




Mannheim curves with modified orthogonal frame in Euclidean 3-space

Mohamd Saleem LONE^{1,*}, Hasan ES², Murat Kemal KARACAN³, Bahaddin BÜKCÜ⁴

¹International Centre for Theoretical Sciences, Tata Institute of Fundamental Research, Bengaluru, India

²Division of Mathematics Education, Faculty of Education, Gazi University, Teknikokullar, Ankara, Turkey

³Department of Mathematics, Faculty of Sciences and Arts, Uşak University, 1 Eylül Campus, Uşak, Turkey

⁴Department of Mathematics, Faculty of Sciences and Arts, Gaziosmanpaşa University, Tokat, Turkey

Received: 24.07.2018

Accepted/Published Online: 14.01.2019

Final Version: 27.03.2019

Abstract: In this paper, we investigate Mannheim pairs, Frenet–Mannheim curves, and weakened Mannheim curves with respect to the modified orthogonal frame in Euclidean 3-space (E^3). We derive some characterization results of these curves.

Key words: Mannheim curves, Mannheim partner curve, Modified orthogonal frame

1. Introduction

In the study of the classical differential geometry of space curves, finding the corresponding relations between different space curves has been an important and interesting characterization problem of space curves. For example, if the normal vector of one space curve φ is normal to another curve ψ , then ψ is called the Bertrand mate of φ . Liu [4] characterized a similar type of curves, weakened Bertrand curves and Frenet–Bertrand curves under weakened conditions. There is another important class of space curves called Mannheim curves, where the normal vector of one curve is the binormal vector to some other curve, and such a pair of curves is called as Mannheim pair. Liu and Wang [5] derived the necessary and sufficient conditions for a curve to possess a Mannheim partner curve in Euclidean and Minkowski spaces. Öztekin and Ergüt [9] studied the null Mannheim curves in Minkowski space and derived some necessary and sufficient conditions. Recently, Tunçer et al. [11] obtained some characterization results of the nonnull weakened Mannheim curves in Minkowski 3-space. Moreover, Karacan [3] characterized the weakened Mannheim curves in Euclidean 3-space. Mostly, researchers have studied Mannheim curves with respect to the classical Frenet–Serret frame of a curve, where we are supposed to consider that the curvature $\kappa(s) \neq 0$. In this paper, we shall drop the condition of $\kappa(s) \neq 0$ and consider a general set of curves with a discrete set of zeros of $\kappa(s)$ to characterize the Mannheim curves according to the modified orthogonal frame in Euclidean 3-space. Bukcu and Karacan studied space curves [2] and spherical curves [1] with respect to the modified orthogonal frame in Minkowski and Euclidean space, respectively. Recently, Lone et al. [6] obtained some characterization results for helices and Bertrand curves with respect to the modified orthogonal frame.

*Correspondence: mohamsaleem.lone@icts.res.in

2010 AMS Mathematics Subject Classification: 53A04, 53A35

2. Preliminaries

Let $\varphi(s)$ be a C^3 space curve in Euclidean 3-space E^3 , parametrized by arc length s . We also assume that its curvature $\kappa(s) \neq 0$ anywhere. Then an orthonormal frame $\{t, n, b\}$ exists satisfying the Frenet–Serret equations

$$\begin{bmatrix} t'(s) \\ n'(s) \\ b'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix}, \tag{2.1}$$

where t is the unit tangent, n is the unit principal normal, b is the unit binormal, and $\tau(s)$ is the torsion. For a given C^1 function $\kappa(s)$ and a continuous function $\tau(s)$, there exists a C^3 curve φ that has an orthonormal frame $\{t, n, b\}$ satisfying the Frenet–Serret frame (2.1). Moreover, any other curve $\tilde{\varphi}$ satisfying the same conditions differs from φ only by a rigid motion.

Now let $\varphi(t)$ be a general analytic curve, which can be reparametrized by its arc length. Assuming that the curvature function has discrete zero points or $\kappa(s)$ is not identically zero, we have an orthogonal frame $\{T, N, B\}$ defined as follows:

$$T = \frac{d\varphi}{ds}, \quad N = \frac{dT}{ds}, \quad B = T \times N,$$

where $T \times N$ is the vector product of T and N . The relations between $\{T, N, B\}$ and previous Frenet frame vectors at nonzero points of κ are

$$T = t, N = \kappa n, B = \kappa b. \tag{2.2}$$

Thus, we see that $N(s_0) = B(s_0) = 0$ when $\kappa(s_0) = 0$ and squares of the length of N and B vary analytically in s . From Eq. (2.2), it is easy to calculate

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2 & \frac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \frac{\kappa'}{\kappa} \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}, \tag{2.3}$$

and

$$\tau = \tau(s) = \frac{\det(\varphi', \varphi'', \varphi''')}{\kappa^2}$$

is the torsion of φ . From Frenet–Serret equations, we know that any point where $\kappa^2 = 0$ is a removable singularity of τ . Let \langle, \rangle be the standard inner product of E^3 , and then $\{T, N, B\}$ satisfies

$$\langle T, T \rangle = 1, \langle N, N \rangle = \langle B, B \rangle = \kappa^2, \langle T, N \rangle = \langle T, B \rangle = \langle N, B \rangle = 0. \tag{2.4}$$

The orthogonal frame defined in Eq. (2.3) satisfying Eq. (2.4) is called the modified orthogonal frame [10]. We see that for $\kappa = 1$, the Frenet–Serret frame coincides with the modified orthogonal frame.

Definition 2.1 *Let φ and ψ be two space curves in Euclidean 3-space such that, at the corresponding points of the curve, the principal normal vectors of φ coincide with the binormal vectors of ψ at the corresponding points. Then ψ is called a Mannheim partner curve of Mannheim curve φ and the pair $\{\varphi, \psi\}$ is called a Mannheim pair [5].*

Definition 2.2 A Mannheim curve $\psi(s^*)$ is a C^∞ regular curve with nonzero curvature for which there exists another C^∞ regular curve $\varphi(s)$ with $\varphi'(s) \neq 0$ and parameterized by arc length and it also has nonvanishing curvature, in bijection with it in such a way that the principal normal to $\psi(s^*)$ and the binormal to $\varphi(s)$ at each pair of corresponding points coincide with the line joining the corresponding points. The curve $\varphi(s)$ is said to be a Mannheim conjugate of $\psi(s^*)$ [3, 8].

Definition 2.3 A Frenet–Mannheim (FM) curve $\psi(s^*)$ is a C^∞ Frenet curve for which there exists another C^∞ Frenet curve $\varphi(s)$ and $\varphi'(s) \neq 0$, in bijection with it so that, by suitable choice of the Frenet frames, the principal normal vector $N_\psi(s^*)$ and the binormal vector $B_\varphi(s)$ at corresponding points on $\psi(s^*)$, $\varphi(s)$, both lie on the line joining the corresponding points. The curve $\varphi(s)$ is called the FM conjugate of $\psi(s^*)$ [3, 8].

Definition 2.4 A weakened Mannheim (WM) curve $\psi(s^*)$, $s^* \in I^*$, is a C^∞ regular curve for which there exists another C^∞ regular curve $\varphi(s)$, $s \in I$, and a homeomorphism $\rho : I \rightarrow I^*$, such that:

- (i) There exist two disjoint closed subsets M, N of I with void interiors such that $\rho \in C^\infty$ on $I \setminus N$, $\frac{ds^*}{ds} = 0$ on M , $\rho^{-1} \in C^\infty$ on $\rho(I \setminus M)$, and $\frac{ds}{ds^*} = 0$ on $\rho(N)$.
- (ii) The line joining the corresponding points s, s^* of $\varphi(s)$ and $\psi(s^*)$ is orthogonal to $\varphi(s)$ and $\psi(s^*)$ at the points s, s^* , respectively, and is along the principal normal to $\varphi(s)$ or $\psi(s^*)$ at the points s, s^* whenever it is well defined.

The curve $\varphi(s)$ is called a WM conjugate of $\psi(s^*)$ [3, 8].

From classical differential geometry, we find that there is a rich literature available on the Bertrand pairs in comparison to Mannheim curves. Thus, in this paper, we study the Mannheim curves according to the modified orthogonal frame in Euclidean 3-space and obtain several conditions for Mannheim partner FM and WM curves.

3. Mannheim partner curves according to modified orthogonal frame in E^3

Theorem 3.1 Let $C : \varphi(s)$ be a Mannheim curve in E^3 parameterized by its arc length s and let $C^* : \psi(s^*)$ be the Mannheim partner curve of C with an arc length parameter s^* . The distance between corresponding points of the Mannheim partner curves in E^3 is $|c| \kappa_\varphi$, where c is a nonzero constant and κ_φ is the curvature of curve φ .

Proof From the definition of Mannheim pair $\{C, C^*\}$, we can write $\overrightarrow{\varphi(s)\psi(s^*)} = \mu(s)N_\varphi(s)$, or

$$\psi(s^*) = \varphi(s) + \mu(s)N_\varphi(s) \tag{3.1}$$

for some function $\mu(s)$. Taking the derivative with respect to s and using Eq. (2.3), we get

$$\psi'(s^*) = T_\varphi + \mu' N_\varphi + \mu(-\kappa_\varphi^2 T_\varphi + \frac{\kappa'_\varphi}{\kappa_\varphi} N_\varphi + \tau_\varphi B_\varphi)$$

or

$$T_\psi \frac{ds^*}{ds} = (1 - \mu \kappa_\varphi^2) T_\varphi + \left(\mu' + \mu \frac{\kappa'_\varphi}{\kappa_\varphi} \right) N_\varphi + \mu \tau_\varphi B_\varphi. \tag{3.2}$$

Taking the inner product of Eq. (3.2) with B_ψ and considering $\frac{N_\varphi}{\kappa_\varphi} = \epsilon \frac{B_\psi}{\kappa_\psi}$ ($\epsilon = \pm 1$), we get

$$\mu' + \mu \frac{\kappa'_\varphi}{\kappa_\varphi} = 0 \text{ or } \mu = \frac{c}{\kappa_\varphi}. \tag{3.3}$$

This means that μ is not constant except $c = 0$. On the other hand, from the distance function between the points of $\varphi(s)$ and $\psi(s^*)$, we have

$$d(\varphi(s), \psi(s^*)) = |c| \kappa_\varphi(s).$$

□

Theorem 3.2 *A space curve $\varphi(s)$ in E^3 with respect to the modified orthogonal frame is a Mannheim curve if and only if its curvature κ_φ and torsion τ_φ satisfy*

$$\kappa_\varphi = c(\kappa_\varphi^2 + \tau_\varphi^2), \tag{3.4}$$

where c is a nonzero constant.

Proof Let $C : \varphi(s)$ be a Mannheim curve in E^3 with arc length parameter s and $C^* : \psi(s^*)$ the Mannheim partner curve of C with arc length parameter s^* . Inserting Eq. (3.3) into Eq. (3.1), we get

$$\psi(s^*) = \varphi(s) + \frac{c}{\kappa_\varphi(s)} N_\varphi(s) \tag{3.5}$$

for some nonzero constant c . Differentiating Eq. (3.5) with respect to s and applying the modified orthogonal frame formulas, we obtain

$$T_\psi \frac{ds^*}{ds} = (1 - c\kappa_\varphi) T_\varphi + \frac{c\tau_\varphi}{\kappa_\varphi} B_\varphi. \tag{3.6}$$

Again differentiating Eq. (3.6) with respect to s and applying the modified orthogonal frame formulas, we get

$$\begin{aligned} N_\psi \left(\frac{ds^*}{ds} \right)^2 + T_\psi \frac{d^2s^*}{ds^2} &= -c\kappa'_\varphi T_\varphi + (1 - c\kappa_\varphi) N_\varphi + \frac{c\tau'_\varphi \kappa_\varphi - c\kappa'_\varphi \tau_\varphi}{\kappa_\varphi^2} B_\varphi \\ &\quad + \frac{c\tau_\varphi}{\kappa_\varphi} \left(-\tau_\varphi N_\varphi + \frac{\kappa'_\varphi}{\kappa_\varphi} B_\varphi \right) \end{aligned}$$

or

$$N_\psi \left(\frac{ds^*}{ds} \right)^2 + T_\psi \frac{d^2s^*}{ds^2} = -c\kappa'_\varphi T + \frac{1}{\kappa_\varphi} (\kappa_\varphi - c\kappa_\varphi^2 - c\tau_\varphi^2) N_\varphi + c\tau'_\varphi \kappa_\varphi B_\varphi. \tag{3.7}$$

Taking the inner product of Eq. (3.7) with B_ψ , we get

$$\kappa_\varphi - c\kappa_\varphi^2 - c\tau_\varphi^2 = 0 \text{ or } \kappa_\varphi = c(\kappa_\varphi^2 + \tau_\varphi^2). \tag{3.8}$$

This completes the proof.

Conversely, if the curvature κ_φ and the torsion τ_φ of curve C satisfy Eq. (3.4) for some nonzero constant c , then define a curve C by Eq. (3.5), and we will prove that C is Mannheim and C^* is the partner curve of C . We have already found the equality below:

$$T_\psi \frac{ds^*}{ds} = (1 - c\kappa_\varphi) T_\varphi + \frac{c\tau_\varphi}{\kappa_\varphi} B_\varphi.$$

Differentiating the last equality with respect to s and with the help of Eq. (3.8), we get

$$N_\psi \left(\frac{ds^*}{ds} \right)^2 + T_\psi \frac{d^2s^*}{ds^2} = -c\kappa'_\varphi T_\varphi + \frac{c\tau'_\varphi}{\kappa_\varphi} B_\varphi. \tag{3.9}$$

Taking the cross product of Eq. (3.6) with Eq. (3.9), we obtain

$$\frac{ds}{ds^*} T_\psi \times \left[N_\psi \left(\frac{ds^*}{ds} \right)^2 + T_\psi \frac{d^2s^{*2}}{ds^2} \right] = \left[(1 - c\kappa_\varphi) T_\varphi + \frac{c\tau_\varphi}{\kappa_\varphi} B_\varphi \right] \times \left(-c\kappa'_\varphi T_\varphi + \frac{c\tau'_\varphi}{\kappa_\varphi} B_\varphi \right)$$

or

$$\left(\frac{ds^*}{ds} \right)^3 B_\psi = c(-\tau'_\varphi + c\tau'_\varphi \kappa_\varphi - c\kappa'_\varphi \tau_\varphi) \frac{N_\varphi}{\kappa_\varphi}. \tag{3.10}$$

Since both $\frac{N_\varphi}{\kappa_\varphi}$ and $\frac{B_\psi}{\kappa_\psi}$ have unit length, we get

$$\left(\frac{ds^*}{ds} \right)^3 = \frac{c(-\tau'_\varphi + c\tau'_\varphi \kappa_\varphi - c\kappa'_\varphi \tau_\varphi)}{\kappa_\psi}. \tag{3.11}$$

Thus, we have

$$\frac{B_\psi}{\kappa_\psi} = \epsilon \frac{N_\varphi}{\kappa_\varphi}, \epsilon = \pm 1,$$

or N_φ and B_ψ are linearly dependent. This completes the proof. □

Theorem 3.3 *A pair of curves (C, C^*) is a Mannheim pair if and only if the curvature κ_ψ and the torsion τ_ψ of curve C^* satisfy:*

$$\tau'_\psi = \frac{d\tau_\psi}{ds^*} = \frac{\kappa_\psi}{a} (1 + a^2 \tau_\psi^2), \tag{3.12}$$

where a is a nonzero constant.

Proof Suppose that $C : \varphi(s)$ is a Mannheim curve. By the definition of $\varphi(s)$, we may write

$$\varphi(s) = \psi(s^*) + \delta(s^*) B_\psi(s^*) \tag{3.13}$$

for some function $\delta(s^*)$. Differentiating Eq. (3.13) with respect to s^* , we get

$$T_\varphi \frac{ds}{ds^*} = T_\psi - \delta \tau_\psi N_\psi + \left(\delta' + \delta \frac{\kappa'_\psi}{\kappa_\psi} \right) B_\psi. \tag{3.14}$$

Since N_φ and B_ψ are linearly dependent, we get

$$\delta' + \delta \frac{\kappa'_\psi}{\kappa_\psi} = 0 \text{ or } \delta(s^*) = \frac{a}{\kappa_\psi}. \tag{3.15}$$

This means that $\delta(s^*)$ is not a constant for each s^* except $a = 0$. Thus, with the help of Eq. (3.15), we can rewrite Eq. (3.14) as follows:

$$T_\varphi \frac{ds}{ds^*} = T_\psi - \frac{a\tau_\psi}{\kappa_\psi} N_\psi. \tag{3.16}$$

Let θ be the angle between T_φ and T_ψ at the corresponding points of C and C^* in Eq. (3.13). Then, taking the inner product of Eq. (3.16) with T_ψ and considering the equality $\cos^2 \theta + \sin^2 \theta = 1$, we get

$$\cos \theta = \frac{ds^*}{ds} \tag{3.17}$$

and

$$\frac{ds^*}{ds} = -a\tau_\psi \sin \theta. \tag{3.18}$$

From Eq. (3.17) and Eq. (3.18), we find

$$\frac{ds}{ds^*} = \frac{1}{\cos \theta} = -\frac{a\tau_\psi}{\sin \theta} \tag{3.19}$$

and

$$\tan \theta = -a\tau_\psi. \tag{3.20}$$

Thus, we can write Eq. (3.16) as follows:

$$T_\varphi = (\cos \theta) T_\psi + \frac{\sin \theta}{\kappa_\psi} N_\psi. \tag{3.21}$$

Differentiating Eq. (3.21) with respect to s^* , we get

$$N_\varphi \frac{ds}{ds^*} = -\sin \theta (\kappa_\psi + \theta') T_\psi + \frac{\cos \theta}{\kappa_\psi} (\kappa_\psi + \theta') N_\psi + \left(\frac{\tau_\psi}{\kappa_\psi} \sin \theta \right) B_\psi. \tag{3.22}$$

From this equation and the fact that the direction of $\frac{N_\varphi}{\kappa_\varphi}$ is coincident with $\frac{B_\psi}{\kappa_\psi}$, we get

$$\begin{cases} -\sin \theta (\kappa_\psi + \theta') = 0 \\ \frac{\cos \theta}{\kappa_\psi} (\kappa_\psi + \theta') = 0 \end{cases} \tag{3.23}$$

or

$$\theta' = -\kappa_\psi. \tag{3.24}$$

Differentiating Eq. (3.20) with respect to s^* and applying Eq. (3.24), we get

$$\kappa_\psi + a^2 \kappa_\psi \tau_\psi^2 - a\tau_\psi' = 0$$

or

$$\tau_\psi' = \frac{\kappa_\psi}{a} (1 + a^2 \tau_\psi^2).$$

Conversely, if the curvature κ_ψ and the torsion τ_ψ of C^* satisfy Eq. (3.12) for some nonzero constant a , then define a curve C by Eq. (3.13) and we will prove that C is a Mannheim and C^* is the partner curve of C . We can easily reduce Eq. (3.13) in the following expression:

$$T_\varphi \frac{ds}{ds^*} = T_\psi - \frac{a\tau_\psi}{\kappa_\psi} N_\psi.$$

Differentiating the above equality with respect to s^* and with the help of Eq. (2.3), we get

$$N_\varphi \left(\frac{ds}{ds^*} \right)^2 + T_\varphi \frac{d^2s}{ds^{*2}} = N_\psi + \frac{a\kappa'_\psi \tau_\psi - a\kappa_\psi \tau'_\psi}{\kappa_\psi^2} N_\psi - \frac{a\tau_\psi}{\kappa_\psi} \left(-\kappa_\psi^2 T_\psi + \frac{\kappa'_\psi}{\kappa_\psi} N_\psi + \tau_\psi B_\psi \right),$$

or noticing Eq. (3.12), we get

$$N_\varphi \left(\frac{ds}{ds^*} \right)^2 + T_\varphi \frac{d^2s}{ds^{*2}} = a\tau_\psi \kappa_\psi T_\psi - a^2 \tau_\psi^2 N_\psi + \frac{c\tau_\psi^2}{\kappa_\psi} B_\psi. \tag{3.25}$$

Taking the cross product of Eq. (3.16) with Eq. (3.25), we have

$$\frac{ds}{ds^*} T_\varphi \times \left[\left(\frac{ds}{ds^*} \right)^2 N_\varphi + T_\varphi \frac{d^2s}{ds^{*2}} \right] = \left(T_\psi - \frac{a\tau_\psi}{\kappa_\psi} N_\psi \right) \times \left(a\tau_\psi \kappa_\psi T_\psi - a^2 \tau_\psi^2 N_\psi + \frac{a\tau_\psi^2}{\kappa_\psi} B_\psi \right)$$

or

$$\left(\frac{ds}{ds^*} \right)^3 B_\varphi = -\frac{a\tau_\psi^2}{\kappa_\psi} \left(\frac{a\tau_\psi}{\kappa_\psi} T_\psi + N_\psi \right). \tag{3.26}$$

Again taking the cross product of Eq. (3.16) with Eq. (3.26), we obtain

$$\left(\frac{ds}{ds^*} \right)^4 N_\varphi = \frac{a\tau_\psi^2}{\kappa_\psi} (\kappa_\psi^2 + a^2 \tau_\psi^2) B_\psi$$

or

$$N_\varphi = \frac{\kappa_\psi}{\kappa_\psi} B_\psi.$$

This means that the principal normal direction $\frac{N_\varphi}{\kappa_\varphi}$ of $C : \varphi(s)$ coincides with the binormal direction $\frac{B_\psi}{\kappa_\psi}$ of $C^* : \psi(s^*)$. Hence, $C : \varphi(s)$ is a Mannheim curve and $C^* : \psi(s^*)$ is its Mannheim partner curve. Therefore, for each Mannheim curve, there is a unique Mannheim partner curve. \square

Proposition 3.4 *A simple parametric transformation reduces the condition*

$$\tau'_\psi = \frac{\kappa_\psi}{a} (1 + a^2 \tau_\psi^2)$$

to

$$\tau_\psi = \frac{1}{a} \tan \left(\int \kappa_\psi ds + c_0 \right).$$

Thus, the existence of a Mannheim partner curve to a Mannheim curve is unique.

Proposition 3.5 *Let $\{\varphi(s), \psi(s^*)\}$ be a Mannheim pair, where both φ and ψ are parameterized by arc length s and s^* , respectively. If $\varphi(s)$ is a generalized helix according to the modified frame in E^3 , then $\psi(s^*)$ is a straight line.*

Proof Let T_φ , N_φ , and B_φ be the tangent, the principal normal, and the binormal vectors of $\varphi(s)$, respectively. From the definition of the Mannheim curve and properties of generalized helices, we have

$$\langle N_\varphi, u \rangle = \langle B_\psi, u \rangle = 0,$$

where u is some constant vector. Differentiating the last equality, we obtain

$$\left\langle -\tau_\psi N_\psi + \frac{\kappa'}{\kappa} B_\psi, u \right\rangle = -\tau_\psi \langle N_\psi, u \rangle + \frac{\kappa'}{\kappa} \langle B_\psi, u \rangle = -\tau_\psi \langle N_\psi, u \rangle = 0.$$

Since $\langle N_\psi, u \rangle \neq 0$, we get

$$\tau_\psi = 0.$$

Thus, using the last equality in Eq. (3.12), we easily obtain

$$\kappa_\psi = 0.$$

□

4. Frenet–Mannheim curves according to modified orthogonal frame in E^3

In this section, we characterize *FM* curves. For that, we begin with a lemma.

Lemma 4.1 Suppose $\psi(s^*)$, $s^* \in I^*$, is a *FM* curve with *FM* conjugate $\varphi(s)$. We mark all the quantities of $\psi(s^*)$ with an asterisk and suppose

$$\psi(s^*) = \varphi(s) + \delta(s)B_\varphi(s). \tag{4.1}$$

Then the distance ($|\delta|$) between corresponding points of $\varphi(s)$, $\psi(s^*)$ is not constant, i.e. $\delta = c\kappa_\varphi$, $c \in R$, and $\langle T_\varphi, T_\psi \rangle = \cos \theta$, where θ is a constant angle and

$$\begin{aligned} (i) \quad & \sin \theta = -a\tau_\varphi \cos \theta, \\ (ii) \quad & (1 + \epsilon a\kappa_\psi) \sin \theta = a\tau_\psi \cos \theta, \\ (iii) \quad & \cos^2 \theta = 1 + \epsilon a\kappa_\psi, \\ (iv) \quad & \sin^2 \theta = a^2\tau_\varphi\tau_\psi. \end{aligned}$$

Proof From Eq. (4.1), we have

$$\delta(s) = \langle \psi(s^*) - \varphi(s), B_\varphi(s) \rangle,$$

where $\delta(s)$ is of class C^∞ . Differentiating Eq. (4.1) with respect to s , we get

$$T_\psi \frac{ds^*}{ds} = T_\varphi + \delta' B_\varphi + \delta(-\tau_\varphi N_\varphi + \frac{\kappa'}{\kappa} B_\varphi)$$

or

$$T_\psi \frac{ds^*}{ds} = T_\varphi - \tau_\varphi \delta N_\varphi + \left(\delta' + \delta \frac{\kappa'_\varphi}{\kappa_\varphi} \right) B_\varphi. \tag{4.2}$$

By the given conditions, we have $B_\varphi = \epsilon N_\psi$ with $\epsilon = \pm 1$. Taking the scalar multiplication of Eq. (4.2) with B_φ , we obtain

$$\frac{\delta'}{\delta} = -\frac{\kappa'_\varphi}{\kappa_\varphi} \Rightarrow \delta = \frac{a}{\kappa_\varphi}, a \in R.$$

Therefore,

$$T_\psi \frac{ds^*}{ds} = T_\varphi - \frac{a\tau_\varphi}{\kappa_\varphi} N_\varphi. \tag{4.3}$$

Now, by the definition of the *FM* curve, we have $\frac{ds^*}{ds} \neq 0$, so T_ψ is a function of s of class C^∞ . Hence,

$$\langle T_\varphi, T_\psi \rangle' = \langle N_\varphi, T_\psi \rangle + \langle T_\varphi, N_\psi \rangle \frac{ds^*}{ds} = 0.$$

This implies that $\langle T_\varphi, T_\psi \rangle$ is constant and thus there exists a constant angle θ , such that

$$T_\psi = T_\varphi \cos \theta + N_\varphi \frac{\sin \theta}{\kappa_\varphi}. \tag{4.4}$$

From Eq. (4.3) and Eq. (4.4), we get

$$\left(\frac{ds}{ds^*} - \cos \theta \right) T_\psi - \left(\frac{ds}{ds^*} \frac{a\tau_\varphi}{\kappa_\varphi} + \frac{\sin \theta}{\kappa_\varphi} \right) N_\psi = 0.$$

Since T_ψ and N_ψ are linearly independent vectors, we have

$$\frac{ds}{ds^*} = \cos \theta \tag{4.5}$$

and

$$\frac{ds^*}{ds} \sin \theta = -a\tau_\varphi.$$

Using Eq. (4.5) in the last equality, we get

$$\sin \theta = -a\tau_\varphi \cos \theta, \tag{4.6}$$

which is (i). Now write

$$\varphi(s) = \psi(s^*) - \epsilon \delta(s) N_\psi(s).$$

The above equation implies that

$$\begin{aligned} T_\varphi &= \frac{ds^*}{ds} \left[T_\psi - \epsilon \delta' N_\psi - \epsilon \delta \left(-\kappa_\psi^2 T_\psi + \frac{\kappa'_\psi}{\kappa_\psi} N_\psi + \tau_\psi B_\psi \right) \right] \\ \text{or} \\ T_\varphi &= \frac{ds^*}{ds} \left[(1 + \epsilon \delta \kappa_\psi^2) T_\psi - \epsilon \left(\delta' + \delta \frac{\kappa'_\psi}{\kappa_\psi} \right) N_\psi - \epsilon \delta \tau_\psi B_\psi \right]. \end{aligned}$$

Using $\frac{\delta'}{\delta} = -\frac{\kappa'_\psi}{\kappa_\psi}$, it follows that

$$T_\varphi = \frac{ds^*}{ds} \left[(1 + \epsilon a \kappa_\psi) T_\psi - \epsilon a \frac{\tau_\psi}{\kappa_\psi} B_\psi \right]. \tag{4.7}$$

On the other hand, Eq. (4.4) gives

$$\begin{aligned} T_\psi \wedge N_\psi &= \left[T_\varphi \cos \theta + N_\varphi \frac{\sin \theta}{\kappa_\varphi} \right] \wedge N_\psi = \left[T_\varphi \cos \theta + N_\varphi \frac{\sin \theta}{\kappa_\varphi} \right] \wedge (\epsilon B_\varphi) \\ &= \epsilon (T_\varphi \wedge B_\varphi) \cos \theta + \epsilon (N_\varphi \wedge B_\varphi) \frac{\sin \theta}{\kappa_\varphi} = -\epsilon N_\varphi \cos \theta + \epsilon T_\varphi \kappa_\varphi \sin \theta \\ &= -\epsilon N_\varphi \cos \theta + \epsilon T_\varphi \kappa_\varphi^2 \frac{\sin \theta}{\kappa_\varphi} = \epsilon T_\varphi \sin \theta - \epsilon N_\varphi \kappa_\varphi \cos \theta. \\ \Rightarrow B_\psi &= \epsilon T_\varphi \sin \theta - \epsilon N_\varphi \cos \theta. \end{aligned}$$

Using Eq. (4.5) again, we get

$$T_\varphi = T_\psi \cos \theta - \epsilon B_\psi \frac{\sin \theta}{\kappa_\psi}. \tag{4.8}$$

Taking the vector product of Eq. (4.5) and Eq. (4.6), we obtain

$$(1 + \epsilon a \kappa_\psi) \sin \theta = a \tau_\psi \cos \theta,$$

which is (ii). On the other hand, from Eq. (4.7) and Eq. (4.8), it follows that

$$\frac{ds^*}{ds} (1 + a \epsilon \kappa_\psi) = \cos \theta, \tag{4.9}$$

$$\frac{ds^*}{ds} (a \tau_\psi) = \sin \theta. \tag{4.10}$$

Thus, inserting Eq. (4.5) into Eq. (4.9) and using Eq. (4.6) in Eq. (4.10), we get (iii) and (iv), respectively. \square

Theorem 4.2 Let $\psi(s^*) \in C^\infty$, $s^* \in I^*$, be a Frenet curve with τ_ψ nowhere vanishing and satisfying

$$(1 + a \epsilon \kappa_\psi) \sin \theta = a \tau_\psi \cos \theta \tag{4.11}$$

for some constant $a \neq 0$. Then $\psi(s^*)$ is a FM curve, which is nonplanar.

Proof Define the position vector of curve $\psi(s^*)$ as follows:

$$\psi(s^*) = \varphi(s) + \frac{a}{\kappa(s)} B_\varphi(s).$$

Let us denote differentiation with respect to s by a dash. We have

$$\psi'(s^*) = T_\varphi - \frac{a \tau_\varphi}{\kappa_\varphi} N_\varphi.$$

Since $\tau_\varphi \neq 0$, we see that $\psi(s^*)$ is a C^∞ regular curve. Suppose all the quantities of $\psi(s^*)$ are marked by an asterisk. Then

$$T_\psi \frac{ds^*}{ds} = T_\varphi - \frac{a \tau_\varphi}{\kappa_\varphi} N_\varphi.$$

Hence, we have

$$\frac{ds^*}{ds} = \sqrt{1 - a^2\tau_\varphi^2}.$$

Using Eq. (4.11), we get

$$T_\psi = T_\varphi \cos \theta + N_\varphi \frac{\sin \theta}{\kappa_\varphi},$$

and notice that from Eq. (4.11), we have $\sin \theta \neq 0$. Therefore,

$$\frac{dT_\psi}{ds^*} \frac{ds^*}{ds} = N_\varphi \cos \theta + \frac{\sin \theta}{\kappa_\varphi} \left(-\kappa_\varphi^2 T_\varphi + \frac{\kappa'_\varphi}{\kappa_\varphi} N_\varphi + \tau_\varphi B_\varphi \right) - \frac{\kappa'_\varphi}{\kappa_\varphi^2} \sin \theta N_\varphi$$

or

$$\kappa_\psi \frac{ds^*}{ds} \frac{N_\psi}{\kappa_\psi} = -(\kappa_\varphi \sin \theta) T_\varphi + N_\varphi \cos \theta + (\tau_\varphi \sin \theta) \frac{B_\varphi}{\kappa_\varphi}. \tag{4.12}$$

Now define $\frac{N_\psi}{\kappa_\psi} = \epsilon \frac{B_\varphi}{\kappa_\varphi}$. Then, from Eq. (4.12), we get

$$\kappa_\psi = \epsilon \frac{ds}{ds^*} \tau_\varphi \sin \theta.$$

These are C^∞ functions of s (and hence of s^*), and

$$\frac{dT_\psi}{ds^*} = N_\psi.$$

Again, define $B_\psi = T_\psi \wedge B_\varphi$ and

$$\left\langle \frac{dB_\psi}{ds^*}, N_\psi \right\rangle = \left\langle -\tau_\psi N_\psi + \frac{\kappa'_\psi}{\kappa_\psi} B_\psi, N_\psi \right\rangle = -\tau_\psi \langle N_\psi, N_\psi \rangle = -\tau_\psi \kappa_\psi^2$$

or

$$\tau_\psi = -\frac{\left\langle \frac{dB_\psi}{ds^*}, N_\psi \right\rangle}{\kappa_\psi^2}.$$

These are also C^∞ functions on I^* . It is then easy to verify that with the modified frame $\{T_\psi, N_\psi, B_\psi\}$ and the functions κ_ψ, τ_ψ , the curve $\psi(s^*)$ becomes a C^∞ Frenet curve. However, B_φ and N_ψ lie on the line joining the corresponding points of $\varphi(s)$ and $\psi(s^*)$. Thus, $\psi(s^*)$ is a FM curve and $\varphi(s)$ is a FM conjugate of $\psi(s^*)$. \square

Lemma 4.3 *A necessary and sufficient condition for a regular curve $\psi \in C^\infty$ to be a FM curve with a FM conjugate is that ψ is either a line or a nonplanar circular helix.*

Proof \Rightarrow : Suppose a line φ is a FM conjugate of ψ . This implies $\kappa_\varphi = 0$. Using Lemma 4.1, (iii) and (i), (ii), we have

$$\cos^2 \theta = 1 + a\epsilon\kappa_\psi \tag{4.13}$$

and then

$$\cos^2 \theta \sin \theta = a\tau_\psi \cos \theta, \tag{4.14}$$

$$\cos \theta = -a\tau_\varphi \sin \theta. \tag{4.15}$$

From Eq. (4.15), it follows that $\cos \theta \neq 0$. Hence, Eq. (4.14) is proportional to

$$a\tau_\psi = \cos \theta \sin \theta. \tag{4.16}$$

Case 1. $\sin \theta = 0$. Then $\cos \theta = \pm 1$, so (4.13) implies $\kappa_\psi = 0$, and ψ is a line. We also note that Eq. (4.16) implies that $\tau_\psi = 0$.

Case 2. $\sin \theta \neq 0$. Then $\cos \theta \neq \pm 1$, and Eqn. (4.13) and (4.16) imply that κ_ψ, τ_ψ are nonvanishing constants, and ψ is a nonplanar circular helix.

\Leftarrow : Suppose that ψ is a nonplanar circular helix given by

$$\psi = (r \cos t, r \sin t, bt), (a, b \in R^+),$$

where $t = \frac{s}{\sqrt{r^2 + b^2}}$.

We may write

$$\psi'(s) = T_\psi = \frac{r}{\sqrt{r^2 + b^2}} (-\sin t, \cos t, b)$$

and

$$\begin{aligned} \psi''(s) = N_\psi &= \frac{-r}{r^2 + b^2} (\cos t, \sin t, 0). \\ \Rightarrow \kappa_\psi = \frac{r}{r^2 + b^2} \text{ and } -r \frac{N_\psi}{\kappa_\psi} &= r (\cos t, \sin t, 0). \end{aligned}$$

Then the curve ψ with

$$\begin{aligned} \psi &= (r \cos t, r \sin t, bt) = r (\cos t, \sin t, 0) + (0, 0, bt) \\ &= -\frac{r}{\kappa_\psi} N_\psi + \varphi = \varphi - \frac{r}{\kappa_\psi} \left(\frac{\kappa_\psi}{\kappa_\varphi} B_\varphi \right) = \varphi - \frac{r}{\kappa_\varphi} B_\varphi, \end{aligned}$$

or putting $\delta = -\frac{r}{\kappa_\varphi}$,

$$\psi = \varphi + \delta B_\varphi$$

will be a line along the z -axis and can be turned into a *FM* conjugate of ψ by defining $\frac{N_\psi}{\kappa_\psi}$ to be equal to $\frac{B_\varphi}{\kappa_\varphi}$. □

Theorem 4.4 *Let $\psi(s^*) \in C^\infty$ be a plane Frenet curve with zero torsion and the curvature being bounded above or bounded below. Then ψ is a FM curve, with FM conjugates that are curves in the plane.*

Proof Suppose ψ is a curve satisfying the given conditions. Then $\kappa_\psi < -\frac{1}{a}$ or $\kappa_\psi > -\frac{1}{a}$ on I for some constant $a \neq 0$. For such a , let φ be the curve with position vector

$$\varphi = \psi - \varepsilon \delta N_\psi.$$

Differentiating the last equality with respect to s^* and considering equality $\delta' + \delta \frac{\kappa'_\varphi}{\kappa_\varphi} = 0$ and $\delta = \frac{a}{\kappa_\varphi}$ ($a \in R$), we get

$$T_\varphi \frac{ds}{ds^*} = T_\psi - \varepsilon \delta' N_\psi - \varepsilon \delta \left(-\kappa_\psi^2 T_\psi + \frac{\kappa'_\psi}{\kappa_\psi} N_\psi \right) = (1 + \varepsilon \delta \kappa_\psi^2) T_\psi - \varepsilon \left(\delta' + \delta \frac{\kappa'_\psi}{\kappa_\psi} \right) N_\psi$$

or

$$T_\varphi = \frac{ds^*}{ds} (1 + \varepsilon a \kappa_\psi) T_\psi.$$

Since $\frac{ds^*}{ds} (1 + \varepsilon a \kappa_\psi) \neq 0$, φ is a C^∞ regular curve, and $T_\varphi = T_\psi$. It is then easy to verify that φ is a *FM* conjugate of ψ . □

5. Weakened Mannheim curves according to modified orthogonal frame in E^3

Definition 5.1 Let \mathcal{D} be a subset of a topological space \mathcal{X} . If a function $\mathcal{X} \mapsto \mathcal{Y}$ is constant for each component of \mathcal{D} , we say that the function is \mathcal{D} -piecewise constant [4].

Lemma 5.2 Suppose \mathcal{D} is an open subset of a proper interval \mathcal{X} of the real line. A necessary and sufficient condition for every \mathcal{D} -piecewise constant real continuous function on \mathcal{X} to be constant is that there is an empty dense-in-itself kernel of $\mathcal{X} \setminus \mathcal{D}$ [4].

We remark, however, that if \mathcal{D} is dense in \mathcal{X} , any \mathcal{D} -piecewise constant, C^1 real function on \mathcal{X} is constant, even if \mathcal{D} has a nonempty dense-in-itself kernel.

Theorem 5.3 A *WM* curve for which M and N (defined in def. 2.4) have void dense-in-itself kernels is a *FM* curve.

Proof Let $\varphi(s)$, $s \in I$, be a *WM* conjugate of *WM* curve $\psi(s^*)$, $s^* \in I^*$. From the definition of $\varphi(s)$ and $\psi(s^*)$, it follows that both the curves have a C^∞ family of the tangent vectors $T_\psi(s^*)$, $T_\varphi(s)$. Let

$$\psi(s) = \psi(\rho(s)) = \varphi(s) + \delta(s)B_\varphi(s), \tag{5.1}$$

where $B_\varphi(s)$ is a vector function and $\delta(s) \geq 0$, $\forall s \in I$. Let $\mathcal{D} = I \setminus N$, $\mathcal{D}^* = I^* \setminus \rho(M)$. Then $s^*(s) \in C^\infty$ on \mathcal{D}^* .

Step 1. Prove $\delta = \frac{a}{\kappa_\varphi(s)}$. Since $\delta (= \|\psi(s) - \varphi(s)\|)$ is C^∞ on I and nowhere zero on every interval of \mathcal{D} , let \mathcal{X} be any component of $\mathcal{P} := \{s \in I : \delta(s) \neq 0\}$. Hence, both \mathcal{P} and \mathcal{X} are open in I . Consider \mathcal{L} as a component interval of $\mathcal{X} \cap \mathcal{D}$. Then $\delta(s)$ and $B_\varphi(s)$ are of class C^∞ on \mathcal{L} , and from Eq. (5.1), we have

$$\psi'(s) = T_\varphi + \delta'(s)B_\varphi(s) + \delta(s)B'_\varphi(s).$$

By the definition of a *WM* curve, we know that $\langle T_\varphi, B_\varphi(s) \rangle = 0 = \langle \psi'(s^*), B_\varphi(s) \rangle$. Hence, using $\langle B'_\varphi(s), B_\varphi(s) \rangle = 0$, we get

$$0 = \left(\delta'(s) + \delta(s) \frac{\kappa'_\varphi(s)}{\kappa_\varphi(s)} \right) \langle B_\varphi(s), B_\varphi(s) \rangle.$$

Therefore, $\delta = \frac{a}{\kappa_\varphi}$ on \mathcal{L} , where a is constant. Thus, δ is not constant on each interval of the set $\mathcal{X} \cap \mathcal{D}$, but by the given conditions $\mathcal{X} \setminus \mathcal{D}$ has a void dense-in-itself kernel. It follows from Lemma 5.2 that δ is not constant (and nonzero) on \mathcal{X} . As δ is continuous on I , \mathcal{X} should be closed in I , but in I , \mathcal{X} is also open. Hence, by connectedness, $\mathcal{X} = I$, i.e. δ is not constant on I .

Step 2. Take the existence of frames

$$\{T_\varphi(s), N_\varphi(s), N_\varphi(s)\}, \{T_\psi(s^*), N_\psi(s^*), B_\psi(s^*)\},$$

which are the modified orthogonal frames for $\varphi(s)$ on \mathcal{D} and $\psi(s^*)$ on \mathcal{D}^* , respectively. Since $\delta = \frac{a}{\kappa_\varphi}$ is a nonzero function, it follows from Eq. (5.1) that $B_\varphi(s)$ is continuous on I and is of class C^∞ on \mathcal{D} , and orthogonal to $T_\varphi(s)$. Define $B_\varphi(s) = T_\varphi(s) \wedge N_\varphi(s)$. Then $\{T_\varphi(s), N_\varphi(s), B_\varphi(s)\}$ forms a right-handed modified orthonormal frame for $\varphi(s)$, which is of class C^∞ on \mathcal{D} and continuous on I .

From the definition of a *WM* curve, it follows that there exists a scalar function $\kappa_\psi(s^*)$ such that $T'_\psi(s^*) = \kappa_\psi(s^*)N_\psi(s^*)$ on I^* . Thus, $\langle T'_\psi(s^*), N_\psi(s^*) \rangle = \kappa_\psi^2(s^*)$ is continuous on I^* and of class C^∞ on \mathcal{D}^* . Hence, the first Frenet formula holds on \mathcal{D}^* . It is now easy to show that there exists a function $\tau_\varphi(s)$ of class C^∞ on \mathcal{D} such that the Frenet formulas hold. Thus, $\{T_\varphi(s), N_\varphi(s), B_\varphi(s)\}$ is a modified orthogonal frame for $\varphi(s)$ on \mathcal{D} .

Similarly, there exists a modified orthogonal frame $\{T_\psi(s^*), N_\psi(s^*), B_\psi(s^*)\}$ for $\psi(s^*)$, which is continuous on I^* and is a Frenet frame for $\psi(s^*)$ on \mathcal{D}^* . Moreover, we may choose

$$B_\varphi(s) = N_\psi(\rho(s)).$$

Step 3. Show that $N = \emptyset$, $M = \emptyset$.

Noticing that on \mathcal{D} , we have

$$\langle T_\psi, T_\varphi \rangle' = \left\langle N_\psi \frac{ds^*}{ds}, T_\varphi \right\rangle + \langle T_\psi, \kappa_\varphi N_\varphi \rangle = 0,$$

on each component of \mathcal{D} , $\langle T_\psi, T_\varphi \rangle$ is constant and hence on I by Lemma 5.2. Thus, there exists an angle θ such that

$$T_\psi = T_\varphi \cos \theta + N_\varphi \sin \theta.$$

Furthermore,

$$B_\varphi(s) = N_\psi(\rho(s)),$$

and so

$$B_\psi(s^*) = -T_\varphi \sin \theta + \frac{N_\varphi}{\kappa_\varphi} \cos \theta.$$

Hence, $\{T_\psi(s^*), N_\psi(s^*), B_\varphi(s)\}$ are also of class C^∞ on \mathcal{D} . On the other hand, with respect to s^* on \mathcal{D}^* , $\{T_\psi(s^*), N_\psi(s^*), B_\psi(s^*)\}$ are of class C^∞ . Writing Eq. (5.1) in the form

$$\varphi = \psi - \frac{a}{\kappa_\psi} N_\psi \text{ or } \varphi = \psi - \delta N_\psi$$

and differentiating with respect to s on $\mathcal{D} \cap \rho^{-1}(\mathcal{D}^*)$, we have

$$T_\varphi = \frac{ds^*}{ds} \left[(1 + a\kappa_\psi) T_\psi - \frac{a\tau_\psi}{\kappa_\psi} B_\psi \right]. \tag{5.2}$$

However,

$$T_\varphi = T_\psi \cos \theta - B_\psi \sin \theta.$$

Hence,

$$\frac{ds^*}{ds} (1 + a\kappa_\psi) = \cos \theta \text{ and } \frac{a\tau_\psi}{\kappa_\psi} = -\sin \theta. \tag{5.3}$$

Since $\kappa_\psi^2(s^*) = \langle T'_\psi, N_\psi \rangle$ is continuous on I^* and $\rho^{-1}(\mathcal{D}^*)$ is dense, it follows by continuity that Eq. (5.3) holds throughout \mathcal{D} .

Case 1. $\cos \theta \neq 0$. Then Eq. (5.3) implies $\frac{ds^*}{ds} \neq 0$ on \mathcal{D} . Hence, $M = \emptyset$. Similarly, $N = \emptyset$.

Case 2. $\cos \theta = 0$. Then

$$T_\psi = \pm \frac{N_\varphi}{\kappa_\varphi}. \tag{5.4}$$

Taking the derivative of Eq. (5.1) with respect to s in \mathcal{D} , we get

$$T_\psi \frac{ds^*}{ds} = \pm \frac{a\tau_\varphi}{\kappa_\varphi} N_\varphi. \tag{5.5}$$

Hence, using Eq. (5.4) in Eq. (5.5), we have

$$\frac{ds^*}{ds} = \pm a\tau_\varphi.$$

Therefore, we get

$$\tau_\varphi = \pm \frac{1}{a} \frac{ds^*}{ds},$$

and so also on I , by Lemma 5.2. It follows that τ_φ is nowhere zero on I . Consequently, $\psi(s^*) = \varphi(s) + \delta(s)B_\varphi(s)$ is of class C^∞ on I^* . Hence, $N = \emptyset$. Similarly, $M = \emptyset$. □

References

- [1] Bukcu B, Karacan MK. Spherical curves with modified orthogonal frame. *Journal of New Results in Science* 2016; 10: 60-68.
- [2] Bukcu B, Karacan MK. On the modified orthogonal frame with curvature and torsion in 3-space. *Mathematical Sciences and Applications E-Notes* 2016; 4: 184-188.
- [3] Karacan MK. Weakened Mannheim curves. *International Journal of Physical Sciences* 2011; 6: 4700-4705.
- [4] Lai HF. Weakened Bertrand curves. *Tohoku Math Journal* 1967; 19: 141-155.
- [5] Liu H, Wang F. Mannheim partner curves in 3-space. *Journal of Geometry* 2008; 88: 120-126.
- [6] Lone MS, Es H, Karacan MK, Bukcu B. On some curves with Modified orthogonal frame in Euclidean 3-space. *Iranian Journal of Science and Technology, Transactions A, Science (in press)*. doi.org/10.1007/s40995-018-0661-2.

- [7] Orbay K, Kasap E. On Mannheim partner curves in E^3 . International Journal of Physical Sciences 2009; 4: 261-264.
- [8] Öztekin HB. Weakened Bertrand curves in the Galilean Space G_3 . Journal of Advanced Mathematical Studies 2009; 2: 69-76.
- [9] Öztekin HB, Ergüt M. Null Mannheim curves in the minkowski 3-space E_1^3 . Turkish Journal of Mathematics 2011; 35: 107-114.
- [10] Sasai T. The fundamental theorem of analytic space curves and apparent singularities of Fuchsian differential equations. Tohoku Math Journal 1984; 36: 17-24.
- [11] Tunçer Y, Karacan MK. Non-null weakened Mannheim curves in Minkowski 3-space. An Ştiinţ Al I Cuza Inşi Mat 2017; 63: 403-412.