i_2 = m$. Otherwise, we have $N = 3$ and $\mathcal{I} = \{i_2, i_3\} = \{n, m\}$. We then obtain

$$dist(f_{a,b}, \mathcal{S}^*_\alpha) = \begin{cases} \frac{(1-\alpha)\sqrt{\pi}}{(n-\alpha)\sqrt{n+1}} \left(\frac{(n+1)(n-\alpha)^2}{(m+1)(1-\alpha)^2}|b|^2 + \left(\frac{n-\alpha}{1-\alpha}|a| - 1\right)^2 \right)^{1/2} & \text{if} \quad |a| \ge \frac{(n-\alpha)(n+1)}{(m-\alpha)(m+1)}|b| + \frac{1-\alpha}{n-\alpha}, \\ \frac{(1-\alpha)\sqrt{\pi}}{(m-\alpha)\sqrt{m+1}} \left(\frac{(m+1)(m-\alpha)^2}{(n+1)(1-\alpha)^2}|a|^2 + \left(\frac{m-\alpha}{1-\alpha}|b| - 1\right)^2 \right)^{1/2} & \text{if} \quad |b| > \frac{(m-\alpha)(m+1)}{(n-\alpha)(n+1)}|a| + \frac{1-\alpha}{m-\alpha}, \\ \frac{\sqrt{\pi}|(n-\alpha)|a| + (m-\alpha)|b| - 1|}{\sqrt{(n+1)(n-\alpha)^2 + (m+1)(m-\alpha)^2}} & \text{if} \quad \text{otherwise}, \end{cases}$$

where the infimum is attained for the function $g_{a,b} : \mathbb{D} \to \mathbb{C}$, defined by

$$g_{a,b}(z) = \begin{cases} z + \frac{1-\alpha}{(n-\alpha)|a|} a z^n & \text{if} \quad |a| \ge \frac{(n-\alpha)(n+1)}{(m-\alpha)(m+1)}|b| + \frac{1-\alpha}{n-\alpha}, \\ z + \frac{1-\alpha}{(m-\alpha)|b|} b z^m & \text{if} \quad |b| > \frac{(m-\alpha)(m+1)}{(n-\alpha)(n+1)}|a| + \frac{1-\alpha}{m-\alpha}, \\ z + c_n e^{i \arg a} z^n + c_m e^{i \arg b} z^m & \text{if} \quad \text{otherwise}, \end{cases}$$

where

$$c_n = \frac{(m-\alpha)^2(m+1)|a| - (m-\alpha)(n-\alpha)(n+1)|b| + (1-\alpha)(n-\alpha)(n+1)}{(n-\alpha)^2(n+1) + (m-\alpha)^2(m+1)},$$

and

$$c_m = \frac{-(m-\alpha)(m+1)(n-\alpha)|a| + (n-\alpha)^2(n+1)|b| + (1-\alpha)(m-\alpha)(m+1)}{(n-\alpha)^2(n+1) + (m-\alpha)^2(m+1)}.$$

Let $n = 3$ and $m = 5$. For $a = -0.1$, it is easy to check that if $b = -0.05$ then $f_{a,b} \in \mathcal{S}^*_{0.5} \subset \mathcal{S}^*$, if $b = -0.1$ then $f_{a,b} \in \mathcal{S}^*$ but $f \notin \mathcal{S}^*_{0.5}$, and $f_{a,b} \notin \mathcal{S}^*$ for $b = -0.5$. Figure 1 shows the starlikeness of $f_{a,b}$ when $b = -0.05, -0.1$ and non-starlikeness when $b = -0.5$. Figure 2 shows the values of $dist(f_{a,b}, \mathcal{S}^*_\alpha)$, when $|a| = 0.1$, $0 \le |b| \le 1$ with $\alpha = 0, 0.25, 0.50, 0.75$.

4.2. Convex functions of order α

By Eq. (40), we have $\beta_n = \frac{n(n-\alpha)}{1-\alpha}$. Applying Theorem 7, we obtain the best convex order α approximation theorem.

Theorem 13 *Suppose that $f \in \mathcal{A}$ and $\lim_{n\to\infty} \frac{|a_n|}{n^3} = 0$.*

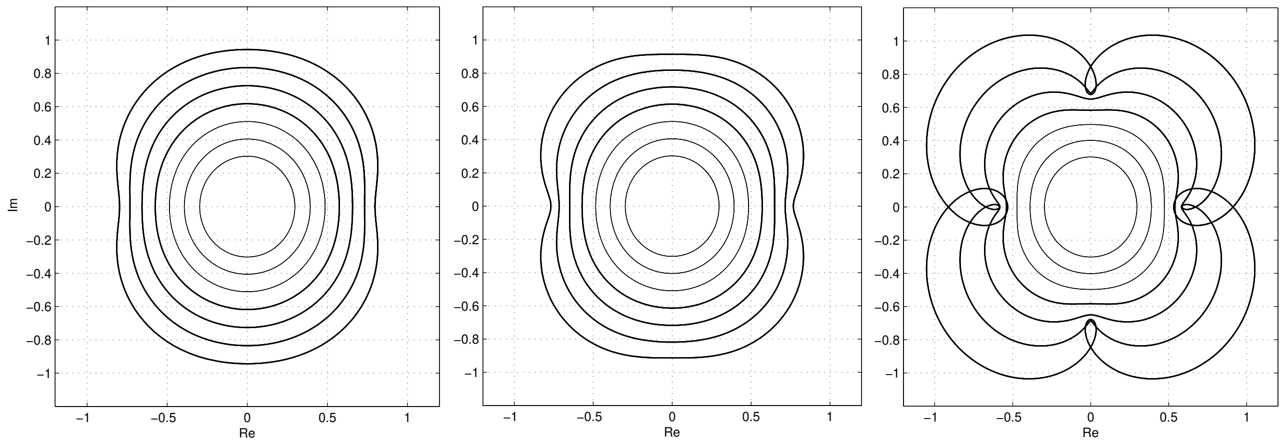

Figure 1. The image of the unit disk on the polynomial $f_{a,b}$ defined in Example 2, for $a = -0.1$ with $b = -0.05$ (left), $b = -0.1$ (center), $b = -0.5$ (right).

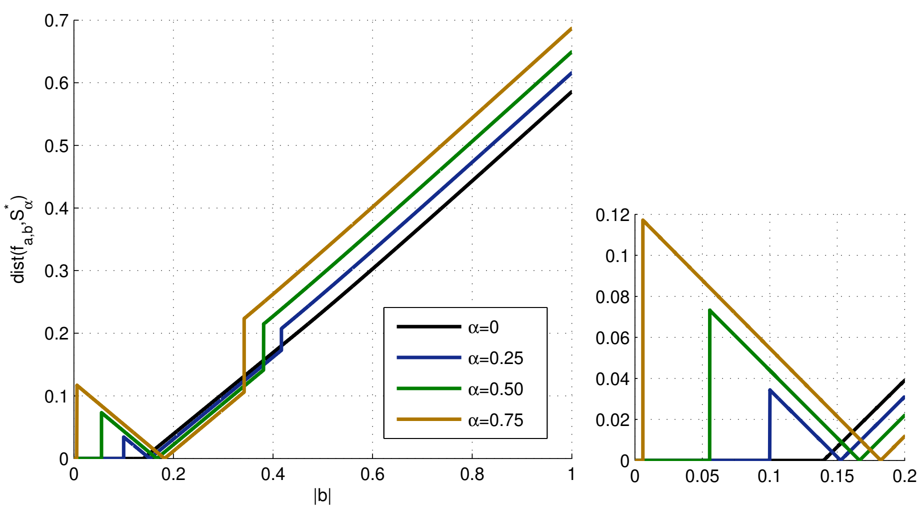

Figure 2. The value of $dist(f_{a,b}, \mathcal{S}^*_\alpha)$ when $|a| = 0.1$ with $\alpha = 0, 0.25, 0.50, 0.75$.

If $\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq 1-\alpha$ *then* $dist(f, \mathcal{K}_\alpha) = 0$, *where the infimum value of* $dist(f, \mathcal{K}_\alpha)$ *is attained for* $g = f \in \mathcal{K}_\alpha$.

If $\sum_{n=2}^{\infty} n(n-\alpha)|a_n| > 1-\alpha$ *then*

$$dist(f, \mathcal{S}^*_\alpha) = \left(\pi \sum_{n \in \mathcal{I}^c} \frac{|a_n|^2}{n+1} + \pi \frac{\left(\sum_{n\in\mathcal{I}}(n(n-\alpha)|a_n|) - (1-\alpha)\right)^2}{\sum_{n\in\mathcal{I}}((n(n-\alpha))^2(n+1)} \right)^{1/2},$$

where the infimum is attained for the function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$,

$$b_n = \begin{cases} \left(|a_n| - (n-\alpha)n(n+1)\frac{\sum_{m\in\mathcal{I}}(m(m-\alpha)|a_m|)-(1-\alpha)}{\sum_{m\in\mathcal{I}}(m(m-\alpha))^2(m+1)}\right) e^{i\arg a_n}, & \textit{for} \quad n \in \mathcal{I}, \\ 0, & \textit{for} \quad n \in \mathcal{I}^c. \end{cases} \tag{42}$$

Remark 14 *By setting $\alpha = 0$ in Theorem 13, we obtain Theorem 5 in [22].*

Example 3 *Let $0 \leq \alpha < 1$. Consider the function $f_{a,b} : \mathbb{D} \to \mathbb{C}$ defined by $f_{a,b}(z) = z + az^n + bz^m$, where $2 \leq n < m$ and $a, b \in \mathbb{C}$ are fixed constants.*

From Theorem 8 and Theorem 11, we obtain the following:

If $n(n-\alpha)|a| + m(m-\alpha)|b| \leq 1-\alpha$, then $f_{a,b} \in \mathcal{S}^*_\alpha$ and $dist(f_{a,b}, \mathcal{S}^*_\alpha) = 0$.

If $n(n-\alpha)|a| + m(m-\alpha)|b| > 1-\alpha$, then we distinguish the following cases. If $|a| \geq \frac{n(n-\alpha)(n+1)}{m(m-\alpha)(m+1)|b|} + \frac{1-\alpha}{n(n-\alpha)}$ we have $N = 2$ and $\mathcal{I} = \{i_2\} = \{n\}$, and if $|b| \geq \frac{m(m-\alpha)(m+1)}{n(n-\alpha)(n+1)}|a| + \frac{1-\alpha}{m(m-\alpha)}$ we have $N = 2$ and $\mathcal{I} = \{i_2\} = \{m\}$. Otherwise, we have $N = 3$ and $\mathcal{I} = \{i_2, i_3\} = \{n, m\}$. We then obtain

$$dist(f_{a,b}, \mathcal{K}_\alpha) = \begin{cases} \frac{(1-\alpha)\sqrt{\pi}}{n(n-\alpha)\sqrt{n+1}} \left(\frac{(n+1)(n(n-\alpha))^2}{(m+1)(1-\alpha)^2}|b|^2 + \left(\frac{n(n-\alpha)}{1-\alpha}|a| - 1\right)^2 \right)^{1/2} & \text{for } |a| \geq \frac{(n-\alpha)n(n+1)}{(m-\alpha)m(m+1)}|b| + \frac{1-\alpha}{n(n-\alpha)}, \\ \frac{(1-\alpha)\sqrt{\pi}}{m(m-\alpha)\sqrt{m+1}} \left(\frac{(m+1)(m(m-\alpha))^2}{(n+1)(1-\alpha)^2}|a|^2 + \left(\frac{m(m-\alpha)}{1-\alpha}|b| - 1\right)^2 \right)^{1/2} & \text{for } |b| > \frac{(m-\alpha)m(m+1)}{(n-\alpha)n(n+1)}|a| + \frac{1-\alpha}{m(m-\alpha)}, \\ \frac{\sqrt{\pi}|n(n-\alpha)|a|+m(m-\alpha)|b|-1|}{\sqrt{(n+1)(n(n-\alpha))^2+(m+1)(m(m-\alpha))^2}} & \text{for otherwise,} \end{cases}$$

where the infimum is attained for the function $g_{a,b} : \mathbb{D} \to \mathbb{C}$, defined by

$$g_{a,b}(z) = \begin{cases} z + \frac{1-\alpha}{n(n-\alpha)|a|}az^n & \text{for} \quad |a| \geq \frac{n(n-\alpha)(n+1)}{m(m-\alpha)(m+1)}|b| + \frac{1-\alpha}{n(n-\alpha)}, \\ z + \frac{1-\alpha}{m(m-\alpha)|b|}bz^m & \text{for} \quad |b| > \frac{m(m-\alpha)(m+1)}{n(n-\alpha)(n+1)}|a| + \frac{1-\alpha}{m(m-\alpha)}, \\ z + c_n e^{i \arg a} z^n + c_m e^{i \arg b} z^m & \text{for} \quad \textit{otherwise}, \end{cases}$$

where

$$c_n = \frac{(m(m-\alpha))^2(m+1)|a| - m(m-\alpha)n(n-\alpha)(n+1)|b| + (1-\alpha)n(n-\alpha)(n+1)}{(n(n-\alpha))^2(n+1) + (m(m-\alpha))^2(m+1)},$$

and

$$c_m = \frac{-m(m-\alpha)(m+1)n(n-\alpha)|a| + (n(n-\alpha))^2(n+1)|b| + (1-\alpha)m(m-\alpha)(m+1)}{(n(n-\alpha))^2(n+1) + (m(m-\alpha))^2(m+1)}.$$

Let $n = 3$ and $m = 5$. For $a = -0.05$, it is easy to check that if $b = -0.005$ then $f_{a,b} \in \mathcal{K}_{0.5} \subset \mathcal{K}$, if $b = -0.01$ then $f_{a,b} \in \mathcal{K}$ but $f \notin \mathcal{K}_{0.5}$, and $f_{a,b} \notin \mathcal{K}^*$ for $b = -0.1$. Figure 3 shows the convexity of $f_{a,b}$ when $b = -0.005, -0.01$ and nonconvexity when $b = -0.1$. Figure 4 shows the values of $dist(f_{a,b}, \mathcal{K}_\alpha)$ when $|a| = 0.05$, $0 \leq |b| \leq 1$ with $\alpha = 0, 0.25, 0.50, 0.75$.

5. Observation and concluding remarks

A general form of subclasses of analytic functions is introduced so that it can be used to study the properties of the class that contains nonunivalent functions. The new subclass of analytic functions is proposed consequently by imposing certain coefficient constraints with a nonnegative sequence of real numbers. By assigning appropriate values to the sequence $(\beta_n)_{n\geq 2}$ and δ, we can derive the corresponding results for several simpler subclasses of the class $\mathcal{A}(\beta_n, \delta)$ from each of our results. In Section 1, we discussed the radii of univalence and gave some sufficient conditions for the univalent property of the class $\mathcal{A}(\beta_n, \delta)$. By using the coefficient inequality, we obtained the distortion and invariant properties for the class $\mathcal{A}(\beta_n, \delta)$. In Section 3, the problem

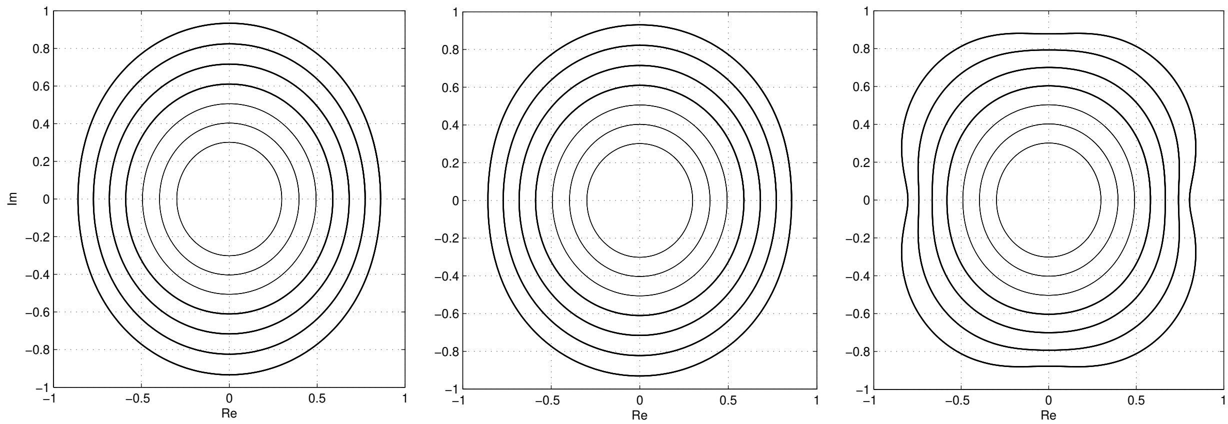

Figure 3. The image of the unit disk on the polynomial $f_{a,b}$ defined in Example 3, for $a = -0.05$ with $b = -0.005$ (left), $b = -0.01$ (center), $b = -0.1$ (right).

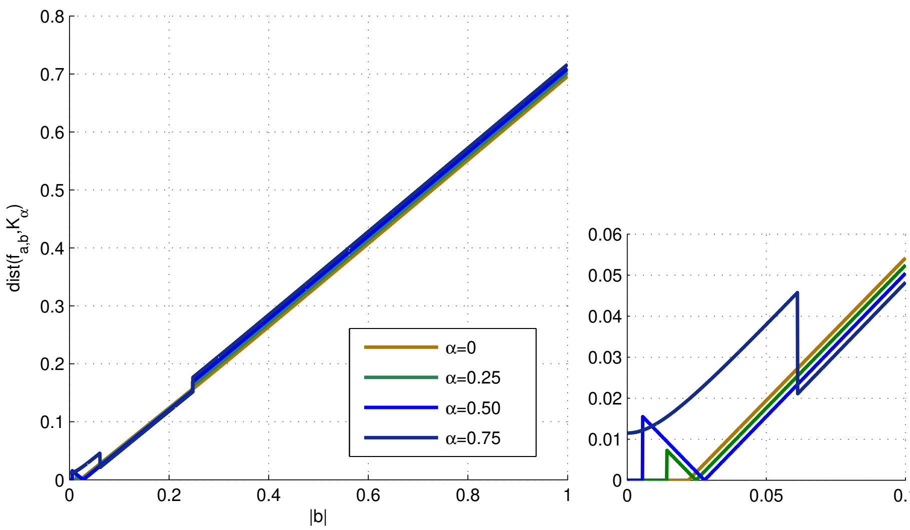

Figure 4. The value of $dist(f_{a,b}, \mathcal{K}_\alpha)$ when $|a| = 0.05$ with $\alpha = 0, 0.25, 0.50, 0.75$.

of finding the best approximation in such a class was solved by using a semiquadratic programming technique. In Section 4, we applied our obtained results to the subclass of starlike and convex functions of order α. Some basic properties and the best approximation problems have been pointed out. By using the presented results, we can obtain the best approximation in other known classes, which gives generalized results and results obtained in [17, 21, 22]. In Section 4, we discussed that the results in [22] and [21] can be obtained by setting $\alpha = 0$ in Theorem 11 and Theorem 13, respectively. In addition, by taking $\beta_n = \frac{n}{\alpha}[2(n-1)-\alpha]$, we immediately obtain Theorem 2 in [17]. We complete this paper by remarking that the presented results can be used to investigate some properties of subclasses of analytic functions satisfying certain coefficient inequality.

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