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Stability analysis for a class of nabla (q,h)-fractional difference equations

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Abstract: This paper investigates stability of the nabla (q,h)-fractional difference equations. Asymptotic stability of the special nabla (q,h)-fractional difference equations are discussed. Stability theorems for discrete fractional Lyapunov direct method are proved. Furthermore, we give some new lemmas (including important comparison theorems) related to the nabla (q,h)-fractional difference operators that allow proving the stability of the nabla (q,h)-fractional difference equations, by means of the discrete fractional Lyapunov direct method, using Lyapunov functions. Some examples are given to illustrate these results.

Key words: Nabla (q,h)-fractional difference equations, stability, discrete fractional Lyapunov direct method, Lyapunov functions

1. Introduction

Fractional calculus plays an important role in modern control areas. Stability theory of fractional differential equations is frequently used in fractional controllers. However, due to fractional operators depend on the value of past state, it is difficult to extend the normal Lyapunov stability results to fractional cases since the Leibniz law becomes very complicated and does not hold in general.

Matignon [15] proposed an explicit stability condition for a linear fractional differential systems. The articles [13, 14] present the fractional Lyapunov direct method to the fractional order differential systems; for the applications of this method, see [20–22]. However, it is a difficult task to find an appropriate Lyapunov function by means of this method. Some authors have proposed Lyapunov functions to prove the stability of the fractional order systems. For the application of this method, we refer to [1–3, 9, 10, 19, 23].

The (q,h)-fractional difference equations have received a lot of attention recently; the basic theory and its applications can be found in [4, 5, 7, 8, 12, 16, 17]. In this paper, we use the idea in [10] to analyse the stability and asymptotical stability of the nabla (q,h)-fractional difference equations. Firstly, we prove the stability theorems of discrete fractional Lyapunov direct method for the special nabla (q,h)-fractional difference equations. Furthermore, we present some new lemmas, which enable us to determine the stability of such equations by establishing Lyapunov functions. Next, using these lemmas and discrete fractional Lyapunov direct method, we give sufficient conditions for these equations to be stable or asymptotically stable. Finally, some examples are given to illustrate our main results.

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2. Preliminaries

We recall some notation of (q,h)-calculus (for details, see [4, 5]). For any real number α and any $q>0$, $q\neq 1$, we set $[\alpha]_q := \frac{q^\alpha-1}{q-1}$. The extension of the q-binomial coefficient to the noninteger value n is given via the $\tilde{q}$-Gamma function $\Gamma_{\tilde{q}}(t)$ defined for $0<\tilde{q}<1$ as follows:

$$\Gamma_{\tilde{q}}(t) := \frac{(\tilde{q},\tilde{q})_\infty (1-\tilde{q})^{1-t}}{(\tilde{q}^t,\tilde{q})_\infty}, \quad 0<\tilde{q}<1,$$

where $(a,\tilde{q})_\infty = \prod_{j=0}^{\infty}(1-a\tilde{q}^j)$ and $t\in\mathbb{R}\backslash\{0,-1,-2,...\}$. It is easy to check that $\Gamma_{\tilde{q}}$ satisfies $\Gamma_{\tilde{q}}(t+1) = [t]_{\tilde{q}}\Gamma_{\tilde{q}}(t)$. The $\tilde{q}$-analogue of the power function is introduced as

$$(t-s)_{\tilde{q}}^{(\alpha)} = t^\alpha \frac{(s/t,\tilde{q})_\infty}{(\tilde{q}^\alpha s/t,\tilde{q})_\infty}, \quad t\neq 0,\ 0<\tilde{q}<1,\ \alpha\in\mathbb{R}.$$

For $\alpha = n$ a positive integer, this expression reduces to

$$(t-s)_{\tilde{q}}^{(n)} = t^n \prod_{j=0}^{n-1}\Big(1-\tilde{q}^j\frac{s}{t}\Big).$$

Here, the (q,h)-set is defined by:

$$\mathbb{T}_{(q,h)}^{t_0} = \{t_0q^k + [k]_q h, k\in\mathbb{Z}\}\cup\Big\{\frac{h}{1-q}\Big\},\ t_0>0,\ q\geq 1,\ h\geq 0,\ q+h>1.$$

Note that if $q=1$, then the cluster point $h/(1-q) = -\infty$ is not involved in $\mathbb{T}_{(q,h)}^{t_0}$. The forward and backward jump operator is the linear function $\sigma(t) = qt+h$ and $\rho(t) = q^{-1}(t-h)$, respectively. Similarly, the forward and backward graininess is given by $\mu(t) = (q-1)t+h$ and $\nu(t) = q^{-1}\mu(t)$, respectively. Observe that

$$\sigma^k(t) = q^k t + [k]_q h, \text{ and } \rho^k(t) = q^{-k}(t-[k]_q h).$$

Let $a\in\mathbb{T}_{(q,h)}^{t_0}$, $a>h/(1-q)$ be fixed. Then we introduce restrictions of the time scale $\mathbb{T}_{(q,h)}^{t_0}$ by the relation

$$\tilde{\mathbb{T}}_{(q,h)}^{\sigma^i(a)} = \{t\in\tilde{\mathbb{T}}_{(q,h)}^{t_0}, t\geq\sigma^i(a)\}, \quad i=0,1,...,$$

where the symbol σ^i stands for the ith iterate of σ (analogously, we use the symbol ρ^i). For the simplicity of notation, we put $\tilde{q} = 1/q$ whenever considering the time scale $\mathbb{T}_{(q,h)}^{t_0}$ or $\tilde{\mathbb{T}}_{(q,h)}^{\sigma^i(a)}$. The nabla (q,h)-difference of the function $x:\mathbb{T}_{(q,h)}^{t_0}\to\mathbb{R}$ is defined by

$$(\nabla_{(q,h)}x)(t) := \frac{x(t)-x(\rho(t))}{\nu(t)} = \frac{x(t)-x(\tilde{q}(t-h))}{(1-\tilde{q})t+\tilde{q}h},$$

where $\tilde{q} = 1/q$. The nabla (q,h)-fractional power functions and the (q,h)-Taylor monomials of degree α are defined by

$$(t-s)_{(\tilde{q},h)}^{(\alpha)} = ([t]-[s])_{\tilde{q}}^{(\alpha)},$$

$$\hat{h}_\alpha(t,s) := \frac{(t-s)^{(\alpha)}_{(\tilde{q},h)}}{\Gamma_{\tilde{q}}(\alpha+1)}, \quad \alpha \in \mathbb{R},$$

respectively, where $[t] = t + h\tilde{q}/(1-\tilde{q})$ and $[s] = s + h\tilde{q}/(1-\tilde{q})$, $0 < \tilde{q} < 1$. The following relations

$$\nu(t) = [t](1-\tilde{q}),$$

$$\nu(\rho^k(t)) = q^{-k}\nu(t),$$

$$\frac{[s]}{[t]} = \tilde{q}^n$$

hold for s, $t \in \mathbb{T}^{t_0}_{(q,h)}$, if there exists $n \in \mathbb{N}_0$ such that $t = \sigma^n(s)$. The nabla (q,h)-integral of $x : [a,t] \cap \tilde{\mathbb{T}}^a_{(q,h)} :\to \mathbb{R}$ is defined by

$$\int_a^t x(\tau)\nabla\tau := \sum_{i=1}^{k} x(\sigma^i(a))\nu(\sigma^i(a)),$$

where $t = \sigma^k(a)$, $k \geq 1$, and by convention $\int_a^a x(\tau)\nabla\tau = 0$.

Definition 2.1 *(See [4, Definition 1]). The Riemann–Liouville nabla (q,h)-fractional sum of order $\alpha > 0$ over the set $\tilde{\mathbb{T}}^a_{(q,h)}$ is defined by*

$$({}_a\nabla^{-\alpha}_{(q,h)}x)(t) = \int_a^t \hat{h}_{\alpha-1}(t,\rho(\tau))x(\tau)\nabla\tau. \tag{2.1}$$

Definition 2.2 *(See [4, Definition 3]). Assume $\alpha > 0$, $n = \lceil \alpha \rceil$, that is, n is the ceiling of α. Then the Riemann–Liouville nabla (q,h)-fractional difference of order α over the set $\tilde{\mathbb{T}}^{\sigma^n(a)}_{(q,h)}$ is defined by*

$$({}_a\nabla^{\alpha}_{(q,h)}x)(t) = (\nabla^n_{(q,h)}({}_a\nabla^{-(n-\alpha)}_{(q,h)}x))(t). \tag{2.2}$$

Lemma 2.1 *Assume $\alpha > 0$, $n = \lceil \alpha \rceil$, that is, n is the ceiling of α. Then the following formula is equivalent to (2.2)*

$$({}_a\nabla^{\alpha}_{(q,h)}x)(t) = \begin{cases} \displaystyle\int_a^t \hat{h}_{-\alpha-1}(t,\rho(\tau))x(\tau)\nabla\tau, & \alpha \in (n-1,n),\ t \in \tilde{\mathbb{T}}^{\sigma^n(a)}_{(q,h)}, \\ (\nabla^n_{(q,h)}x)(t), & \alpha = n,\ t \in \tilde{\mathbb{T}}^{\sigma^n(a)}_{(q,h)}. \end{cases} \tag{2.3}$$

Proof If $\alpha = n$, we have

$$({}_a\nabla^{\alpha}_{(q,h)}x)(t) = (\nabla^n_{(q,h)}({}_a\nabla^{-(n-\alpha)}_{(q,h)}x))(t) = (\nabla^n_{(q,h)}({}_a\nabla^{-0}_{(q,h)}x))(t) = (\nabla^n_{(q,h)}x)(t).$$

If $\alpha \in (n-1,n)$, we have

$$({}_a\nabla^{\alpha}_{(q,h)}x)(t) = (\nabla^n_{(q,h)}({}_a\nabla^{-(n-\alpha)}_{(q,h)}x))(t) = \Big(\nabla^{n-1}_{(q,h)}\nabla_{(q,h)}\Big(\int_a^t \hat{h}_{n-\alpha-1}(t,\rho(\tau))x(\tau)\nabla\tau\Big)\Big).$$

Taking the difference with respect to t, and using (see [5, Lemma 2.3]) ${}_t\nabla_{(q,h)}\hat{h}_{-\alpha}(t,\rho(\tau)) = \hat{h}_{-\alpha-1}(t,\rho(\tau))$, we get

$$\begin{aligned}\nabla_{(q,h)}\Big(\int_a^t \hat{h}_{n-\alpha-1}(t,\rho(\tau))x(\tau)\nabla\tau\Big) &= \frac{1}{\nu(t)}\Big(\int_a^t \hat{h}_{n-\alpha-1}(t,\rho(\tau))x(\tau)\nabla\tau - \int_a^{\rho(t)} \hat{h}_{n-\alpha-1}(\rho(t),\rho(\tau))x(\tau)\nabla\tau\Big)\\ &= \int_a^t {}_t\nabla_{(q,h)}(\hat{h}_{n-\alpha-1}(t,\rho(\tau))x(\tau))\nabla\tau + \hat{h}_{n-\alpha-1}(\rho(t),\rho(t))x(t)\\ &= \int_a^t \hat{h}_{n-\alpha-2}(t,\rho(\tau))x(\tau)\nabla\tau.\end{aligned}$$

Hence, we have

$$({}_a\nabla^{\alpha}_{(q,h)}x)(t) = \nabla^{n-1}_{(q,h)}\int_a^t \hat{h}_{n-\alpha-2}(t,\rho(\tau))x(\tau)\nabla\tau.$$

Repeating the similar procedure $n-1$ times, we obtain

$$({}_a\nabla^{\alpha}_{(q,h)}x)(t) = \int_a^t \hat{h}_{-\alpha-1}(t,\rho(\tau))x(\tau)\nabla\tau.$$

The proof is complete. □

Definition 2.3 *(See [17, p. 2218]). Assume $\alpha > 0$, $n = \lceil\alpha\rceil$, that is, n is the ceiling of α. Then the Caputo nabla (q,h)-fractional difference of order α over the set $\tilde{\mathbb{T}}^{\sigma^n(a)}_{(q,h)}$ is defined by*

$$({}^C_a\nabla^{\alpha}_{(q,h)}x)(t) = ({}_a\nabla^{-(n-\alpha)}_{(q,h)}(\nabla^n_{(q,h)}x))(t) = \int_a^t \hat{h}_{n-\alpha-1}(t,\rho(\tau))(\nabla^n_{(q,h)}x)(\tau)\nabla\tau. \tag{2.4}$$

Lemma 2.2 *(See [17, Theorem 3.9]). Assume $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}_1$ so that $n-1 < \alpha \le n$. Then*

$${}_a\nabla^{-\alpha}_{(q,h)}{}^C_a\nabla^{\alpha}_{(q,h)}x(t) = x(t) - \sum_{k=0}^{n-1}\hat{h}_k(t,a)\nabla^k_{(q,h)}x(a), \quad t \in \tilde{\mathbb{T}}^a_{(q,h)}. \tag{2.5}$$

The following corollary appears in Du et al. [7, Corollary 4.6].

Corollary 2.1 *Assume $x : \tilde{\mathbb{T}}^a_{(q,h)} \to \mathbb{R}$, $q > 1$, and $0 < \alpha < 1$. Then*

$$({}_{\sigma(a)}\nabla^{-\alpha}_{(q,h)}(\nabla_{(q,h)}x))(t) = (\nabla_{(q,h)}({}_a\nabla^{-\alpha}_{(q,h)}x))(t) - x(\sigma(a))\hat{h}_{\alpha-1}(t,a), \quad t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}. \tag{2.6}$$

Lemma 2.3 *Assume x, $y : \tilde{\mathbb{T}}^a_{(q,h)} \to \mathbb{R}$ and b, $c \in \tilde{\mathbb{T}}^a_{(q,h)}$, $b < c$. Then we have the integration by parts formula:*

$$\int_b^c x(\rho(t))(\nabla_{(q,h)}y)(t)\nabla t = x(t)y(t)|_{t=b}^c - \int_b^c y(t)(\nabla_{(q,h)}x)(t)\nabla t. \tag{2.7}$$

Proof From the definition of nabla (q,h)-difference, we have

$$\begin{aligned}\nabla_{(q,h)}(x(t)y(t)) &= \frac{x(t)y(t)-x(\rho(t))y(\rho(t))}{\nu(t)} \\ &= \frac{x(\rho(t))[y(t)-y(\rho(t))]+y(t)[x(t)-x(\rho(t))]}{\nu(t)} \\ &= x(\rho(t))(\nabla_{(q,h)}y)(t)+y(t)(\nabla_{(q,h)}x)(t).\end{aligned}$$

Integrating from b to c on both sides of the above formula, we have (2.7) holds. The proof is complete. □

Now, we give the following remark, it is essential for our main results.

Remark 2.1 *For* $0<\alpha<1$, $0<\tilde{q}<1$, $1\le j\le k_1$, $k_1+1\le k_2$, *we have*

$$\begin{aligned}&\hat{h}_{-\alpha}(\sigma^{k_2}(a),\sigma^{j-1}(a))-\hat{h}_{-\alpha}(\sigma^{k_1}(a),\sigma^{j-1}(a)) \\ &= \frac{(\sigma^{k_2}(a)-\sigma^{j-1}(a))^{(-\alpha)}_{(\tilde{q},h)}}{\Gamma_{\tilde{q}}(-\alpha+1)}-\frac{(\sigma^{k_1}(a)-\sigma^{j-1}(a))^{(-\alpha)}_{(\tilde{q},h)}}{\Gamma_{\tilde{q}}(-\alpha+1)} \\ &= \frac{([\sigma^{k_2}(a)]-[\sigma^{j-1}(a)])^{(-\alpha)}_{\tilde{q}}}{\Gamma_{\tilde{q}}(-\alpha+1)}-\frac{([\sigma^{k_1}(a)]-[\sigma^{j-1}(a)])^{(-\alpha)}_{\tilde{q}}}{\Gamma_{\tilde{q}}(-\alpha+1)} \\ &= \frac{[\sigma^{k_2}(a)]^{-\alpha}(\frac{[\sigma^{j-1}(a)]}{[\sigma^{k_2}(a)]},\tilde{q})_\infty}{\Gamma_{\tilde{q}}(-\alpha+1)(\tilde{q}^{-\alpha}\frac{[\sigma^{j-1}(a)]}{[\sigma^{k_2}(a)]},\tilde{q})_\infty}-\frac{[\sigma^{k_1}(a)]^{-\alpha}(\frac{[\sigma^{j-1}(a)]}{[\sigma^{k_1}(a)]},\tilde{q})_\infty}{\Gamma_{\tilde{q}}(-\alpha+1)(\tilde{q}^{-\alpha}\frac{[\sigma^{j-1}(a)]}{[\sigma^{k_1}(a)]},\tilde{q})_\infty} \\ &= \frac{[\sigma^{k_2}(a)]^{-\alpha}(\tilde{q}^{k_2-j+1},\tilde{q})_\infty}{\Gamma_{\tilde{q}}(-\alpha+1)(\tilde{q}^{-\nu+k_2-j+1},\tilde{q})_\infty}-\frac{[\sigma^{k_1}(a)]^{-\alpha}(\tilde{q}^{k_1-j+1},\tilde{q})_\infty}{\Gamma_{\tilde{q}}(-\alpha+1)(\tilde{q}^{-\alpha+k_1-j+1},\tilde{q})_\infty} \\ &= \frac{[\sigma^{k_2}(a)]^{-\alpha}\prod_{i=0}^{\infty}(1-\tilde{q}^{k_2-j+1+i})}{\Gamma_{\tilde{q}}(-\alpha+1)\prod_{i=0}^{\infty}(1-\tilde{q}^{-\alpha+k_2-j+1+i})}-\frac{[\sigma^{k_1}(a)]^{-\alpha}\prod_{i=0}^{\infty}(1-\tilde{q}^{k_1-j+1+i})}{\Gamma_{\tilde{q}}(-\alpha+1)\prod_{i=0}^{\infty}(1-\tilde{q}^{-\alpha+k_1-j+1+i})} \\ &= \frac{\tilde{q}^{k_2\alpha}(a+\frac{h}{q-1})^{-\alpha}\prod_{i=0}^{\infty}(1-\tilde{q}^{k_2-j+1+i})}{\Gamma_{\tilde{q}}(-\alpha+1)\prod_{i=0}^{\infty}(1-\tilde{q}^{-\alpha+k_2-j+1+i})}-\frac{\tilde{q}^{k_1\alpha}(a+\frac{h}{q-1})^{-\alpha}\prod_{i=0}^{\infty}(1-\tilde{q}^{k_1-j+1+i})}{\Gamma_{\tilde{q}}(-\alpha+1)\prod_{i=0}^{\infty}(1-\tilde{q}^{-\alpha+k_1-j+1+i})} \\ &= \Big[\tilde{q}^{k_2\alpha-k_1\alpha}-\frac{(1-\tilde{q}^{k_1-j+1})\cdots(1-\tilde{q}^{k_2-j})}{(1-\tilde{q}^{-\alpha+k_1-j+1})\cdots(1-\tilde{q}^{-\alpha+k_2-j})}\Big] \\ &\quad\times\frac{\tilde{q}^{k_1\alpha}(a+\frac{h}{q-1})^{-\alpha}\prod_{i=0}^{\infty}(1-\tilde{q}^{k_2-j+1+i})}{\Gamma_{\tilde{q}}(-\alpha+1)\prod_{i=0}^{\infty}(1-\tilde{q}^{-\alpha+k_2-j+1+i})}<0.\end{aligned}$$

For $0<\alpha<1$, $0<\tilde{q}<1$, $1\le j\le k$, *we have*

$$\begin{aligned}\hat{h}_{-\alpha}(\sigma^k(a),\sigma^{j-1}(a)) &= \frac{(\sigma^k(a)-\sigma^{j-1}(a))^{(-\alpha)}_{(\tilde{q},h)}}{\Gamma_{\tilde{q}}(-\alpha+1)} \\ &= \frac{([\sigma^k(a)]-[\sigma^{j-1}(a)])^{(-\alpha)}_{\tilde{q}}}{\Gamma_{\tilde{q}}(-\alpha+1)} \\ &= \frac{[\sigma^k(a)]^{-\alpha}(\frac{[\sigma^{j-1}(a)]}{[\sigma^k(a)]},\tilde{q})_\infty}{\Gamma_{\tilde{q}}(-\alpha+1)(\tilde{q}^{-\alpha}\frac{[\sigma^{j-1}(a)]}{[\sigma^k(a)]},\tilde{q})_\infty}\end{aligned}$$

$$
\begin{aligned}
&= \frac{[\sigma^k(a)]^{-\alpha}(\tilde{q}^{k-j+1},\tilde{q})_\infty}{\Gamma_{\tilde{q}}(-\alpha+1)(\tilde{q}^{-\alpha+k-j+1},\tilde{q})_\infty}\\
&= \frac{[\sigma^k(a)]^{-\alpha}\prod_{i=0}^{\infty}(1-\tilde{q}^{k-j+1+i})}{\Gamma_{\tilde{q}}(-\alpha+1)\prod_{i=0}^{\infty}(1-\tilde{q}^{-\alpha+k-j+1+i})} > 0.
\end{aligned}
$$

For $0<\alpha\le 1,\ 0<\tilde{q}<1,\ 1\le j\le k$, *we have*

$$
\begin{aligned}
\hat{h}_{\alpha-1}(\sigma^k(a),\sigma^{j-1}(a)) &= \frac{(\sigma^k(a)-\sigma^{j-1}(a))_{(\tilde{q},h)}^{(\alpha-1)}}{\Gamma_{\tilde{q}}(\alpha)}\\
&= \frac{([\sigma^k(a)]-[\sigma^{j-1}(a)])_{\tilde{q}}^{(\alpha-1)}}{\Gamma_{\tilde{q}}(\alpha)}\\
&= \frac{[\sigma^k(a)]^{\alpha-1}(\frac{[\sigma^{j-1}(a)]}{[\sigma^k(a)]},\tilde{q})_\infty}{\Gamma_{\tilde{q}}(\alpha)(\tilde{q}^{\alpha-1}\frac{[\sigma^{j-1}(a)]}{[\sigma^k(a)]},\tilde{q})_\infty}\\
&= \frac{[\sigma^k(a)]^{\alpha-1}(\tilde{q}^{k-j+1},\tilde{q})_\infty}{\Gamma_{\tilde{q}}(\alpha)(\tilde{q}^{\alpha+k-j},\tilde{q})_\infty}\\
&= \frac{[\sigma^k(a)]^{\alpha-1}\prod_{i=0}^{\infty}(1-\tilde{q}^{k-j+1+i})}{\Gamma_{\tilde{q}}(\alpha)\prod_{i=0}^{\infty}(1-\tilde{q}^{\alpha+k-j+i})} > 0.
\end{aligned}
$$

For $0<\alpha<1,\ 0<\tilde{q}<1,\ 1\le j\le k-1$, *we have*

$$
\begin{aligned}
\hat{h}_{-\alpha-1}(\sigma^k(a),\sigma^{j-1}(a)) &= \frac{(\sigma^k(a)-\sigma^{j-1}(a))_{(\tilde{q},h)}^{(-\alpha-1)}}{\Gamma_{\tilde{q}}(-\alpha)}\\
&= \frac{([\sigma^k(a)]-[\sigma^{j-1}(a)])_{\tilde{q}}^{(-\alpha-1)}}{\Gamma_{\tilde{q}}(-\alpha)}\\
&= \frac{[\sigma^k(a)]^{-\alpha-1}(\frac{[\sigma^{j-1}(a)]}{[\sigma^k(a)]},\tilde{q})_\infty}{\Gamma_{\tilde{q}}(-\alpha)(\tilde{q}^{-\alpha-1}\frac{[\sigma^{j-1}(a)]}{[\sigma^k(a)]},\tilde{q})_\infty}\\
&= \frac{[\sigma^k(a)]^{-\alpha-1}(\tilde{q}^{k-j+1},\tilde{q})_\infty}{\Gamma_{\tilde{q}}(-\alpha)(\tilde{q}^{-\alpha+k-j},\tilde{q})_\infty}\\
&= \frac{[\sigma^k(a)]^{-\alpha-1}\prod_{i=0}^{\infty}(1-\tilde{q}^{k-j+1+i})}{\Gamma_{\tilde{q}}(-\alpha)\prod_{i=0}^{\infty}(1-\tilde{q}^{-\alpha+k-j+i})} < 0,
\end{aligned}
$$

$$
\begin{aligned}
\hat{h}_{-\alpha-1}(\sigma^k(a),\sigma^{k-1}(a)) &= \frac{(\sigma^k(a)-\sigma^{k-1}(a))_{(\tilde{q},h)}^{(-\alpha-1)}}{\Gamma_{\tilde{q}}(-\alpha)}\\
&= \frac{([\sigma^k(a)]-[\sigma^{k-1}(a)])_{\tilde{q}}^{(-\alpha-1)}}{\Gamma_{\tilde{q}}(-\alpha)}\\
&= \frac{[\sigma^k(a)]^{-\alpha-1}(\frac{[\sigma^{k-1}(a)]}{[\sigma^k(a)]},\tilde{q})_\infty}{\Gamma_{\tilde{q}}(-\alpha)(\tilde{q}^{-\alpha-1}\frac{[\sigma^{k-1}(a)]}{[\sigma^k(a)]},\tilde{q})_\infty}
\end{aligned}
$$

$$= \frac{[\sigma^k(a)]^{-\alpha-1}(\tilde{q},\tilde{q})_\infty}{\Gamma_{\tilde{q}}(-\alpha)(\tilde{q}^{-\alpha},\tilde{q})_\infty}$$
$$= \frac{[\sigma^k(a)]^{-\alpha-1}\prod_{i=0}^{\infty}(1-\tilde{q}^{1+i})}{\Gamma_{\tilde{q}}(-\alpha)\prod_{i=0}^{\infty}(1-\tilde{q}^{-\alpha+i})} > 0.$$

For $q > 1$, $1 \le j \le k$, *we have*

$$\begin{aligned}
\nu(\sigma^j(a)) &= \sigma^j(a) - \rho(\sigma^j(a)) \\
&= \sigma^j(a) - \sigma^{j-1}(a) \\
&= \Big(q^j a + \frac{q^j-1}{q-1}h\Big) - \Big(q^{j-1}a + \frac{q^{j-1}-1}{q-1}h\Big) \\
&= q^{j-1}a(q-1) + q^{j-1}h \\
&> q^{j-1}(q-1)\frac{h}{1-q} + q^{j-1}h \\
&= 0,
\end{aligned}$$

where we used $a > \frac{h}{1-q}$.

3. Basic definitions and lemmas

In this section, we will present some basic definitions and lemmas, which are important for our main results.

Consider the following nonlinear nabla (q,h)-fractional difference equations

$$\begin{cases} ({}_a^C\nabla^\alpha_{(q,h)}x)(t) = f(t,x(t)), & t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}, \\ x(a) = x_0, \end{cases} \tag{3.1}$$

where $f : \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)} \times \mathbb{R} \to \mathbb{R}$, $x : \tilde{\mathbb{T}}^{a}_{(q,h)} \to \mathbb{R}$, and $\alpha \in (0,1]$, and

$$\begin{cases} ({}_a\nabla^\alpha_{(q,h)}x)(t) = f(t,x(t)), & t \in \tilde{\mathbb{T}}^{\sigma^2(a)}_{(q,h)}, \\ x(\sigma(a)) = x_0, \end{cases} \tag{3.2}$$

where $f : \tilde{\mathbb{T}}^{\sigma^2(a)}_{(q,h)} \times \mathbb{R} \to \mathbb{R}$, $x : \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)} \to \mathbb{R}$, and $\alpha \in (0,1]$. It is easy to see that equations (3.1) and (3.2) has a unique solution.

The constant x_{eq} is an *equilibrium point* of equation (3.1) (or (3.2)) if and only if $({}_a^C\nabla^\alpha_{(q,h)}x_{eq})(t) = f(t,x_{eq}(t)) = 0$ $(({}_a\nabla^\alpha_{(q,h)}x_{eq})(t) = f(t,x_{eq}(t))$ in the case of the Riemann–Liouville nabla (q,h)-fractional difference equation) for all $t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$.

Assume that $f(t,0) = 0$ so that the trivial solution $x \equiv 0$ is an equilibrium point of equation (3.1) (or (3.2)). Note that there is no loss of generality in doing so because any equilibrium point can be shifted to the origin via a change of variables.

First, we present the following simple definitions and important facts.

Definition 3.1 *The equilibrium point* $x = 0$ *of equation* (3.1) *(or* (3.2)*) is said to be*

(a) stable, if for each $\varepsilon > 0$*, there exists* $\delta = \delta(\varepsilon) > 0$ *such that* $\|x(a)\| < \delta$ *(or* $\|x(\sigma(a))\| < \delta$ *implies* $\|x(\sigma^k(a))\| < \varepsilon$ *for all* $k \in \mathbb{N}_0$*.*

(b) attractive, if there exists $\delta > 0$ *such that* $\|x(a)\| < \delta$ *(or* $\|x(\sigma(a))\| < \delta$ *implies* $\lim_{k\to\infty} x(\sigma^k(a)) = 0$*.*

(c) asymptotically stable, if it is stable and attractive.

The equation (3.1) (or (3.2)) is called stable (asymptotically stable) if their equilibrium point $x = 0$ is stable (asymptotically stable).

Definition 3.2 *(See [11, Definition 3.2]). A function* $\phi(r)$ *is said to belong to the class* $\mathcal{K}$ *if and only if* $\phi \in C[[0,\rho),\mathbb{R}_+]$*,* $\phi(0) = 0$*, and* $\phi(r)$ *is strictly monotonically increasing in* r*.*

Definition 3.3 *A real valued function* $V(t,x)$ *defined on* $\tilde{\mathbb{T}}^a_{(q,h)} \times S_\rho$*, where* $S_\rho = \{x \in \mathbb{R}^n : \|x\| \le \rho\}$*, is said to be positive definite if and only if* $V(t,0) = 0$ *for all* $t \in \tilde{\mathbb{T}}^a_{(q,h)}$ *and there exists* $\phi \in \mathcal{K}$ *such that* $\phi(r) \le V(t,x)$*,* $\|x\| = r$*,* $(t,x) \in \tilde{\mathbb{T}}^a_{(q,h)} \times S_\rho$*.*

Definition 3.4 *A real valued function* $V(t,x)$ *defined on* $\tilde{\mathbb{T}}^a_{(q,h)} \times S_\rho$*, where* $S_\rho = \{x \in \mathbb{R}^n : \|x\| \le \rho\}$*, is said to be decrescent if and only if* $V(t,0) = 0$ *for all* $t \in \tilde{\mathbb{T}}^a_{(q,h)}$ *and there exists* $\phi \in \mathcal{K}$ *such that* $V(t,x) \le \phi(r)$*,* $\|x\| = r$*,* $(t,x) \in \tilde{\mathbb{T}}^a_{(q,h)} \times S_\rho$*.*

Now, we give the following lemmas for the Caputo nabla (q,h)-fractional difference, which will be useful for proving the stability of equation (3.1). The proof of Lemmas 3.2–3.4 is motivated by the proof in [2, Lemmas 2.7–2.9].

Lemma 3.1 *Assume* $({}^C_a\nabla^\alpha_{(q,h)}x)(t) \ge ({}^C_a\nabla^\alpha_{(q,h)}y)(t)$*,* $t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$*,* $x(a) \ge y(a)$*, and* $\alpha \in (0,1]$*. Then we have* $x(t) \ge y(t)$ *for* $t \in \tilde{\mathbb{T}}^a_{(q,h)}$*.*

Proof Let $F(t) := x(t) - y(t)$. For $\alpha = 1$, we have

$$({}^C_a\nabla^\alpha_{(q,h)}F)(t) = (\nabla_{(q,h)}F)(t) \ge 0,$$

it is easy to see $x(t) \ge y(t)$ for $t \in \tilde{\mathbb{T}}^a_{(q,h)}$.

For $\alpha \in (0,1)$, since $({}^C_a\nabla^\alpha_{(q,h)}x)(t) \ge ({}^C_a\nabla^\alpha_{(q,h)}y)(t)$, we have

$$({}^C_a\nabla^\alpha_{(q,h)}F)(t) \ge 0,$$

which can be written as

$$\int_a^t \hat{h}_{-\alpha}(t,\rho(\tau))(\nabla_{(q,h)}F)(\tau)\nabla\tau \ge 0.$$

By the integration by parts formula (2.7), we have

$$\hat{h}_{-\alpha}(t,\tau)F(\tau)|_{\tau=a}^t + \int_a^t \hat{h}_{-\alpha-1}(t,\rho(\tau))F(\tau)\nabla\tau \ge 0.$$

Letting $t=\sigma^k(a)$, $k\geq 1$, we have

$$-\hat{h}_{-\alpha}(\sigma^k(a),a)F(a)+\sum_{j=1}^{k}\hat{h}_{-\alpha-1}(\sigma^k(a),\sigma^{j-1}(a))F(\sigma^j(a))\nu(\sigma^j(a))\geq 0.$$

Since $\hat{h}_{-\alpha}(\sigma^k(a),a)>0$, $\hat{h}_{-\alpha-1}(\sigma^k(a),\sigma^{k-1}(a))>0$, $\hat{h}_{-\alpha-1}(\sigma^k(a),\sigma^{j-1}(a))<0$, $1\leq j\leq k-1$, $\nu(\sigma^j(a))>0$, $1\leq j\leq k$, and $x(a)\geq y(a)$ are true. When $k=1$, we have $x(\sigma(a))\geq y(\sigma(a))$. Suppose $F(\sigma^j(a))\geq 0$, $0\leq j\leq k-1$, by strong induction, we obtain $F(t)\geq 0$, that is, $x(t)\geq y(t)$ for $t\in\tilde{\mathbb{T}}^a_{(q,h)}$. The proof is complete. □

Consider the following fractional difference equation

$$({}_a^C\nabla^\alpha_{(q,h)}x)(t)=-\gamma(x(t)),\ x(a)=x_0,\quad \alpha\in(0,1],\quad t\in\tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}, \tag{3.3}$$

where $\gamma\in\mathcal{K}$ and $x(t)$ is a positive definite and decrescent function. We can easily show this equation has a unique solution.

Lemma 3.2 *Assume $x(t)$ is a solution of equation (3.3), and $x(a)>0$. Then $(\nabla_{(q,h)}x)(t)<0$ for $t\in\tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$.*

Proof We assume that there exists a first point t_1 such that $(\nabla_{(q,h)}x)(t)\geq 0$ on $[\sigma(t_1),t_2]\cap\tilde{\mathbb{T}}^a_{(q,h)}$, where $t_1\in\tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$, $t_2\in\tilde{\mathbb{T}}^{\sigma^2(a)}_{(q,h)}$, and $(\nabla_{(q,h)}x)(t)<0$ on $[\sigma(a),t_1]\cap\tilde{\mathbb{T}}^a_{(q,h)}$. For $\alpha=1$, we have

$$({}_a^C\nabla^\alpha_{(q,h)}x)(t_2)-({}_a^C\nabla^\alpha_{(q,h)}x)(t_1)=(\nabla_{(q,h)}x)(t_2)-(\nabla_{(q,h)}x)(t_1)>0.$$

For $\alpha\in(0,1)$, by Definition 2.3, we have

$$\begin{aligned}
({}_a^C\nabla^\alpha_{(q,h)}x)(t_2)-({}_a^C\nabla^\alpha_{(q,h)}x)(t_1) &= \int_a^{t_2}\hat{h}_{-\alpha}(t_2,\rho(\tau))(\nabla_{(q,h)}x)(\tau)\nabla\tau-\int_a^{t_1}\hat{h}_{-\alpha}(t_1,\rho(\tau))(\nabla_{(q,h)}x)(\tau)\nabla\tau\\
&\overset{t_1=\sigma^{k_1}(a),\ t_2=\sigma^{k_2}(a)}{\underset{k_1\geq 1,\ k_2\geq 2,\ k_2\geq k_1+1}{=\!=\!=\!=}} \sum_{j=1}^{k_2}\hat{h}_{-\alpha}(\sigma^{k_2}(a),\sigma^{j-1}(a))(\nabla_{(q,h)}x)(\sigma^j(a))\nu(\sigma^j(a))\\
&\quad-\sum_{j=1}^{k_1}\hat{h}_{-\alpha}(\sigma^{k_1}(a),\sigma^{j-1}(a))(\nabla_{(q,h)}x)(\sigma^j(a))\nu(\sigma^j(a))\\
&= \sum_{j=1}^{k_1}(\hat{h}_{-\alpha}(\sigma^{k_2}(a),\sigma^{j-1}(a))-\hat{h}_{-\alpha}(\sigma^{k_1}(a),\sigma^{j-1}(a)))(\nabla_{(q,h)}x)(\sigma^j(a))\nu(\sigma^j(a))\\
&\quad+\sum_{j=k_1+1}^{k_2}\hat{h}_{-\alpha}(\sigma^{k_2}(a),\sigma^{j-1}(a))(\nabla_{(q,h)}x)(\sigma^j(a))\nu(\sigma^j(a))>0,
\end{aligned}$$

where $\hat{h}_{-\alpha}(\sigma^{k_2}(a),\sigma^{j-1}(a))-\hat{h}_{-\alpha}(\sigma^{k_1}(a),\sigma^{j-1}(a))<0$, $1\leq j\leq k_1$, $\hat{h}_{-\alpha}(\sigma^{k_2}(a),\sigma^{j-1}(a))>0$, $k_1+1\leq j\leq k_2$, and $\nu(\sigma^j(a))>0$, $1\leq j\leq k_2$.

On the other hand, we have

$$({}_a^C\nabla^\alpha_{(q,h)}x)(t_2)-({}_a^C\nabla^\alpha_{(q,h)}x)(t_1)=-\gamma(x(t_2))+\gamma(x(t_1))\leq 0,$$

which is a contradiction. Hence, we have $(\nabla_{(q,h)}x)(t) < 0$ for $t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$. The proof is complete. □

Lemma 3.3 *Assume $x(a) > 0$. Then the solution of equation (3.3) is positive on $\tilde{\mathbb{T}}^{a}_{(q,h)}$.*

Proof According to Lemma 3.2, we can see that $(\nabla_{(q,h)}x)(t) < 0$ leads to

$$({}_a^C\nabla^{\alpha}_{(q,h)}x)(t) < 0, \quad \alpha \in (0,1], \quad t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}.$$

Hence, by equation (3.3) and the monotonicity of the function γ, we have $x(t) > 0$ for $t \in \tilde{\mathbb{T}}^{a}_{(q,h)}$. The proof is complete. □

Lemma 3.4 *Assume $x(t)$ is a solution of equation (3.3), and $x(a) > 0$. Then the solution of equation (3.3) has a limit and*

$$\lim_{t\to\infty} x(t) = 0, \quad t \in \tilde{\mathbb{T}}^{a}_{(q,h)}.$$

Proof From Lemmas 3.2 and 3.3, we can see the limit exists. Arguing by contradiction, we assume $\lim_{t\to\infty} x(t) = c > 0$ for $t \in \tilde{\mathbb{T}}^{a}_{(q,h)}$. For $\alpha \in (0,1]$, taking the operator ${}_a\nabla^{-\alpha}_{(q,h)}$ on both side of equation (3.3), and using (2.5), we have

$$\begin{aligned}
x(t) - x(a) &= -({}_a\nabla^{-\alpha}_{(q,h)}\gamma)(x(t)) \\
&= -\int_a^t \hat{h}_{\alpha-1}(t,\rho(\tau))\gamma(x(\tau))\nabla\tau \\
&\xlongequal[k\geq 1]{t=\sigma^k(a)} -\sum_{j=1}^{k} \hat{h}_{\alpha-1}(\sigma^k(a),\sigma^{j-1}(a))\gamma(x(\sigma^j(a)))\nu(\sigma^j(a)) \\
&\leq -\gamma(x(\sigma^k(a)))\sum_{j=1}^{k} \hat{h}_{\alpha-1}(\sigma^k(a),\sigma^{j-1}(a))\nu(\sigma^j(a)) \\
&= -\gamma(x(\sigma^k(a)))\hat{h}_{\alpha}(\sigma^k(a),a),
\end{aligned}$$

where we used $\hat{h}_{\alpha-1}(\sigma^k(a),\sigma^{j-1}(a)) > 0$, $\nu(\sigma^j(a)) > 0$, $1 \leq j \leq k$. Due to the fact that

$$\lim_{t\to\infty} (x(t) - x(a)) = c - x(a) < 0,$$

while

$$\lim_{k\to\infty} -\gamma(x(\sigma^k(a)))\hat{h}_{\alpha}(\sigma^k(a),a) = -\infty,$$

because of the fact that

$$\begin{aligned}
\lim_{k\to\infty}\gamma(x(\sigma^k(a)))\hat{h}_\alpha(\sigma^k(a),a) &= \gamma(c)\lim_{k\to\infty}\frac{([\sigma^k(a)]-[a])_{\tilde{q}}^{(\alpha)}}{\Gamma_{\tilde{q}}(\alpha+1)}\\
&= \gamma(c)\lim_{k\to\infty}\frac{[\sigma^k(a)]^\alpha(\tilde{q}^k,\tilde{q})_\infty}{\Gamma_{\tilde{q}}(\alpha+1)(\tilde{q}^{k+\alpha},\tilde{q})_\infty}\\
&= \gamma(c)\lim_{k\to\infty}\frac{\tilde{q}^{-k\alpha}(a+\frac{h\tilde{q}}{1-\tilde{q}})^\alpha}{\Gamma_{\tilde{q}}(\alpha+1)}\frac{\prod_{i=0}^\infty(1-\tilde{q}^{k+i})}{\prod_{i=0}^\infty(1-\tilde{q}^{k+\alpha+i})}\\
&= \gamma(c)\lim_{k\to\infty}\frac{\tilde{q}^{-k\alpha}(a+\frac{h\tilde{q}}{1-\tilde{q}})^\alpha}{\Gamma_{\tilde{q}}(\alpha+1)}\frac{(1-\tilde{q}^\alpha)\cdots(1-\tilde{q}^{k+\alpha-1})}{(1-\tilde{q})\cdots(1-\tilde{q}^{k-1})}\frac{\prod_{i=0}^\infty(1-\tilde{q}^{1+i})}{\prod_{i=0}^\infty(1-\tilde{q}^{\alpha+i})}\\
&= \gamma(c)\lim_{k\to\infty}\frac{\tilde{q}^{-k\alpha}(a+\frac{h\tilde{q}}{1-\tilde{q}})^\alpha}{\Gamma_{\tilde{q}}(\alpha+1)}\frac{(1-\tilde{q}^\alpha)\cdots(1-\tilde{q}^{k+\alpha-1})}{(1-\tilde{q})\cdots(1-\tilde{q}^{k-1})}\frac{\Gamma_{\tilde{q}}(\alpha)}{(1-\tilde{q})^{1-\alpha}}\\
&= \infty,
\end{aligned}$$

where we used $\lim_{k\to\infty}\tilde{q}^{-k\alpha}=\infty$, and

$$\lim_{k\to\infty}\frac{(1-\tilde{q}^\alpha)\cdots(1-\tilde{q}^{k+\alpha-1})}{(1-\tilde{q})\cdots(1-\tilde{q}^{k-1})}=\frac{\prod_{i=0}^\infty(1-\tilde{q}^{\alpha+i})}{\prod_{i=0}^\infty(1-\tilde{q}^{1+i})}=\frac{(\tilde{q}^\alpha,\tilde{q})}{(\tilde{q},\tilde{q})}=\frac{(1-\tilde{q})^{1-\alpha}}{\Gamma_{\tilde{q}}(\alpha)}.$$

This yields a contradiction. Hence, we have

$$\lim_{t\to\infty}x(t)=0,\quad t\in\tilde{\mathbb{T}}_{(q,h)}^a.$$

The proof is complete. □

Lemma 3.5 *Assume* $x(t)$, $y(t)$ *satisfy*

$$({}_a^C\nabla_{(q,h)}^\alpha x)(t)\le-\gamma(x(t)),\quad t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)},$$

and

$$({}_a^C\nabla_{(q,h)}^\alpha y)(t)\ge-\gamma(y(t)),\quad t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}.$$

If $x(a)\le y(a)$, *then* $x(t)\le y(t)$ *for* $t\in\tilde{\mathbb{T}}_{(q,h)}^a$.

Proof We assume that there exists a first point t_1 such that $x(t_1)>y(t_1)$, and $x(t)\le y(t)$ on $[a,\rho(t_1)]\cap\tilde{\mathbb{T}}_{(q,h)}^a$, $t_1\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$. For $\alpha=1$, we have

$$({}_a^C\nabla_{(q,h)}^\alpha x)(t_1)-({}_a^C\nabla_{(q,h)}^\alpha y)(t_1)=(\nabla_{(q,h)}x)(t_1)-(\nabla_{(q,h)}y)(t_1)>0.$$

For $\alpha\in(0,1)$, using Definition 2.3, we have

$$\begin{aligned}
&({}_a^C\nabla^{\alpha}_{(q,h)}x)(t_1) - ({}_a^C\nabla^{\alpha}_{(q,h)}y)(t_1) \\
&= \int_a^{t_1} \hat{h}_{-\alpha}(t_1,\rho(\tau))\nabla_{(q,h)}(x(\tau)-y(\tau))\nabla\tau \\
&= \hat{h}_{-\alpha}(t_1,\tau)(x(\tau)-y(\tau))|_{\tau=a}^{t_1} + \int_a^{t_1} \hat{h}_{-\alpha-1}(t_1,\rho(\tau))(x(\tau)-y(\tau))\nabla\tau \\
&\xlongequal[k_1\geq 1]{t_1=\sigma^{k_1}(a)} -\hat{h}_{-\alpha}(\sigma^{k_1}(a),a)(x(a)-y(a)) \\
&\quad +\hat{h}_{-\alpha-1}(\sigma^{k_1}(a),\sigma^{k_1-1}(a))(x(\sigma^{k_1}(a))-y(\sigma^{k_1}(a)))\nu(\sigma^{k_1}(a)) \\
&\quad +\sum_{j=1}^{k_1-1}\hat{h}_{-\alpha-1}(\sigma^{k_1}(a),\sigma^{j-1}(a))(x(\sigma^{j}(a))-y(\sigma^{j}(a)))\nu(\sigma^{j}(a)) > 0,
\end{aligned}$$

where $\hat{h}_{-\alpha}(\sigma^{k_1}(a),a) > 0$, $\hat{h}_{-\alpha-1}(\sigma^{k_1}(a),\sigma^{k_1-1}(a)) > 0$, $\hat{h}_{-\alpha-1}(\sigma^{k_1}(a),\sigma^{j-1}(a)) < 0$, $1 \leq j \leq k_1-1$, and $\nu(\sigma^j(a)) > 0$, $1 \leq j \leq k_1$.

On the other hand, we have

$$({}_a^C\nabla^{\alpha}_{(q,h)}x)(t_1) - ({}_a^C\nabla^{\alpha}_{(q,h)}y)(t_1) \leq -\gamma(x(t_1)) + \gamma(y(t_1)) < 0,$$

which is a contradiction. Hence, we have $x(t) \leq y(t)$ for $t \in \tilde{\mathbb{T}}^a_{(q,h)}$. The proof is complete.

□

Theorem 3.1 *Assume $x=0$ is an equilibrium point of equation (3.1). If there exists a positive definite and decrescent scalar function $V(t,x)$, and class-$\mathcal{K}$ functions γ_1, γ_2, and γ_3 such that*

$$\gamma_1(\|x(t)\|) \leq V(t,x(t)) \leq \gamma_2(\|x(t)\|), \quad t \in \tilde{\mathbb{T}}^a_{(q,h)}, \tag{3.4}$$

and

$$({}_a^C\nabla^{\alpha}_{(q,h)}V)(t,x(t)) \leq -\gamma_3(\|x(t)\|), \quad t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}. \tag{3.5}$$

Then equation (3.1) is asymptotically stable.

Proof From the inequalities (3.4), (3.5), we have

$$({}_a^C\nabla^{\alpha}_{(q,h)}V)(t,x(t)) \leq -\gamma_3(\gamma_2^{-1}(V(t,x(t)))), \quad t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}.$$

Consider the fractional difference equation

$$({}_a^C\nabla^{\alpha}_{(q,h)}U)(t,x(t)) = -\gamma_3(\gamma_2^{-1}(U(t,x(t)))), \quad t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)},$$

when $V(a,x(a)) \leq U(a,x(a))$. By Lemma 3.5, we have $V(t,x(t)) \leq U(t,x(t))$, $t \in \tilde{\mathbb{T}}^a_{(q,h)}$. According to Lemma 3.2, we obtain $U(t,x(t)) \leq U(a,x(a))$, $t \in \tilde{\mathbb{T}}^a_{(q,h)}$. Using (3.4), we get $\|x(t)\| \leq \gamma_1^{-1}(V(t,x(t)))$. Hence, we have $\|x(t)\| \leq \gamma_1^{-1}(U(a,x(a)))$. Then, it follows from the definition of stability that equation (3.1) is stable. Furthermore, from Lemma 3.4, we have $\lim_{t\to\infty} V(t,x(t)) = 0$. Since $\gamma_1 \in \mathcal{K}$, and the fact that

$\gamma_1(\|x(t)\|) \leq V(t, x(t))$, we have $\lim_{t\to\infty} x(t) = 0$. Hence, equation (3.1) is asymptotically stable. The proof is complete. □

In what follows, we will present results concerning the Riemann–Liouville nabla (q,h)-fractional difference, which are important to prove the stability of equation (3.2).

Lemma 3.6 *Assume that* $({}_a\nabla^{\alpha}_{(q,h)}x)(t) \geq ({}_a\nabla^{\alpha}_{(q,h)}y)(t)$, $t \in \tilde{\mathbb{T}}^{\sigma^2(a)}_{(q,h)}$, $x(\sigma(a)) \geq y(\sigma(a))$, *and* $\alpha \in (0,1]$. *Then we have* $x(t) \geq y(t)$ *for* $t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$.

Proof Let $F(t) := x(t) - y(t)$. For $\alpha = 1$, we have

$$({}_a\nabla^{\alpha}_{(q,h)}F)(t) = (\nabla_{(q,h)}F)(t) \geq 0,$$

it is easy to see $x(t) \geq y(t)$ for $t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$.

For $\alpha \in (0,1)$, since $({}_a\nabla^{\alpha}_{(q,h)}x)(t) \geq ({}_a\nabla^{\alpha}_{(q,h)}y)(t)$, we have

$$({}_a\nabla^{\alpha}_{(q,h)}F)(t) \geq 0,$$

which can be written as

$$\int_a^t \hat{h}_{-\alpha-1}(t, \rho(\tau))F(\tau)\nabla\tau \geq 0.$$

Letting $t = \sigma^k(a)$, $k \geq 2$, we have

$$\sum_{j=1}^{k} \hat{h}_{-\alpha-1}(\sigma^k(a), \sigma^{j-1}(a))F(\sigma^j(a))\nu(\sigma^j(a)) \geq 0.$$

Since $\hat{h}_{-\alpha-1}(\sigma^k(a), \sigma^{k-1}(a)) > 0$, $\hat{h}_{-\alpha-1}(\sigma^k(a), \sigma^{j-1}(a)) < 0$, $1 \leq j \leq k-1$, $\nu(\sigma^j(a)) > 0$, $1 \leq j \leq k$, and $x(\sigma(a)) \geq y(\sigma(a))$ are true. When $k = 2$, we have $x(\sigma^2(a)) \geq y(\sigma^2(a))$. Suppose $F(\sigma^j(a)) \geq 0$, $1 \leq j \leq k-1$, by strong induction, we obtain $F(t) \geq 0$, that is, $x(t) \geq y(t)$ for $t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$. The proof is complete. □

Consider the following fractional difference equation

$$({}_a\nabla^{\alpha}_{(q,h)}x)(t) = -\gamma(x(t)),\ x(\sigma(a)) = x_0, \quad \alpha \in (0,1], \quad t \in \tilde{\mathbb{T}}^{\sigma^2(a)}_{(q,h)}, \tag{3.6}$$

where $\gamma \in \mathcal{K}$ and $x(t)$ is a positive definite and decrescent function. We can easily show this equation has a unique solution.

Lemma 3.7 *Assume* $x(\sigma(a)) > 0$. *Then the solution of equation* (3.6) *is positive on* $\tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$.

Proof In order to show $x(t) > 0$ for $t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$. Arguing by contradiction, we assume that there exists a first point $t_1 = \sigma^{k_1}(a)$, $k_1 \geq 2$ such that $x(t_1) \leq 0$, and $x(t) > 0$ on $[a, \rho(t_1)] \cap \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$. For $\alpha = 1$, and $t = t_1$, the equation (3.6) can be written as

$$(\nabla_{(q,h)}x)(t_1) = -\gamma(x(t_1)), \tag{3.7}$$

we can see easily the L.H.S. of equation (3.7) is negative, while the R.H.S. of equation (3.7) is nonnegative, which is a contradiction.

For $\alpha \in (0,1)$, and $t = t_1$, the equation (3.6) can be written as

$$\int_a^{t_1} \hat{h}_{-\alpha-1}(t_1, \rho(\tau))x(\tau)\nabla\tau = -\gamma(x(t_1)). \tag{3.8}$$

Taking $t_1 = \sigma^{k_1}(a)$, $k_1 \geq 2$ in (3.8), we have

$$\sum_{j=1}^{k_1} \hat{h}_{-\alpha-1}(\sigma^{k_1}(a), \sigma^{j-1}(a))x(\sigma^j(a))\nu(\sigma^j(a)) = -\gamma(x(\sigma^{k_1}(a))), \tag{3.9}$$

that is,

$$\begin{aligned} &\sum_{j=1}^{k_1-1} \hat{h}_{-\alpha-1}(\sigma^{k_1}(a), \sigma^{j-1}(a))x(\sigma^j(a))\nu(\sigma^j(a)) \\ &\quad = -\hat{h}_{-\alpha-1}(\sigma^{k_1}(a), \sigma^{k_1-1}(a))x(\sigma^{k_1}(a))\nu(\sigma^{k_1}(a)) - \gamma(x(\sigma^{k_1}(a))). \end{aligned} \tag{3.10}$$

Since $\hat{h}_{-\alpha-1}(\sigma^{k_1}(a), \sigma^{j-1}(a)) < 0$, $1 \leq j \leq k_1 - 1$, $\hat{h}_{-\alpha-1}(\sigma^{k_1}(a), \sigma^{k_1-1}(a)) > 0$, and $\nu(\sigma^j(a)) > 0$, $1 \leq j \leq k_1$, we can obtain the L.H.S. of equation (3.10) is negative, while the R.H.S. of equation (3.10) is nonnegative, which is a contradiction. Thus, we conclude $x(t) > 0$ for $t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$. The proof is complete. □

Lemma 3.8 *Assume $x(t)$ is a solution of equation (3.6), and $x(\sigma(a)) > 0$. Then the solution of equation (3.6) has a limit and*

$$\lim_{t\to\infty} x(t) = 0, \quad t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}.$$

Proof For $\alpha = 1$, the equation (3.6) can be written as

$$(\nabla_{(q,h)}x)(t) = -\gamma(x(t)),$$

so, by taking $t = \sigma^k(a)$, we obtain

$$\begin{aligned} x(\sigma^k(a)) - x(\sigma(a)) &= -\nu(\sigma^k(a))\gamma(x(\sigma^k(a))) - \tilde{q}\nu(\sigma^k(a))\gamma(x(\sigma^{k-1}(a))) - \cdots \\ &\quad -\tilde{q}^{k-2}\nu(\sigma^k(a))\gamma(x(\sigma^2(a))) \\ &\leq -(1 + \tilde{q} + \cdots + \tilde{q}^{k-2})\nu(\sigma^k(a))\gamma(x(\sigma^k(a))) \\ &= -\frac{1 - \tilde{q}^{k-1}}{1 - \tilde{q}}[aq^{k-1}(q-1) + q^{k-1}h]\gamma(x(\sigma^k(a))) \\ &= -\left[aq(q^{k-1} - 1) + qh\frac{q^{k-1} - 1}{q - 1}\right]\gamma(x(\sigma^k(a))) \\ &= -\left(a + \frac{h}{q-1}\right)q(q^{k-1} - 1)\gamma(x(\sigma^k(a))). \end{aligned}$$

Due to the fact that $x(t)$ is positive and decreasing, so $\lim_{t\to\infty} x(t)$ exists. Assume $\lim_{t\to\infty} x(t) = c > 0$ for $t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma^2(a)}$, we have

$$\lim_{k\to\infty}(x(\sigma^k(a)) - x(\sigma(a))) = c - x(\sigma(a)) < 0,$$

while

$$\lim_{k\to\infty}\Big[-\Big(a+\frac{h}{q-1}\Big)q(q^{k-1}-1)\gamma(x(\sigma^k(a)))\Big] = -\infty.$$

This yields a contradiction. So, we have

$$\lim_{t\to\infty} x(t) = 0, \quad t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}.$$

For $\alpha \in (0,1)$, applying the operator ${}_{\sigma(a)}\nabla_{(q,h)}^{-\alpha}$ to both sides of equation (3.6), we obtain

$$({}_{\sigma(a)}\nabla_{(q,h)}^{-\alpha}({}_a\nabla_{(q,h)}^{\alpha}x))(t) = -({}_{\sigma(a)}\nabla_{(q,h)}^{-\alpha}\gamma)(x(t)).$$

Using (2.6), we get

$$x(t) - x(\sigma(a))\hat{h}_{\alpha-1}(t,a)[\sigma(a)]^{1-\alpha}(1-\tilde{q})^{1-\alpha} = -{}_{\sigma(a)}\nabla_{(q,h)}^{-\alpha}\gamma(x(t)).$$

Since $\hat{h}_{\alpha-1}(\sigma^k(a),\sigma^{j-1}(a)) > 0$, $\nu(\sigma^j(a)) > 0$, $1 \le j \le k$, we obtain

$$\begin{aligned} x(t) &= x(\sigma(a))\hat{h}_{\alpha-1}(t,a)[\sigma(a)]^{1-\alpha}(1-\tilde{q})^{1-\alpha} \\ &\quad - \int_a^t \hat{h}_{\alpha-1}(t,\rho(\tau))\gamma(x(\tau))\nabla\tau \\ &\xlongequal[k\ge 1]{t=\sigma^k(a)} x(\sigma(a))\hat{h}_{\alpha-1}(\sigma^k(a),a)[\sigma(a)]^{1-\alpha}(1-\tilde{q})^{1-\alpha} \\ &\quad - \sum_{j=1}^{k}\hat{h}_{\alpha-1}(\sigma^k(a),\sigma^{j-1}(a))\gamma(x(\sigma^j(a)))\nu(\sigma^j(a)) \\ &< x(\sigma(a))\hat{h}_{\alpha-1}(\sigma^k(a),a)[\sigma(a)]^{1-\alpha}(1-\tilde{q})^{1-\alpha}. \end{aligned}$$

Due to the fact that

$$\begin{aligned}
\lim_{k\to\infty}\hat{h}_{\alpha-1}(\sigma^k(a),a) &= \lim_{k\to\infty}\frac{([\sigma^k(a)]-[a])_{\tilde{q}}^{(\alpha-1)}}{\Gamma_{\tilde{q}}(\alpha)}\\
&= \lim_{k\to\infty}\frac{[\sigma^k(a)]^{\alpha-1}(\tilde{q}^k,\tilde{q})_\infty}{\Gamma_{\tilde{q}}(\alpha)(\tilde{q}^{k+\alpha-1},\tilde{q})_\infty}\\
&= \lim_{k\to\infty}\frac{\tilde{q}^{k(1-\alpha)}(a+\frac{h\tilde{q}}{1-\tilde{q}})^{\alpha-1}}{\Gamma_{\tilde{q}}(\alpha)}\frac{\prod_{i=0}^{\infty}(1-\tilde{q}^{k+i})}{\prod_{i=0}^{\infty}(1-\tilde{q}^{k+\alpha-1+i})}\\
&= \lim_{k\to\infty}\frac{\tilde{q}^{k(1-\alpha)}(a+\frac{h\tilde{q}}{1-\tilde{q}})^{\alpha-1}}{\Gamma_{\tilde{q}}(\alpha)}\frac{(1-\tilde{q}^{\alpha})\cdots(1-\tilde{q}^{k+\alpha-2})}{(1-\tilde{q})\cdots(1-\tilde{q}^{k-1})}\frac{\prod_{i=0}^{\infty}(1-\tilde{q}^{1+i})}{\prod_{i=0}^{\infty}(1-\tilde{q}^{\alpha+i})}\\
&= \lim_{k\to\infty}\frac{\tilde{q}^{k(1-\alpha)}(a+\frac{h\tilde{q}}{1-\tilde{q}})^{\alpha-1}}{\Gamma_{\tilde{q}}(\alpha)}\frac{(1-\tilde{q}^{\alpha})\cdots(1-\tilde{q}^{k+\alpha-2})}{(1-\tilde{q})\cdots(1-\tilde{q}^{k-1})}\frac{\Gamma_{\tilde{q}}(\alpha)}{(1-\tilde{q})^{1-\alpha}}\\
&= 0,
\end{aligned}$$

where we used $\lim_{k\to\infty}\tilde{q}^{k(1-\alpha)}=0$, and

$$\lim_{k\to\infty}\frac{(1-\tilde{q}^{\alpha})\cdots(1-\tilde{q}^{k+\alpha-2})}{(1-\tilde{q})\cdots(1-\tilde{q}^{k-1})} = \frac{\prod_{i=0}^{\infty}(1-\tilde{q}^{\alpha+i})}{\prod_{i=0}^{\infty}(1-\tilde{q}^{1+i})} = \frac{(\tilde{q}^{\alpha},\tilde{q})}{(\tilde{q},\tilde{q})} = \frac{(1-\tilde{q})^{1-\alpha}}{\Gamma_{\tilde{q}}(\alpha)}.$$

Thus, we conclude

$$\lim_{t\to\infty}x(t)=0,\quad t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}.$$

The proof is complete. □

Lemma 3.9 *Assume $x(t)$, $y(t)$ satisfy*

$$({}_a\nabla_{(q,h)}^{\alpha}x)(t)\le-\gamma(x(t)),\quad t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma^2(a)},$$

and

$$({}_a\nabla_{(q,h)}^{\alpha}y)(t)\ge-\gamma(y(t)),\quad t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma^2(a)}.$$

If $x(\sigma(a))\le y(\sigma(a))$, then $x(t)\le y(t)$ for $t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$.

Proof The proof is similar to Lemma 3.5, and so we omit the details. □

Theorem 3.2 *Assume $x=0$ is an equilibrium point of equation (3.2). Assume there exists a positive definite and decrescent scalar function $V(t,x)$, and class-$\mathcal{K}$ functions γ_1, γ_2, and γ_3 such that*

$$\gamma_1(\|x(t)\|)\le V(t,x(t))\le\gamma_2(\|x(t)\|),\quad t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)},\tag{3.11}$$

and

$$({}_a\nabla_{(q,h)}^{\alpha}V)(t,x(t))\le-\gamma_3(\|x(t)\|),\quad t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma^2(a)}.\tag{3.12}$$

Then equation (3.2) is asymptotically stable.

Proof From the inequalities (3.11), (3.12), we have

$$({}_a\nabla^{\alpha}_{(q,h)}V)(t,x(t)) \leq -\gamma_3(\gamma_2^{-1}(V(t,x(t)))), \quad t \in \tilde{\mathbb{T}}^{\sigma^2(a)}_{(q,h)}.$$

Consider the fractional difference equation

$$({}_a\nabla^{\alpha}_{(q,h)}U)(t,x(t)) = -\gamma_3(\gamma_2^{-1}(U(t,x(t)))), \quad t \in \tilde{\mathbb{T}}^{\sigma^2(a)}_{(q,h)},$$

when $V(\sigma(a),x(\sigma(a))) \leq U(\sigma(a),x(\sigma(a)))$. By Lemma 3.9, we have $V(t,x(t)) \leq U(t,x(t))$, $t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$. From the proof of Lemma 3.8, we obtain $U(t,x(t)) \leq U(\sigma(a),x(\sigma(a)))\hat{h}_{\alpha-1}(\sigma^k(a),a)[\sigma(a)]^{1-\alpha}(1-\tilde{q})^{1-\alpha} \leq U(\sigma(a),x(\sigma(a)))\hat{h}_{\alpha-1}(\sigma(a),a)[\sigma(a)]^{1-\alpha}(1-\tilde{q})^{1-\alpha}$, $t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$. Using (3.11), we get $\|x(t)\| \leq \gamma_1^{-1}\left(U\left(\sigma\left(a\right),x\left(\sigma\left(a\right)\right)\right)\hat{h}_{\alpha-1}\left(\sigma\left(a\right),a\right)\left[\sigma\left(a\right)\right]^{1-\alpha}\left(1-\tilde{q}\right)^{1-\alpha}\right)$. Then, according to the definition of stability, we conclude that equation (3.2) is stable. Furthermore, from Lemma 3.8, we have $\lim_{t\to\infty} V(t,x(t)) = 0$. Since $\gamma_1 \in \mathcal{K}$, and the fact that $\gamma_1(\|x(t)\|) \leq V(t,x(t))$, we have $\lim_{t\to\infty} x(t) = 0$. So, equation (3.2) is asymptotically stable. The proof is complete. □

4. Stability analysis of fractional difference equations

In this section, we will introduce some relevant results for the nabla (q,h)-fractional difference equations. Initially, we will present some new lemmas, which will subsequently allow us to extend the Lyapunov type results for the nabla (q,h)-fractional difference equations. Then, the sufficient conditions for stability of the nabla (q,h)-fractional difference equations are presented.

Lemma 4.1 *(See [6, Theorem 2.2]). Assume* a, $b \geq 0$, *and* p, $q > 1$ *are such that* $\frac{1}{p}+\frac{1}{q}=1$. *Then the following inequality holds*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q, \tag{4.1}$$

where equality holds if and only if $a^p = b^q$.

Lemma 4.2 *Assume* $\alpha \in (0,1]$, $x \in \mathbb{R}$, $t \in \tilde{\mathbb{T}}^{a}_{(q,h)}$, *and* $\beta = \frac{m}{n} \geq 1$, *where* $m \in \{2k, k \in \mathbb{N}_1\}$ *and* $n \in \mathbb{N}_1$. *Then the following inequality holds*

$$({}^C_a\nabla^{\alpha}_{(q,h)}x^{\beta})(t) \leq \beta x^{\beta-1}(t)({}^C_a\nabla^{\alpha}_{(q,h)}x)(t), \quad t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}. \tag{4.2}$$

Proof For $\beta = 1$, the inequality (4.2) is clearly true. For $\beta > 1$, we need to equivalently prove

$$\beta x^{\beta-1}(t)({}^C_a\nabla^{\alpha}_{(q,h)}x)(t) - ({}^C_a\nabla^{\alpha}_{(q,h)}x^{\beta})(t) \geq 0. \tag{4.3}$$

For $\alpha = 1$, we have

$$\begin{aligned}
&\beta x^{\beta-1}(t)(\nabla_{(q,h)}x)(t) - (\nabla_{(q,h)}x^{\beta})(t)\\
&\quad= \beta x^{\beta-1}(t)\frac{x(t)-x(\rho(t))}{\nu(t)} - \frac{x^{\beta}(t)-x^{\beta}(\rho(t))}{\nu(t)}\\
&\quad= \frac{(\beta-1)x^{\beta}(t) - \beta x^{\beta-1}(t)x(\rho(t)) + x^{\beta}(\rho(t))}{\nu(t)}\\
&\quad\geq 0,
\end{aligned}$$

where we used the following inequality

$$\begin{aligned}
x^{\beta-1}(t)x(\tau) &\leq |x^{\beta-1}(t)|\cdot|x(\tau)|\\
&\overset{(4.1)}{\leq} \frac{\beta-1}{\beta}|x^{\beta-1}(t)|^{\frac{\beta}{\beta-1}} + \frac{1}{\beta}|x(\tau)|^{\beta}\\
&= \frac{\beta-1}{\beta}x^{\beta}(t) + \frac{1}{\beta}x^{\beta}(\tau), \quad t,\tau \in \tilde{\mathbb{T}}^{a}_{(q,h)}.
\end{aligned} \tag{4.4}$$

For $\alpha \in (0,1)$, using the integration by parts formula (2.7), we have

$$\begin{aligned}
&\beta x^{\beta-1}(t)({}^{C}_{a}\nabla^{\alpha}_{(q,h)}x)(t) - ({}^{C}_{a}\nabla^{\alpha}_{(q,h)}x^{\beta})(t)\\
&\quad= \beta x^{\beta-1}(t)\int_a^t \hat{h}_{-\alpha}(t,\rho(\tau))(\nabla_{(q,h)}x)(\tau)\nabla\tau - \int_a^t \hat{h}_{-\alpha}(t,\rho(\tau))(\nabla_{(q,h)}x^{\beta})(\tau)\nabla\tau\\
&\quad= \beta x^{\beta-1}(t)\Big[\hat{h}_{-\alpha}(t,\tau)x(\tau)|_{\tau=a}^{t} + \int_a^t \hat{h}_{-\alpha-1}(t,\rho(\tau))x(\tau)\nabla\tau\Big]\\
&\qquad-\Big[\hat{h}_{-\alpha}(t,\tau)x^{\beta}(\tau)|_{\tau=a}^{t} + \int_a^t \hat{h}_{-\alpha-1}(t,\rho(\tau))x^{\beta}(\tau)\nabla\tau\Big]\\
&\quad\overset{t=\sigma^k(a)}{\underset{k\geq 1}{=\!=\!=}} -\beta\hat{h}_{-\alpha}(\sigma^k(a),a)x^{\mu-1}(\sigma^k(a))x(a) + \hat{h}_{-\alpha}(\sigma^k(a),a)x^{\beta}(a)\\
&\qquad+(\beta-1)\hat{h}_{-\alpha-1}(\sigma^k(a),\sigma^{k-1}(a))x^{\beta}(\sigma^k(a))\nu(\sigma^k(a))\\
&\qquad+\sum_{j=1}^{k-1}\hat{h}_{-\alpha-1}(\sigma^k(a),\sigma^{j-1}(a))[\beta x^{\beta-1}(\sigma^k(a))x(\sigma^j(a)) - x^{\beta}(\sigma^j(a))]\nu(\sigma^j(a))\\
&\quad\overset{(4.4)}{\geq} -\beta\hat{h}_{-\alpha}(\sigma^k(a),a)x^{\beta-1}(\sigma^k(a))x(a) + \hat{h}_{-\alpha}(\sigma^k(a),a)x^{\beta}(a)\\
&\qquad+(\beta-1)x^{\beta}(\sigma^k(a))\sum_{j=1}^{k}\hat{h}_{-\alpha-1}(\sigma^k(a),\sigma^{j-1}(a))\nu(\sigma^j(a))\\
&\quad= \hat{h}_{-\alpha}(\sigma^k(a),a)(-\mu x^{\mu-1}(\sigma^k(a))x(a) + x^{\beta}(a))\\
&\qquad+(\beta-1)\hat{h}_{-\alpha}(\sigma^k(a),a)x^{\beta}(\sigma^k(a))\\
&\quad\overset{(4.4)}{\geq} 0,
\end{aligned}$$

where $\hat{h}_{-\alpha}(\sigma^k(a),a) > 0$, $\hat{h}_{-\alpha-1}(\sigma^k(a),\sigma^{j-1}(a)) < 0$, $1 \leq j \leq k-1$, and $\nu(\sigma^j(a)) > 0$, $1 \leq j \leq k$. The proof is complete. □

Corollary 4.1 *Assume* $\alpha \in (0,1]$, $x(t) \geq 0$, $t \in \tilde{\mathbb{T}}^a_{(q,h)}$, *and* $n \in \{2k+1, k \in \mathbb{N}_1\}$. *Then the following inequality holds*

$$({}^C_a\nabla^\alpha_{(q,h)} x^n)(t) \leq n x^{n-1}(t)({}^C_a\nabla^\alpha_{(q,h)} x)(t), \quad t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}. \tag{4.5}$$

Corollary 4.2 *Assume* $\alpha \in (0,1]$, *and* $m \in \mathbb{N}_1$. *Then the following inequality holds*

$$({}^C_a\nabla^\alpha_{(q,h)} x^{2^m})(t) \leq 2^m x^{(2^m-1)}(t)({}^C_a\nabla^\alpha_{(q,h)} x)(t), \quad t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}. \tag{4.6}$$

Theorem 4.1 *Assume* $x = 0$ *is an equilibrium point of equation* (3.1). *Then, for* $\beta = \frac{m}{n} \geq 1$, *where* $m \in \{2k, k \in \mathbb{N}_1\}$ *and* $n \in \mathbb{N}_1$, *if the following condition is satisfied*

$$x^{\beta-1}(t) f(t,x(t)) \leq 0, \quad t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)},$$

then equation (3.1) *is stable. Also, if*

$$x^{\beta-1}(t) f(t,x(t)) < 0, \quad t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}, \ \forall\, x \neq 0,$$

then equation (3.1) *is asymptotically stable.*

Proof Let us consider the following Lyapunov function, which is positive definite:

$$V(t) = \frac{x^\beta(t)}{\beta}.$$

Using Lemma 4.2 gives us

$$({}^C_a\nabla^\alpha_{(q,h)} V)(t) \leq x^{\beta-1}(t)({}^C_a\nabla^\alpha_{(q,h)} x)(t) = x^{\beta-1}(t) f(t,x(t)) \leq 0.$$

Hence, by Lemma 3.1, we have

$$V(t,x(t)) \leq V(a,x(a)), \quad t \in \tilde{\mathbb{T}}^a_{(q,h)},$$

that is,

$$\frac{x^\beta(t)}{\beta} \leq \frac{x^\beta(a)}{\beta}.$$

According to the definition of stability in the sense of Lyapunov, we obtain equation (3.1) is stable in the sense of Lyapunov.

If

$$x^{\beta-1}(t) f(t,x(t)) < 0, \quad t \in \tilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}, \ \forall\, x \neq 0,$$

similar to the above step, we can show equation (3.1) is stable. Then, according to Lemma 4.2, we have $({}^C_a\nabla^\alpha_{(q,h)} V)(t) \leq x^{\beta-1}(t)({}^C_a\nabla^\alpha_{(q,h)} x)(t) < 0$, that is, the fractional order (q,h)-difference of V is negative definite. According to Theorem 3.1 and the relationship between positive definite functions and class-$\mathcal{K}$ functions in [18]. We obtain that equation (3.1) is asymptotically stable. The proof is complete. □

Lemma 4.3 *Assume* $\alpha \in (0,1]$, $x \in \mathbb{R}$, $t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$, *and* $\beta = \frac{m}{n} \geq 1$, *where* $m \in \{2k, k \in \mathbb{N}_1\}$ *and* $n \in \mathbb{N}_1$. *Then the following inequality holds*

$$({}_a\nabla_{(q,h)}^{\alpha} x^{\beta})(t) \leq \beta x^{\beta-1}(t)({}_a\nabla_{(q,h)}^{\alpha} x)(t), \quad t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}. \tag{4.7}$$

Proof For $\beta = 1$, the inequality (4.7) is clearly true. For $\beta > 1$, we need to equivalently prove

$$\beta x^{\beta-1}(t)({}_a\nabla_{(q,h)}^{\alpha} x)(t) - ({}_a\nabla_{(q,h)}^{\alpha} x^{\beta})(t) \geq 0. \tag{4.8}$$

For $\alpha = 1$, the proof of this result is similar to the proof of Lemma 4.2. For $\alpha \in (0,1)$, using Lemma 2.1, we have

$$\begin{aligned}
&\beta x^{\beta-1}(t)({}_a\nabla_{(q,h)}^{\alpha} x)(t) - ({}_a\nabla_{(q,h)}^{\alpha} x^{\beta})(t)\\
&= \beta x^{\beta-1}(t)\int_a^t \hat{h}_{-\alpha-1}(t,\rho(\tau))x(\tau)\nabla\tau - \int_a^t \hat{h}_{-\alpha-1}(t,\rho(\tau))x^{\beta}(\tau)\nabla\tau\\
&\underset{k\geq 1}{\overset{t=\sigma^k(a)}{=\!=\!=}} (\beta-1)\hat{h}_{-\nu-1}(\sigma^k(a),\sigma^{k-1}(a))x^{\beta}(\sigma^k(a))\nu(\sigma^k(a))\\
&\quad + \sum_{j=1}^{k-1}\hat{h}_{-\alpha-1}(\sigma^k(a),\sigma^{j-1}(a))\big[\beta x^{\beta-1}(\sigma^k(a))x(\sigma^j(a)) - x^{\beta}(\sigma^j(a))\big]\nu(\sigma^j(a))\\
&\overset{(4.4)}{\geq} (\beta-1)x^{\beta}(\sigma^k(a))\sum_{j=1}^{k}\hat{h}_{-\alpha-1}(\sigma^k(a),\sigma^{j-1}(a))\nu(\sigma^j(a))\\
&= (\beta-1)x^{\beta}(\sigma^k(a))\hat{h}_{-\alpha}(\sigma^k(a),a)\\
&\geq 0,
\end{aligned}$$

where $\hat{h}_{-\alpha}(\sigma^k(a),a) > 0$, $\hat{h}_{-\alpha-1}(\sigma^k(a),\sigma^{j-1}(a)) < 0$, $1 \leq j \leq k-1$, and $\nu(\sigma^j(a)) > 0$, $1 \leq j \leq k$. The proof is complete. □

Corollary 4.3 *Assume* $\alpha \in (0,1]$, $x(t) \geq 0$, $t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$, *and* $n \in \{2k+1, k \in \mathbb{N}_1\}$. *Then the following inequality holds*

$$({}_a\nabla_{(q,h)}^{\alpha} x^{n})(t) \leq n x^{n-1}(t)({}_a\nabla_{(q,h)}^{\alpha} x)(t), \quad t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}. \tag{4.9}$$

Corollary 4.4 *Assume* $\alpha \in (0,1]$, *and* $m \in \mathbb{N}_1$. *Then the following inequality holds*

$$({}_a\nabla_{(q,h)}^{\alpha} x^{2^m})(t) \leq 2^m x^{(2^m-1)}(t)({}_a\nabla_{(q,h)}^{\alpha} x)(t), \quad t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}. \tag{4.10}$$

Theorem 4.2 *Assume* $x = 0$ *is an equilibrium point of equation* (3.2). *Then, for* $\beta = \frac{m}{n} \geq 1$, *where* $m \in \{2k, k \in \mathbb{N}_1\}$ *and* $n \in \mathbb{N}_1$, *if the following condition is satisfied*

$$x^{\beta-1}(t)f(t,x(t)) \leq 0, \quad t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma^2(a)},$$

then equation (3.2) is stable. Also, if

$$x^{\beta-1}(t)f(t,x(t))<0,\quad t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma^2(a)},\ \forall\ x\neq 0,$$

then equation (3.2) is asymptotically stable.

Proof Let us consider the following Lyapunov function, which is positive definite:

$$V(t)=\frac{x^{\beta}(t)}{\beta}.$$

Using Lemma 4.3 gives us

$$({}_a\nabla_{(q,h)}^{\alpha}V)(t)\leq x^{\beta-1}(t)({}_a\nabla_{(q,h)}^{\alpha}x)(t)=x^{\beta-1}(t)f(t,x(t))\leq 0.$$

By Lemma 3.6, we have

$$V(t,x(t))\leq V(\sigma(a),x(\sigma(a))),\quad t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)},$$

that is,

$$\frac{x^{\beta}(t)}{\beta}\leq\frac{x^{\beta}(\sigma(a))}{\beta}.$$

According to the definition of stability in the sense of Lyapunov, we obtain equation (3.2) is stable in the sense of Lyapunov.

If

$$x^{\beta-1}(t)f(t,x(t))<0,\quad t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma^2(a)},\ \forall\ x\neq 0,$$

similar to the above step, we can show equation (3.2) is stable. Then, using Lemma 4.3, we have $({}_a\nabla_{(q,h)}^{\alpha}V)(t)\leq x^{\beta-1}(t)({}_a\nabla_{(q,h)}^{\alpha}x)(t)<0$, that is, the fractional order (q,h)-difference of V is negative definite. Using Theorem 3.2 and the relationship between positive definite functions and class-$\mathcal{K}$ functions in [18]. We conclude that equation (3.2) is asymptotically stable. The proof is complete. □

Remark 4.1 *If $x(t)\geq 0$, then the power rules in Lemmas 4.2 and 4.3 hold for $\beta\geq 1$. In particular, the assumption $\beta=\frac{m}{n}$ ($m\in\{2k,k\in\mathbb{N}_1\}$ and $n\in\mathbb{N}_1$) is no longer required.*

5. Numerical results

Now, we give some numerical examples to illustrate the application of the results established in the previous sections.

Example 5.1 *Consider the following nabla (q,h)-fractional difference equation*

$$({}_a^C\nabla_{(q,h)}^{\alpha}x)(t)=-x^3(t),\quad x(0)=0.4,\tag{5.1}$$

where $\alpha=0.9$, $a=0$, $q=h=1$, $x\in\mathbb{R}$, $t\in\tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$, and this difference equation has the trivial solution $x(t)=0$.

We can see that

$$x^{\beta-1}(t)({}_a^C\nabla^{\alpha}_{(q,h)}x)(t) = x^{\beta-1}(t)(-x^3(t)) \\ = -x^{\frac{12}{5}}(t) \le 0$$

for $\beta = \frac{2}{5}$. *Thus, from Theorem 4.1, equation (5.1) is stable, as it can be seen from Figure 1.*

Example 5.2 *Consider the following nabla* (q,h)*-fractional difference equation*

$$({}_a\nabla^{\alpha}_{(q,h)}x)(t) = -x^3(t), \quad x(1) = 0.4, \tag{5.2}$$

where $\alpha = 0.9$, $a = 0$, $q = h = 1$, $x \in \mathbb{R}$, $t \in \tilde{\mathbb{T}}^{\sigma^2(a)}_{(q,h)}$, *and this difference equation has the trivial solution* $x(t) = 0$.

We can see that

$$x^{\beta-1}(t)({}_a\nabla^{\alpha}_{(q,h)}x)(t) = x^{\beta-1}(t)(-x^3(t)) \\ = -x^{\frac{12}{5}}(t) \le 0$$

for $\beta = \frac{2}{5}$. *Thus, from Theorem 4.2, equation (5.2) is stable, as can be seen from Figure 2.*

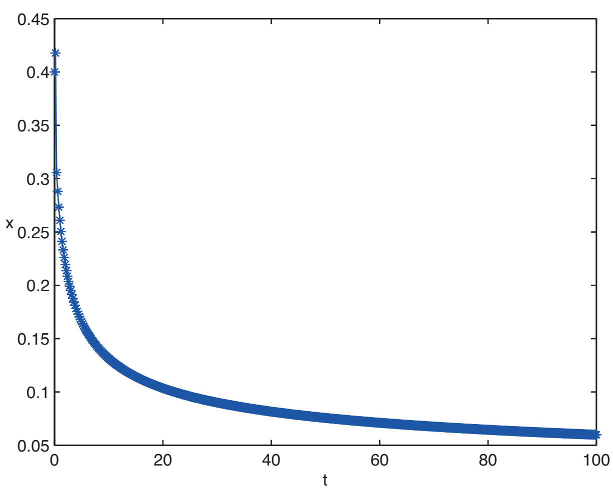

Figure 1. Stability of x for $\alpha = 0.9$.

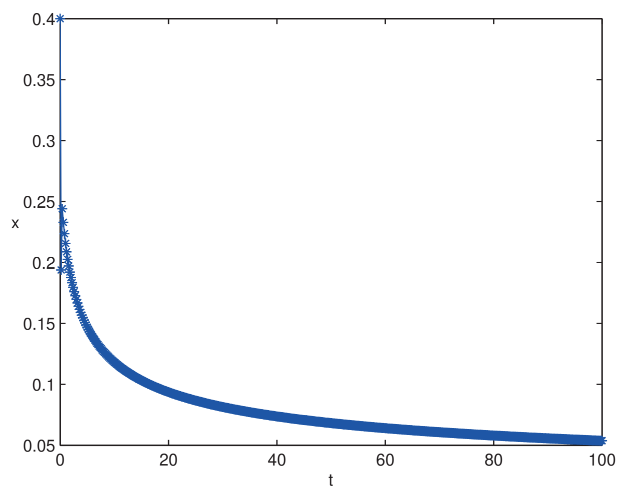

Figure 2. Stability of x for $\alpha = 0.9$.

6. Conclusion

This paper gives stability theorems for discrete fractional Lyapunov direct method for the special nabla (q,h)-fractional difference equations. Furthermore, some new lemmas are presented that allows establishing a broader family of Lyapunov functions to determine the stability of the nabla (q,h)-fractional difference equations. As a result, we give sufficient conditions for these equations to be stable or asymptotically stable. In addition, some examples are given to show the established results.

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