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n-T-torsionfree modules

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Abstract: As a generalization of the Auslander-Reiten transpose, Xi introduced and studied a more general transpose, called the relative transpose (or, T-transpose). Based on this notion, the notion of relative n-torsionfree modules (or, n-T-torsionfree modules) is introduced in this paper, which is a generalization of the n-torsionfree modules introduced by Auslander and Bridger. We show that relative n-torsionfree modules have many similar properties of n-torsionfree modules.

Key words: n-torsionfree modules, n-T-torsionfree module, n-spherical, T-grade

1. Introduction and preliminaries

It is well known that the Auslander–Reiten theory is very important for representation theory of Artin algebra and honological algebra. The transpose plays an important role in this theory. The transpose was studied by many authors. For example, let C be a semidualizing R-bimodule, a transpose Tr_C M of an R-module M with respect to C was introduced in [6]. Later, Geng [5] used Tr_C M to further develop the generalized Gorenstein dimension with respect to C in the two-sided Noetherian setting. Especially, he generalized the Auslander–Bridger formula to the generalized Gorenstein dimension case. The dual of Auslander transpose was studied in [7], and the relative transpose of an R-module was considered in [8].

Auslander and Bridger introduced n-torsionfree modules and obtained an approximation theory for finitely generated modules when n-syzygy modules and n-torsionfree modules coincide in [1]. Tang and Huang [7] introduced and studied the cotranspose of modules with respect to a semidualizing module C, and using it, they introduced n-C-cotorsionfree modules and showed that n-C-cotorsionfree modules have many dual properties of n-torsionfree modules.

Based on [8], we introduce the notion of n-T-torsionfree modules. It turns out that many important results on the n-C-torsion module are still true in this paper. We mainly prove the following two conclusions:

Theorem 1.1 *Let T be self-orthogonal (i.e. $\mathrm{Ext}_A^{i\geq 1}(T,T)=0$). Assume that M has an add(T)-resolution and $n\geq 1$. Then the following statements are equivalent:*

(1) *$\Omega_T^n(M)$ is n-T-torsionfree*

(2) *There exists an exact sequence $0\to L\to N\to M\to 0$ with N n-T-spherical and add(T)-pd $(L)\leq n-1$.*

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Theorem 1.2 *Assume that M has an $add(T)$-resolution and $n \geq 1$. Then $\Omega_T^i(M)$ is i-T-torsionfree for all $1 \leq i \leq n$ if and only if T-grade $\mathrm{Ext}^i(M,T) \geq i-1$ for all $1 \leq i \leq n$.*

We note that the above theorems extend two interesting theorems proved by Auslander–Bridger [1].

Let A be an Artin R-algebra, that is, R is a commutative Artin ring and A is an R-algebra which is finitely generated as an R-module. The category of finitely generated left A-modules will be denoted by A-mod. Throughout this paper, we assume that all modules are always finitely generated.

This paper is organized as follows. In Section 2, we introduce the definition of n-T-torsionfree mdules as a generalization of n-torsionfree modules and give some characterizations of these modules (Theorem 2.7). In particular, the proof of Theorem 1.1 (i.e. Theorem 2.9 in this section) is presented. In Section 3, we give the definition of T-grade and prove Theorem 1.2 (i.e. Theorem 3.3 in this section)

Let $\mathcal{X}$ be a subcategory of A-mod and M be a left A-module. A homomorphism $f\colon\ X \to M$ with $X \in \mathcal{X}$ is called a right $\mathcal{X}$-approximation (or, $\mathcal{X}$-precover) of M if the induced morphism $\mathrm{Hom}(X^{'}, f)$ is surjective for all $X^{'} \in \mathcal{X}$. Dually, a homomorphism $f\colon\ M \to X$ with $X \in \mathcal{X}$ is called a left $\mathcal{X}$-approximation (or, $\mathcal{X}$-preenvelope) of M if the induced morphism $\mathrm{Hom}(f, X^{'})$ is surjective for all $X^{'} \in \mathcal{X}$ in [2, 3]. An $\mathcal{X}$-resolution of M is an exact sequence

$$\cdots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

with $X_i \in \mathcal{X}$ for all $i \geq 0$. In addition, if the exact sequence is $\mathrm{Hom}(\mathcal{X}, -)$-exact, then the exact sequence is called a proper $\mathcal{X}$-resolution of M. Dually, we can define $\mathcal{X}$-coresolution and proper $\mathcal{X}$-coresolution. We say that M has $\mathcal{X}$-projective dimension $M \leq m$, denoted by $\mathcal{X}$-pd $(M) \leq m$, if there is an $\mathcal{X}$-resolution of M of the form $0 \to X_m \to \cdots \to X_1 \to X_0 \to M \to 0$. Let T be a module in A-mod. We denoted by B the endomorphism algebra of T, thus T is a A-B bimodule in the natural manner. Throughout this paper, we shall fix such a triple $(A,\ T,\ B)$. Denoted by $\mathrm{Pre}({}_AT)$ the class whose objects are those left A-modules M which posses an exact sequence of the form $T_1 \to T_0 \to M \to 0$ with T_0, $T_1 \in \mathrm{add}(T)$, here $\mathrm{add}(T)$ stands for the additive category generated by T. For simplicity, we will denote the functor $\mathrm{Hom}(-,T)$ by $(-)^*$.

2. n-T-torsionfree modules

In this section, we introduce the definition of n-T-torsionfree module and give a characterization of n-T-torsionfree modules (Theorem 2.7) and the relationship between n-T-torsionfree modules and n-T-spherical modules (Theorem 2.9). Firstly, we recall the definition of transpose [4] and relative transpose [8]

Let $P_1 \to P_0 \to M \to 0$ be a minimal projective presentation of M. Applying the functor $\mathrm{Hom}_A(-,A)$, we obtain an exact sequence of right A-modules

$$0 \longrightarrow \mathrm{Hom}_A(M,A) \longrightarrow \mathrm{Hom}_A(P_0,A) \stackrel{f}{\longrightarrow} \mathrm{Hom}_A(P_1,A) \longrightarrow C \longrightarrow 0$$

We denote the Cokernel of f by $\mathrm{Tr}M$ and call it the transpose of M, i.e. $C{=}\mathrm{Tr}M$.

Let M be a left A-module in $\mathrm{Pre}({}_AT)$. Then we have an exact sequence

$$T_1 \stackrel{f_1}{\longrightarrow} T_0 \stackrel{f_0}{\longrightarrow} M \longrightarrow 0$$

Applying $\mathrm{Hom}_A(-,T)$ to exact sequence above, we have an exact sequence in mod-B:

$$0 \longrightarrow M^* \longrightarrow T_0^* \longrightarrow T_1^* \longrightarrow \Sigma_T(M) \longrightarrow 0\ , \tag{$\natural$}$$

where $\Sigma_T(M)$ stands for the cokernel of $\mathrm{Hom}(f_1,T)$. We call the $\Sigma_T(M)$ the transpose of M with respect to T, or T-transpose of M. Note the T-transpose is a right B-module and depends on the exact sequence above.

Theorem 2.1 *[8, Theorem 3.9] If M lies in $\mathrm{Pre}({}_AT)$, then we have an exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1_B(\textstyle\sum_T(M),T) \longrightarrow M \xrightarrow{\alpha_M} M^{**} \longrightarrow \mathrm{Ext}^2_B(\textstyle\sum_T(M),T) \longrightarrow 0,$$

where α_M is the natural homomorphism, given by $m \mapsto (f \mapsto f(m))$.

We introduce the following definition of n-T-torsionfree modules.

Definition 2.2 *Let M be a finitely generated left A-module in $\mathrm{Pre}({}_AT)$. Then M is called n-T-torsionfree if $\mathrm{Ext}^i_B(\Sigma_T(M),T)=0$ for all $1 \le i \le n$. If $\mathrm{Ext}^i_B(\Sigma_T(M),T)=0$ for all $i \ge 1$, then M is ∞-T-torsionfree.*

Remark 2.3 (1) *If $T=A$, then n-T-torsionfree module coincide with n-torsionfree;*

(2) *If M is in $add({}_AT)$, then M is ∞-T-torsionfree. This is very useful for the rest of the discussion;*

(3) *If M is n-T-torsionfree, then M is m-T-torsionfree for any $m \le n$.*

The following lemma will be used frequently in this paper.

Lemma 2.4 *Let M be a finitely generated left A-module in $\mathrm{Pre}({}_AT)$. Then M is n-T-torsionfree if and only if α_M is an isomorphism and $\mathrm{Ext}^i_B(M^*,T)=0$ for all $1 \le i \le n-2$.*

Proof ($\Rightarrow$) Assume that M is n-T-torsionfree, then α_M is an isomorphism by Theorem 2.1 and Definition 2.2. By dimension shifting for the exact sequence ($\natural$), we can obtain that $\mathrm{Ext}^i_B(\Sigma_T(M),T) \cong \mathrm{Ext}^{i-2}_B(M^*,T)$ for all $i \ge 3$. And $\mathrm{Ext}^i_B(\Sigma_T(M),T)=0$ for all $1 \le i \le n$ since M is n-T-torsionfree, thus, $\mathrm{Ext}^i_B(M^*,T)=0$ for all $1 \le i \le n-2$.

($\Leftarrow$) By the assumption, we have $\mathrm{Ext}^i_B(\Sigma_T(M),T) \cong \mathrm{Ext}^{i-2}_B(M^*,T) = 0$ for all $3 \le i \le n$, but $\mathrm{Ext}^{1,2}_B(\Sigma_T(M),T)=0$ by Theorem 2.1. Thus, the proof is completed. □

Proposition 2.5 *Let the exact sequence $0 \to X \to Y \to Z \to 0$ be $\mathrm{Hom}(-,T)$-exact and Z be n-T-torsionfree. Then X is n-T-torsionfree if and only if Y is n-T-torsionfree.*

Proof Applying $\mathrm{Hom}(-,T)$ to the exact sequence $0 \to X \to Y \to Z \to 0$, we can obtain a new exact sequence $0 \to Z^* \to Y^* \to X^* \to 0$ since this exact sequence is $\mathrm{Hom}(-,T)$-exact. In a similar way, we can obtain a new exact sequence $0 \to X^{**} \to Y^{**} \to Z^{**}$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} & & \downarrow{\scriptstyle \alpha_Z} & & \\
0 & \longrightarrow & X^{**} & \longrightarrow & Y^{**} & \xrightarrow{f} & Z^{**} & \longrightarrow & \mathrm{Ext}^1_B(X^*,T)
\end{array}$$

Since Z is n-T-torsionfree, we have that α_Z is an isomorphism by Lemma 2.4. Thus α_X is an isomorphism if and only if α_Y is an isomorphism by Snake lemma. Now it is enough to prove that $\mathrm{Ext}^i_B(X^*,T)=0$ if and only if $\mathrm{Ext}^i_B(Y^*,T)=0$ for all $1\le i\le n-2$ by Lemma 2.4.

($\Rightarrow$) It follows from the long exact sequence theorem and Lemma 2.4.

($\Leftarrow$) We only prove that $\mathrm{Ext}^1_B(X^*,T)=0$ by the long exact sequence theorem and Lemma 2.4. Consider the exact sequence

$$0 \longrightarrow X^{**} \longrightarrow Y^{**} \xrightarrow{f} Z^{**} \longrightarrow \mathrm{Ext}^1_B(X^*,T) \longrightarrow \mathrm{Ext}^1_B(Y^*,T)$$

From the commutative diagram above, it is easy to verify that f is surjective, but $\mathrm{Ext}^1_B(Y^*,T)=0$ since Y is n-T-torsionfree. Thus, $\mathrm{Ext}^1_B(X^*,T)=0$. □

Lemma 2.6 *Let M be in $\mathrm{Pre}({}_AT)$, then the following conclusions hold:*

(1) *M is 1-T-torsionfree if and only if M admits an injective add$({}_AT)$-preenvelope.*

(2) *M is 2-T-torsionfree if and only if there is an exact sequence $0\to M\to T_0\to T_1$, where T_0 and T_1 are in add$({}_AT)$ and this exact sequence is $\mathrm{Hom}_A(-,{}_AT)$-exact.*

Proof (1), ($\Rightarrow$) Assume that M is 1-T-torsionfree, so α_M is an injection by Theorem 2.1. Note that there is an exact sequence $B^{(X)}\to M^*\to 0$ for some set X. By functor $\mathrm{Hom}_B(-,T_B)$, we obtain a new exact sequence $0\to M^{**}\xrightarrow{f}\mathrm{Hom}_B(B^{(X)},T_B)\cong T^X$. Note that X is a finite set. Thus, we obtain a monomorphism $f\alpha_M$: $M\to T^X\cong T^{(X)}$ since f and α_M are injective. Hence, we obtain an monomorphic add$({}_AT)$-preenvelope: $M\xrightarrow{v}T^{(Y)}$ with $Y=\mathrm{Hom}_A(M,T)$ finite set and v evaluation map.

($\Leftarrow$) Assume that M admits an injective add$({}_AT)$-preenvelope, then we have an exact sequence $0\to M\to T_0$. Consider the following commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \longrightarrow & T_0 \\ & & \downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_{T_0}} \\ 0 & \longrightarrow & M^{**} & \longrightarrow & T_0^{**} \end{array}$$

It follows from Snake lemma and α_{T_0} is an isomorphism that α_M is a monomorphism. i.e. M is 1-T-torsionfree.

(2), ($\Rightarrow$) Suppose that M is 2-T-torsionfree, then we can obtain an exact sequence $0\to M\to T_0\to C\to 0$ with $\mathrm{Hom}_A(-,T)$-exact by (1). Now we only prove that C is 1-T-torsionfree by (1) again. We consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & T_0 & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_{T_0}} & & \downarrow{\scriptstyle \alpha_C} & & \\ 0 & \longrightarrow & M^{**} & \longrightarrow & T_0^{**} & \xrightarrow{f} & C^{**} & & \end{array}$$

Since M is 2-T-torsionfree, α_M is an isomorphism by Lemma 2.4. It follows from Snake lemma that α_C is monomorphic. i.e. C is 1-T-torsionfree.

($\Leftarrow$) Set C is the cokernel of $M \to T_0$. Thus M and C are 1-T-torsionfree by (1). i.e. α_M and α_C are injective. Based on the above commutative diagram, we easily verify that α_M is surjective by the Snake lemma. Consequently, α_M is an isomorphism. i.e. M is 2-T-torsionfree. □

Theorem 2.7 *Let M be in $\mathrm{Pre}({}_AT)$ and $n \geq 1$. Then M is n-T-torsionfree if and only if there exists an exact sequence*

$$0 \longrightarrow M \longrightarrow T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_{n-1},$$

where T_i are in $add({}_AT)$ for any $0 \leq i \leq n-1$ and this exact sequence is $\mathrm{Hom}_A(-,{}_AT)$-exact.

Proof We proceed by induction on n. By Lemma 2.6, the cases $n \leq 2$ is clear. Suppose that $n \geq 3$ and that the conclusion holds for the case $n-1$.

($\Rightarrow$) There is an exact sequence $0 \to M \xrightarrow{f} T_0 \to M_1 \to 0$ with f an $\mathrm{add}(T)$-preenvelope by Lemma 2.6. Then we have a new exact sequence $0 \to M_1^* \to T_0^* \to M^* \to 0$. Note that T_0^* is a projective B-module. By dimension shifting, we have $\mathrm{Ext}_B^i(M_1^*, T) \cong \mathrm{Ext}_B^{i+1}(M^*, T)$ for all $i \geq 1$. Since M is n-T-torsionfree, α_M is an isomorphism and $\mathrm{Ext}_B^i(M^*, T) = 0$ for all $1 \leq i \leq n-2$ by Lemma 2.4. We consider the following commutative with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & T_0 & \longrightarrow & M_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_{T_0}} & & \downarrow{\scriptstyle \alpha_{M_1}} & & \\
0 & \longrightarrow & M^{**} & \longrightarrow & T_0^{**} & \longrightarrow & M_1^{**} & \longrightarrow & 0
\end{array}$$

Since α_{T_0} and α_M are isomorphisms, α_{M_1} is also an isomorphism by the Five Lemma. Note that $\mathrm{Ext}_B^i(M_1^*, T) \cong \mathrm{Ext}_B^{i+1}(M^*, T) = 0$ for all $1 \leq i \leq n-3$, so M_1 is (n-1)-T-torsionfree by Lemma 2.4. By the induction hypothesis, there exists an exact sequence, $0 \to M_1 \to T_1 \to T_2 \to \cdots \to T_{n-1}$. So combining it with the exact sequence $0 \to M \to T_0 \to M_1 \to 0$, we can get the exact sequence we desired.

($\Leftarrow$) Set M_1 to be the cokernel of $M \to T_0$. By the induction hypothesis, M_1 is (n-1)-T-torsionfree. We have $\mathrm{Ext}_B^i(M^*, T) \cong \mathrm{Ext}_B^{i-1}(M_1^*, T) = 0$ for all $2 \leq i \leq n-2$ by Lemma 2.4. Consider the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \longrightarrow & T_0 & \longrightarrow & M_1 & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle \alpha_M} & & \downarrow{\scriptstyle \alpha_{T_0}} & & \downarrow{\scriptstyle \alpha_{M_1}} & & & & \\
0 & \longrightarrow & M^{**} & \longrightarrow & T_0^{**} & \xrightarrow{f} & M_1^{**} & \longrightarrow & \mathrm{Ext}_B^1(M^*, T) & \longrightarrow & 0
\end{array}$$

Since α_{T_0} and α_{M_1} are isomorphisms, α_M is also an isomorphism by the Five Lemma. It follows from the commutative diagram above that f is surjective; thus, $\mathrm{Ext}_B^1(M^*, T) = 0$. So M is n-T-torsionfree by Lemma 2.4 again. □

Corollary 2.8 *Let M be in $\mathrm{Pre}({}_AT)$. The following statements are equivalent:*

(1) *M is 1-T-torsionfree;*

(2) *there is an exact sequence $0 \to M \to T_0 \to N \to 0$ with $T_0 \in add(T)$ and $\mathrm{Ext}_A^1(N, T) = 0$;*

(3) *there exists a monomorphic $add(T)$-preenvelope of M.*

Assume that M has an add(T)-resolution, that is, there is an exact sequence

$$\cdots \longrightarrow T_n \xrightarrow{f_n} T_{n-1} \xrightarrow{f_{n-1}} \cdots \longrightarrow T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \longrightarrow 0 \qquad (\sharp)$$

with $T_i \in \mathrm{add}(T)$ for all $i \geq 0$. $\Omega_T^i(M) = \mathrm{Im} f_i$ is called an n-th T-syzygy of M for any $i \geq 0$. In particular, put $\Omega_T^0(M) = M$. In the following part of this section, we always assume that M has an add(T)-resolution. A module M is called n-T-spherical if $\mathrm{Ext}_A^i(M, T) = 0$ for all $1 \leq i \leq n$, and M is called ∞-T-spherical if it is n-T-spherical for all $n \geq 1$.

Theorem 2.9 *If T is self-orthogonal and M has an add(T)-resolution $(\sharp)$, then the following statements are equivalent:*

(1) *$\Omega_T^n(M)$ is n-T-torsionfree*

(2) *There exists an exact sequence $0 \to L \to N \to M \to 0$ such that N is n-T-spherical and add(T)-pd $(L) \leq n-1$.*

Proof (1) $\Rightarrow$ (2) Suppose that $\Omega_T^n(M)$ is n-T-torsionfree. By Theorem 2.7, there is an exact sequence $0 \to \Omega_T^n(M) \to T^0 \to L \to 0$, where $T^0 \in add(T)$, L is $(n-1)$-T-torsionfree and $\mathrm{Ext}^1(L, T) = 0$. Consider the push-out of $\Omega_T^n(M) \to T^0$ and $\Omega_T^n(M) \to T_{n-1}$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega_T^n(M) & \longrightarrow & T^0 & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & T_{n-1} & \longrightarrow & N_0 & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega_T^{n-1}(M) & = & \Omega_T^{n-1}(M) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

When n=1, it follows from the second row in diagram above that $\mathrm{Ext}_A^1(N_0, T) = 0$ since $\mathrm{Ext}^1(L, T) = 0 = \mathrm{Ext}^1(T_{n-1}, T)$. Hence, the conclusion follows from the middle column. For the cases of $n \geq 2$. Since $\mathrm{Ext}_A^1(L, T) = 0$, the second row in diagram above is $\mathrm{Hom}(-, T)$-exact. Then N_0 is $(n-1)$-T-torsionfree by Proposition 2.5, since L is $(n-1)$-T-torsinfree. By Theorem 2.7, there is an exact sequence $0 \to N_0 \to T^1 \to V_0 \to 0$, where $T^1 \in add(T)$, V_0 is $(n-2)$-T-torsionfree and $\mathrm{Ext}_A^1(V_0, T) = 0$. Consider the push-out of $N_0 \to T^1$ and $N_0 \to \Omega_T^{n-1}(M)$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & T^0 & = & T^0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N_0 & \longrightarrow & T^1 & \xrightarrow{f} & V_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega_T^{n-1}(M) & \longrightarrow & L_0 & \longrightarrow & V_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Note that $\mathrm{Ext}^1(N_0, T) = 0$. Thus $\mathrm{Ext}^2(V_0, T) = 0$ by dimension shifting applied to the second row in the diagram above. From the second column, we have $\mathrm{add}(T)$-pd $(L_0) \leq 1$. Consider the push-out of $\Omega_T^{n-1}(M) \to L_0$ and $\Omega_T^{n-1}(M) \to T_{n-2}$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega_T^{n-1}(M) & \longrightarrow & L_0 & \longrightarrow & V_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & T_{n-2} & \longrightarrow & N_1 & \longrightarrow & V_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega_T^{n-2}(M) & = & \Omega_T^{n-2}(M) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

When $n = 2$, it is easy to show that N_1 is 2-T-spherical from the second row. Then the middle column in the diagram above is just the exact sequence we desired.

Now, we assume that $n \geq 3$. Then N_1 is $(n-2)$-T-torsionfree by Proposition 2.5, since V_0 is $(n-2)$-T-torsionfree. By Theorem 2.7, there is an exact sequence $0 \to N_1 \to T_1^{'} \to V_1 \to 0$, where $T_1^{'} \in add(T)$, V_1 is $(n-3)$-T-torsionfree and $\mathrm{Ext}_A^1(V_1, T) = 0$. Repeating the above discussion, and so on, we can obtain the exact sequence we desired.

(2) $\Rightarrow$ (1) Since $\mathrm{add}(T)$-pd $(L) \leq n-1$, we have the following exact sequence:

$$0 \longrightarrow T_{n-1}^{'} \xrightarrow{f_{n-1}^{'}} \cdots \longrightarrow T_1^{'} \xrightarrow{f_1^{'}} T_0^{'} \xrightarrow{f_0^{'}} L \longrightarrow 0.$$

with $T_i^{'} \in \mathrm{add}(T)$ for all $0 \leq i \leq n-1$. Set $\mathrm{Im} f_i^{'} = L_i$ for all $0 \leq i \leq n-1$. It is clear that $\mathrm{Ext}^i(T, L_j) = 0$ for all $i \geq 1$ and $0 \leq j \leq n-1$ since T is self-orthogonal. Consider the pull-back of $T_0 \to M$ and $N \to M$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L & = & L & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_T^1(M) & \longrightarrow & H_0 & \longrightarrow & N & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_T^1(M) & \longrightarrow & T_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since $\mathrm{Ext}^1(T, L) = 0$, we have that the second column is split in the diagram above and $H_0 \cong L \bigoplus T_0$. Hence, we can obtain a new short exact sequence $0 \to L_1 \to T_0^{'} \bigoplus T_0 \to L \bigoplus T_0 \to 0$. Consider the pull-back of $T_0^{'} \bigoplus T_0 \to L \bigoplus T_0$ and $\Omega_T^1(M) \to L \bigoplus T_0$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_1 & \longrightarrow & N_1 & \longrightarrow & \Omega_T^1(M) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_1 & \longrightarrow & T_0^{'} \bigoplus T_0 & \longrightarrow & L \bigoplus T_0 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N & = & N & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Then consider further the pull-back of $N_1 \to \Omega_T^1(M)$ and $T_1 \to \Omega_T^1(M)$:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega_T^2(M) & = & \Omega_T^2(M) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_1 & \longrightarrow & H_1 & \longrightarrow & T_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_1 & \longrightarrow & N_1 & \longrightarrow & \Omega_T^1(M) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since $\mathrm{Ext}^1(T,L_1)=0$, we have $H_1 \cong L_1 \bigoplus T_1$. Similar to the discussion above, we can get some exact sequences $0 \to N_i \to T'_{i-1} \bigoplus T_{i-1} \to N_{i-1} \to 0$ with $1 \le i \le n$ and $N_0 = N$. By dimension shifting, we have $\mathrm{Ext}^{1\le j \le n-i}(N_i,T)=0$ for all $1 \le i \le n-1$. There is an exact sequence $0 \to N^*_{i-1} \to (T'_{i-1} \bigoplus T_{i-1})^* \to N^*_i \to 0$. Next, we will prove that N_i is i-T-torsionfree.

When i=1, we consider the following natural commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_1 & \longrightarrow & T'_0 \bigoplus T_0 & \longrightarrow & N_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_{N_1}} & & \downarrow{\scriptstyle \alpha_{T'_0 \oplus T_0}} & & \downarrow{\scriptstyle \alpha_{N_0}} & & \\ 0 & \longrightarrow & N_1^{**} & \longrightarrow & (T'_0 \bigoplus T_0)^{**} & \longrightarrow & N_0^{**} & & \end{array}$$

It follows from Snake lemma that α_{N_1} is injective. i.e. N_1 is 1-T-torsionfree.

When i=2, we consider the following natural commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_2 & \longrightarrow & T'_1 \bigoplus T_1 & \longrightarrow & N_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_{N_2}} & & \downarrow{\scriptstyle \alpha_{T'_1 \oplus T_1}} & & \downarrow{\scriptstyle \alpha_{N_1}} & & \\ 0 & \longrightarrow & N_2^{**} & \longrightarrow & (T'_1 \bigoplus T_1)^{**} & \longrightarrow & N_1^{**} & & \end{array}$$

We have proved that α_{N_1} is injective, it follows from Snake lemma that α_{N_2} is an isomorphism. i.e. N_2 is 2-T-torsionfree.

For the case i=3, we consider the following natural commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & N_3 & \longrightarrow & T'_2 \bigoplus T_2 & \longrightarrow & N_2 & \longrightarrow & 0 & & \\ & & \downarrow{\scriptstyle \alpha_{N_3}} & & \downarrow{\scriptstyle \alpha_{T'_2 \oplus T_2}} & & \downarrow{\scriptstyle \alpha_{N_2}} & & & & \\ 0 & \longrightarrow & N_3^{**} & \longrightarrow & (T'_2 \bigoplus T_2)^{**} & \longrightarrow & N_2^{**} & \longrightarrow & \mathrm{Ext}^1_B(N_3^*,T) & \longrightarrow & 0 \end{array}$$

It follows from the diagram above that α_{N_3} is an isomorphism and $\mathrm{Ext}^1_B(N_3^*,T)=0$. Thus, N_3 is 3-T-torsionfree by Lemma 2.4. Iterating the argument above, we can finally get that N_n is n-T-torsionfree. It is clear to see that $\Omega^n_T(M) \cong N_n$. Thus $\Omega^n_T(M)$ is n-T-torsionfree. □

Proposition 2.10 *If T is self-orthogonal and $\Omega^n_T(M)$ is ∞-T-torsionfree for some $n \ge 1$, then there exists an exact sequence $0 \to L \to N \to M \to 0$ such that $N \in \infty$-T-torsionfree and add(T)-pd $(L) \le n-1$.*

Proof We will prove the result by induction on n.

When $n=1$, since $\Omega^1_T(M)$ is ∞-T-torsionfree, there exists an exact sequence $0 \to \Omega^1_T(M) \to T'_0 \to X_0 \to 0$, where $T'_0 \in add(T)$, X_0 is ∞-T-torsionfree and $\mathrm{Ext}^1(X_0,T)=0$ by Theorem 2.7. Consider the push-out of $\Omega^1_T(M) \to T_0$ and $\Omega^1_T(M) \to T'_0$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega_T^1(M) & \longrightarrow & T_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \Vert & & \\
0 & \longrightarrow & T_0^{'} & \longrightarrow & N & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & X_0 & = & X_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

It is easy to show that the middle row in the diagram above is just the exact sequence we desired.

Now suppose that $n \geq 2$. Since $\Omega_T^n(M) = \Omega_T^{n-1}(\Omega_T^1(M))$, we obtain an exact sequence $0 \to L^{'} \to N^{'} \to \Omega_T^1(M) \to 0$ with $N^{'}$ ∞-T-torsionfree and $\mathrm{add}(T)$-pd $(L^{'}) \leq n-2$ by induction hypothesis. And there is an exact sequence $0 \to N^{'} \to T_0^{'} \to N^{''} \to 0$ with $N^{''}$ ∞-T-torsionfree and $T_0^{'} \in \mathrm{add}(T)$ and $\mathrm{Ext}^1(N^{''}, T) = 0$ by Theorem 2.7. Consider the push-out of $N^{'} \to T_0^{'}$ and $N^{'} \to \Omega_T^1(M)$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L^{'} & \longrightarrow & N^{'} & \longrightarrow & \Omega_T^1(M) & \longrightarrow & 0 \\
 & & \Vert & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L^{'} & \longrightarrow & T_0^{'} & \longrightarrow & L & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N^{''} & = & N^{''} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

It follows from the middle row in the diagram above that $\mathrm{add}(T)$-pd $(L) \leq n-1$. We consider the push-out of $\Omega_T^1(M) \to T_0$ and $\Omega_T^1(M) \to L$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega_T^1(M) & \longrightarrow & T_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L & \longrightarrow & N & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & N^{''} & = & N^{''} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $\mathrm{Ext}^1(N^{''}, T) = 0$, we have that the second column in this diagram is $\mathrm{Hom}(-, T)$-exact. From Proposition 2.5, we get that N is ∞-T-torsionfree. Thus, the middle row in the diagram above is just desired. □

3. T-grade

In this section, let M be in A-mod, we give the definition of T-grade and mainly show that $\Omega_T^i(M)$ is i-T-torsionfree for all $1 \le i \le n$ if and only if T-grade $\mathrm{Ext}^i(M, T) \ge i-1$ for any $1 \le i \le n$.

Assume that M has an add(T)-resolution,

$$\cdots \longrightarrow T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0 \qquad (\S)$$

with $T_i \in \mathrm{add}(T)$ for all $i \ge 0$. Applying $\mathrm{Hom}(-, T)$ to the exact sequence:

$$0 \longrightarrow \Omega_T^n(M) \longrightarrow T_{n-1} \longrightarrow \Omega_T^{n-1}(M) \longrightarrow 0,$$

we can obtain the following exact sequence

$$0 \longrightarrow (\Omega_T^{n-1}(M))^* \longrightarrow T_{n-1}^* \xrightarrow{f} (\Omega_T^n(M))^* \longrightarrow \mathrm{Ext}^1(\Omega_T^{n-1}(M), T) \longrightarrow 0.$$

It is easy to show that $\mathrm{Ext}^1(\Omega_T^{n-1}(M), T) \cong \mathrm{Ext}^n(M, T)$. Set $Q = \mathrm{Im} f$. We get two new exact sequences

$$0 \longrightarrow (\Omega_T^{n-1}(M))^* \longrightarrow T_{n-1}^* \longrightarrow Q \longrightarrow 0. \qquad (1)$$

and

$$0 \longrightarrow Q \longrightarrow (\Omega_T^n(M))^* \longrightarrow \mathrm{Ext}^n(M, T) \longrightarrow 0. \qquad (2)$$

Applying the functor $\mathrm{Hom}(-, T)$ to the exact sequence (1), we have the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \Omega_T^n(M) & \longrightarrow & T_{n-1} & \longrightarrow & \Omega_T^{n-1}(M) & \longrightarrow & & & 0 \\
 & & \downarrow g & & \downarrow \alpha_{T_{n-1}} & & \downarrow \alpha_{\Omega_T^{n-1}(M)} & & & & \\
0 & \longrightarrow & Q^* & \longrightarrow & T_{n-1}^{**} & \xrightarrow{h} & (\Omega_T^{n-1}(M))^{**} & \longrightarrow & \mathrm{Ext}^1(Q, T) & \longrightarrow & 0
\end{array} \qquad (3)$$

Similarly, applying the functor $\mathrm{Hom}(-,T)$ to the exact sequence (2), we have the following diagram with exact row:

$$\begin{array}{ccccccc} & & & & \Omega_T^n(M) & = & \Omega_T^n(M) \\ & & & & \downarrow{\scriptstyle \alpha_{\Omega_T^n(M)}} & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & (\mathrm{Ext}^n(M,T))^* & \longrightarrow & (\Omega_T^n(M))^{**} & \xrightarrow{i^*} & Q^* \end{array} \tag{4}$$

From the left square in diagram (3), it is easy to verify that the square in diagram (4) is commutative.

Lemma 3.1 *Assume that M has an $add(T)$-resolution $(\S)$, then the following conclusions hold:*

(1) $\Omega_T^1(M)$ *is* 1-T*-torsionfree*

(2) *For any* $n \geq 2$, $\mathrm{Coker}(\alpha_{\Omega_T^n(M)}) \cong \mathrm{Hom}_A(\mathrm{Ext}_A^n(M,T),T)$.

Proof (1) We have the following commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & \Omega_T^1(M) & \longrightarrow & T_0 \\ & & \downarrow{\scriptstyle \alpha_{\Omega_T^1(M)}} & & \downarrow{\scriptstyle \alpha_{T_0}} \\ & & (\Omega_T^1(M))^{**} & \longrightarrow & T_0^{**} \end{array}$$

It follows that $\alpha_{\Omega_T^1(M)}$ is injective, i.e. $\Omega_T^1(M)$ is 1-T-torsionfree.

(2) If $n \geq 2$, then $\alpha_{\Omega_T^{n-1}(M)}$ is injective by (1). Hence g is an isomorphism in diagram (3) since $\alpha_{T_{n-1}}$ is an isomorphism. It follows from the diagram (3.4) by the Snake lemma that $\mathrm{Coker}(\alpha_{\Omega_T^n(M)}) \cong \mathrm{Hom}_A(\mathrm{Ext}_A^n(M,T),T)$. □

Definition 3.2 *Let N be in A-mod. the T-grade of N with respect to T, denoted by T-grade N, is defined to be the integer n=inf$\{i|\mathrm{Ext}^i(N,T) \neq 0\}$, and ∞ if such integer doesn't exist.*

Theorem 3.3 *Assume that M has an $add(T)$-resolution $(\S)$ and $n \geq 1$. Then $\Omega_T^i(M)$ is i-T-torsionfree for all $1 \leq i \leq n$ if and only if T-grade $\mathrm{Ext}^i(M,T) \geq i-1$ for all $1 \leq i \leq n$.*

Proof We will prove the result by induction on n.

For the case of $n = 1$, the conclusion follows from Lemma 3.1.

Suppose that $n = 2$. Then $\Omega_T^2(M)$ is 2-T-torsionfree if and only if $\alpha_{\Omega_T^2(M)}$ is an isomorphism. By Lemma 3.1 (1), $\alpha_{\Omega_T^2(M)}$ is injective. So $\Omega_T^2(M)$ is 2-T-torsionfree if and only if $\alpha_{\Omega_T^2(M)}$ is surjective, but

$$\mathrm{Coker}(\alpha_{\Omega_T^n(M)}) \cong \mathrm{Hom}_A(\mathrm{Ext}_A^n(M,T),T)$$

by Lemma 3.1 (2). Hence, $\Omega_T^2(M)$ is 2-T-torsionfree if and only if $\mathrm{Hom}_A(\mathrm{Ext}_A^2(M,T),T) = 0$, i.e., T-grade $\mathrm{Ext}^2(M,T) \geq 1$. Now we assume that $n \geq 3$.

($\Rightarrow$) Assume that $\Omega_T^i(M)$ is i-T-torsionfree for all $1 \leq i \leq n$, we only need to prove that T-grade $\mathrm{Ext}^n(M,T) \geq n-1$. By Lemma 3.1 (2), we have that $0 = \mathrm{Coker}(\alpha_{\Omega_T^n(M)}) \cong \mathrm{Hom}_A(\mathrm{Ext}_A^n(M,T),T)$. Applying the functor $\mathrm{Hom}(-,T)$ to the exact sequence (2), we get the following new exact sequence:

$$0 \longrightarrow (\mathrm{Ext}^n(M,T))^* \longrightarrow (\Omega_T^n(M))^{**} \xrightarrow{i^*} Q^* \longrightarrow \mathrm{Ext}^1(\mathrm{Ext}^n(M,T),T) \longrightarrow \mathrm{Ext}^1((\Omega_T^n(M))^*,T).$$

By induction hypothesis, we know that $\alpha_{\Omega_T^{n-1}(M)}$ is an isomorphism. So g in diagram (3) is also an isomorphism, thus, i^* in diagram (4) is surjective. Since $\mathrm{Ext}^1((\Omega_T^n(M))^*,T) = 0$ by Lemma 2.4, we have $\mathrm{Ext}^1(\mathrm{Ext}^n(M,T),T) = 0$ from the exact sequence above. Hence, T-grade $\mathrm{Ext}^n(M,T) \geq 2$. Consider the two exact sequences (1) and (2), we have

$$0 = \mathrm{Ext}^i((\Omega_T^{n-1}(M))^*,T) \cong \mathrm{Ext}^{i+1}(Q,T)$$

for all $1 \leq i \leq n-3$ by the assumption and Lemma 2.4. By dimension shifting, we obtain that $\mathrm{Ext}^i(Q,T) \cong \mathrm{Ext}^{i+1}(\mathrm{Ext}^n(M,T),T)$ for $1 \leq i \leq n-3$, but $\mathrm{Ext}^{n-2}(Q,T) \ncong \mathrm{Ext}^{n-1}(\mathrm{Ext}^n(M,T),T)$. Hence, $\mathrm{Ext}^j(\mathrm{Ext}^n(M,T),T) = 0$ for any $3 \leq j \leq n-2$. For the case of $j=2$, by the assumption, we have that $\alpha_{\Omega_T^{n-1}(M)}$ is an isomorphism. It follows that h in diagram (3) is surjective, and $\mathrm{Ext}^1(Q,T) = 0$. So $0 = \mathrm{Ext}^1(Q,T) \cong \mathrm{Ext}^2(\mathrm{Ext}^n(M,T),T)$. Consequently, $\mathrm{Ext}^k(\mathrm{Ext}^n(M,T),T) = 0$ for all $0 \leq k \leq n-2$. i.e. T-grade $\mathrm{Ext}^n(M,T) \geq n-1$

($\Leftarrow$) Assume that the assertion holds for the case $n-1$. i.e. if T-grade $\mathrm{Ext}^i(M,T) \geq i-1$ for all $1 \leq i \leq n-1$, then $\Omega_T^i(M)$ is i-T-torsionfree for all $1 \leq i \leq n-1$. Suppose that T-grade $\mathrm{Ext}^i(M,T) \geq i-1$ for all $1 \leq i \leq n$, it suffices to show that $\Omega_T^n(M)$ is n-T-torsionfree by induction hypothesis. Note that $\alpha_{\Omega_T^{n-1}(M)}$ is an isomorphism by Lemma 2.4. It follows that g is an isomorphism and $\mathrm{Ext}^1(Q,T) = 0$ in diagram (3). Because T-grade $\mathrm{Ext}^n(M,T) \geq n-1$, i^* is an isomorphism in the diagram (4); thus, $\alpha_{\Omega_T^n(M)}$ is an isomorphism by Snake lemma.

Next, we only need to prove that $\mathrm{Ext}^i((\Omega_T^n(M))^*,T) = 0$ for all $1 \leq i \leq n-2$ by Lemma 2.4. From the exact sequence (1), we have that

$$0 = \mathrm{Ext}^i((\Omega_T^{n-1}(M))^*,T) \cong \mathrm{Ext}^{i+1}(Q,T)$$

for all $1 \leq i \leq n-3$ by the assumption and Lemma 2.4. Since T-grade $\mathrm{Ext}^n(M,T) \geq n$-1, we have that $\mathrm{Ext}^j(\mathrm{Ext}^n(M,T),T) = 0$ for any $1 \leq j \leq n$-2, and that $\mathrm{Ext}^j(Q,T) \cong \mathrm{Ext}^j((\Omega_T^n(M))^*,T)$ for $1 \leq j \leq n-3$ from the exact sequence (2). Consequently, $\mathrm{Ext}^j((\Omega_T^n(M))^*,T) = 0$ for $2 \leq j \leq n-3$. It follows from the assumption and Lemma 2.4 that $\mathrm{Ext}^{n-3}((\Omega_T^{n-1}(M))^*,T) = 0$, so we have that $0 = \mathrm{Ext}^{n-3}((\Omega_T^{n-1}(M))^*,T) \cong \mathrm{Ext}^{n-2}(Q,T)$ from the exact sequence (1). Thus, $\mathrm{Ext}^{n-2}((\Omega_T^n(M))^*,T) = 0$ from the exact sequence (2) since $\mathrm{Ext}^{n-2}(\mathrm{Ext}^n(M,T),T) = 0$. In former portion, we proved $\mathrm{Ext}^1(Q,T) = 0$, so we have that $0 = \mathrm{Ext}^1(Q,T) \cong \mathrm{Ext}^1((\Omega_T^n(M))^*,T)$ from the exact sequence (2). Thus $\mathrm{Ext}^i((\Omega_T^n(M))^*,T) = 0$ for all $1 \leq i \leq n-2$. □

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