

Quantum integral equations of Volterra type in terms of discrete-time normal martingale

Jinshu CHEN^{1*}, Yuling TANG²

¹School of Science, Lanzhou University of Technology, Gansu, P.R. China

²School of Mathematics and Statistics, Hexi University, Gansu, P.R. China

Received: 31.05.2018

Accepted/Published Online: 22.02.2019

Final Version: 29.05.2019

Abstract: In this paper, we aim to introduce a quantum linear stochastic Volterra integral equation of convolution type with operator-valued kernels in a nuclear topological algebra. We first establish the existence and uniqueness of the solutions and give the explicit expression of the solutions. Then we prove the continuity, continuous dependence on free terms and other properties of the solution.

Key words: Discrete-time normal martingale, Volterra integral equation, operator, existence and uniqueness

1. Introduction

Integral equations of Volterra type are used in the areas of physical and biological sciences [1, 2, 6, 7, 17]. In mathematics, the Volterra integral equations are a special type of integral equations. In fact, the Volterra equations are not only simple special cases of the Fredholm equations but represent a class of equations with their own specific problems. In the past three decades, much attention has been paid to the theory of Volterra equations; there has been a big development particularly in the fractional ones. The increasing interest in these equations comes from their application to problems in physics and other subjects. This paper is devoted to studying the existence and some kind of regularity of solutions to stochastic Volterra equations in a nuclear topological algebra of discrete-time normal martingales.

Hida's white noise analysis is essentially a theory of infinite dimensional calculus on generalized functionals of Brownian motion, which linked up the applications to the study of random processes and stochastic differential equations. And considering the space of slowly growing Schwartz distributions as the space of trajectories of white noise, Hida constructed a new theory of white noise functionals [10, 11, 14], which proved to be useful in the study of stochastic equations. In [4, 13] the authors studied the following integral equation of Volterra type with Hida distribution-valued kernels

$$\varphi(t) = \varphi_0(t) + \int_0^t \eta(t, s) \diamond \varphi(s) ds, \quad 0 \leq t \leq l, \quad (1.1)$$

where \diamond stands for Wick product, $\{\eta(t, s) | 0 \leq t, s \leq l\}$ and $\{\varphi_0(t) | 0 \leq t \leq l\}$ are given Hida distribution-valued processes. It was shown in [13] that many interesting stochastic differential equations can look as the special

*Correspondence: emily-chen@126.com

2010 *AMS Mathematics Subject Classification*: Primary: 60H40; Secondary: 47B38

cases of equation (1.1). In this paper, we will generalize equation (1.1) to the level of discrete-time normal martingale operators.

Discrete-time normal martingales [15], which also play an important role in many theoretical and applied fields, have attracted much attention in recent years. For example, the classical random walk is just such a discrete-time normal martingale [12, 16]. Let $M = (M_n)_{n \in \mathbb{N}}$ be a discrete-time normal martingale satisfying some mild conditions. In [18], the authors constructed the testing functional space $\mathcal{S}(M)$ and generalized functional space $\mathcal{S}^*(M)$ of M by using a specific orthonormal basis for square integrable functionals of M . In [21], we introduced a transform, called 2D-Fock transform, for operators from $\mathcal{S}(M)$ to $\mathcal{S}^*(M)$, characterized the continuous linear operators from $\mathcal{S}(M)$ to $\mathcal{S}^*(M)$, applied the 2D-Fock transform, and introduced a new product, called convolution for operators from $\mathcal{S}(M)$ to $\mathcal{S}^*(M)$. Let \mathcal{L} be the set of all continuous linear operators from $\mathcal{S}(M)$ to $\mathcal{S}^*(M)$, we verified that $(\mathcal{L}, *)$ forms a commutative algebra with an involution and a unit.

Motivated by what is mentioned above, we investigate the following quantum integral equation of Volterra type with operator-valued kernels

$$T(t) = J(t) + \int_0^t K(t, s) * T(s) ds, \quad 0 \leq t \leq l, \quad (1.2)$$

where $l > 0$, $*$ stands for the convolution of \mathcal{L} -valued function. $\{K(t, s) \mid 0 \leq t, s \leq l\} \subset \mathcal{L}$ and $\{J(t) \mid 0 \leq t \leq l\} \subset \mathcal{L}$ is a given \mathcal{L} -valued quantum stochastic process.

As we know, in the mathematical description of quantum physics, many observables and quantities are not bona fide operators in the Hilbert space but can only be considered as operators on a certain framework of white noise analysis. The equation (1.2) can be interpreted as a quantum stochastic differential equation for quantum stochastic processes to describe the evolution of a quantum system at a level of operators. We study the existence and some kind of regularity of solutions to equation (1.2) and give the explicit expression of the solutions. Moreover, we also prove the continuity, the continuous dependence on initial values as well as other properties of the solution.

The paper is organized as follows. In Section 2, we briefly recall the construction and characterization of continuous linear operators in \mathcal{L} . In Section 3, we establish the existence and uniqueness of solution to equation (1.2) and give the explicit expression of the solutions. In Sections 4 and 5, we prove the continuity, the continuous dependence on initial values as well as other properties of the solution.

Notation and conventions: Throughout the paper, \mathbb{N} designates the set of all nonnegative integers, Γ denotes the finite power set of \mathbb{N} , namely

$$\Gamma = \{ \sigma \mid \sigma \subset \mathbb{N} \text{ and } \#(\sigma) < \infty \}, \quad (1.3)$$

where $\#(\sigma)$ means the cardinality of σ as a set. In addition, we always assume that (Ω, \mathcal{F}, P) is a given probability space with \mathbb{E} denoting the expectation with respect to P . We denote by $L^2(\Omega, \mathcal{F}, P)$ the usual Hilbert space of square integrable complex-valued functions on (Ω, \mathcal{F}, P) and use $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ to mean its inner product and norm, respectively. By convention, $\langle \cdot, \cdot \rangle$ is conjugate-linear in its first argument and linear in its second argument.

2. Generalized functionals of discrete-time normal martingales

Definition 2.1 A sequence $M = (M_n)_{n \in \mathbb{N}}$ of random variables on (Ω, \mathcal{F}, P) is called a discrete-time normal martingale if it is square integrable and satisfies:

- (1) $\mathbb{E}[M_0 | \mathcal{F}_{-1}] = 0$ and $\mathbb{E}[M_n | \mathcal{F}_{n-1}] = M_{n-1}$ for $n \geq 1$;
- (2) $\mathbb{E}[M_0^2 | \mathcal{F}_{-1}] = 1$ and $\mathbb{E}[M_n^2 | \mathcal{F}_{n-1}] = M_{n-1}^2 + 1$ for $n \geq 1$,

where $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$, $\mathcal{F}_n = \sigma(M_k; 0 \leq k \leq n)$ for $n \in \mathbb{N}$ and $\mathbb{E}[\cdot | \mathcal{F}_k]$ means the conditional expectation.

Let $M = (M_n)_{n \in \mathbb{N}}$ be a discrete-time normal martingale on (Ω, \mathcal{F}, P) . Then one can construct from M a process $Z = (Z_n)_{n \in \mathbb{N}}$ as

$$Z_0 = M_0, \quad Z_n = M_n - M_{n-1}, \quad n \geq 1.$$

It can be verified that Z admits the following properties:

$$\mathbb{E}[Z_n | \mathcal{F}_{n-1}] = 0 \quad \text{and} \quad \mathbb{E}[Z_n^2 | \mathcal{F}_{n-1}] = 1, \quad n \in \mathbb{N}.$$

Thus, it can be viewed as a discrete-time noise, we called the discrete-time normal noise associated with M .

It is known [15] that Z has the chaotic representation property. Then the system $\{Z_\sigma \mid \sigma \in \Gamma\}$ defined by $Z_\emptyset = 1$ and

$$Z_\sigma = \prod_{i \in \sigma} Z_i, \quad \sigma \in \Gamma, \sigma \neq \emptyset \tag{2.1}$$

is actually an orthonormal basis for $L^2(\Omega, \mathcal{F}_\infty, P)$, where $\mathcal{F}_\infty = \sigma(M_n; n \in \mathbb{N})$, the σ -field over Ω generated by M , which is a closed subspace of $L^2(\Omega, \mathcal{F}, P)$ as is known.

And, for brevity, we use $L^2(M) = L^2(\Omega, \mathcal{F}_\infty, P)$, which shares the same inner product and norm with $L^2(\Omega, \mathcal{F}, P)$, namely $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$.

Lemma 2.1 [20] Let $\sigma \mapsto \lambda_\sigma$ be the \mathbb{N} -valued function on Γ given by

$$\lambda_\sigma = \begin{cases} \prod_{k \in \sigma} (k + 1), & \sigma \neq \emptyset, \sigma \in \Gamma; \\ 1, & \sigma = \emptyset, \sigma \in \Gamma. \end{cases} \tag{2.2}$$

Then, for $p > 1$, the positive term series $\sum_{\sigma \in \Gamma} \lambda_\sigma^{-p}$ converges; moreover,

$$\sum_{\sigma \in \Gamma} \lambda_\sigma^{-p} \leq \exp \left[\sum_{k=1}^{\infty} k^{-p} \right] < \infty. \tag{2.3}$$

Using the \mathbb{N} -valued function defined by (2.2), we can construct a chain of Hilbert spaces consisting of functionals of M as follows. For $p \geq 0$, we define a norm $\|\cdot\|_p$ on $L^2(M)$ through

$$\|\xi\|_p^2 = \sum_{\sigma \in \Gamma} \lambda_\sigma^{2p} |\langle Z_\sigma, \xi \rangle|^2, \quad \xi \in L^2(M) \tag{2.4}$$

and put

$$\mathcal{S}_p(M) = \{ \xi \in L^2(M) \mid \|\xi\|_p < \infty \}. \tag{2.5}$$

It is not hard to check that $\|\cdot\|_p$ is a Hilbert norm and $\mathcal{S}_p(M)$ becomes a Hilbert space with $\|\cdot\|_p$.

It is easy to see that $\lambda_\sigma \geq 1$ for all $\sigma \in \Gamma$. This implies that $\|\cdot\|_p \leq \|\cdot\|_q$ and $\mathcal{S}_q(M) \subset \mathcal{S}_p(M)$ whenever $0 \leq p \leq q$. Thus we actually get a chain of Hilbert spaces of functionals of M :

$$\cdots \subset \mathcal{S}_{p+1}(M) \subset \mathcal{S}_p(M) \subset \cdots \subset \mathcal{S}_1(M) \subset \mathcal{S}_0(M) = L^2(M). \tag{2.6}$$

We now put

$$\mathcal{S}(M) = \bigcap_{p=0}^{\infty} \mathcal{S}_p(M) \tag{2.7}$$

and endow it with the topology generated by the norm sequence $\{\|\cdot\|_p\}_{p \geq 0}$. Note that, for each $p \geq 0$, $\mathcal{S}_p(M)$ is just the completion of $\mathcal{S}(M)$ with respect to $\|\cdot\|_p$. Thus, $\mathcal{S}(M)$ is a countably Hilbert space [8].

For $p \geq 0$, we denote by $\mathcal{S}_p^*(M)$ the dual of $\mathcal{S}_p(M)$ and $\|\cdot\|_{-p}$ the norm of $\mathcal{S}_p^*(M)$. Then $\mathcal{S}_p^*(M) \subset \mathcal{S}_q^*(M)$ and $\|\cdot\|_{-p} \geq \|\cdot\|_{-q}$ whenever $0 \leq p \leq q$. The lemma below is then an immediate consequence of the general theory of countably Hilbert spaces (see [8]).

Lemma 2.2 [18] *Let $\mathcal{S}^*(M)$ be the dual of $\mathcal{S}(M)$ and endow it with the strong topology. Then*

$$\mathcal{S}^*(M) = \bigcup_{p=0}^{\infty} \mathcal{S}_p^*(M). \tag{2.8}$$

Moreover, the inductive limit topology on $\mathcal{S}^*(M)$ given by space sequence $\{\mathcal{S}_p^*(M)\}_{p \geq 0}$ coincides with the strong topology.

We mention that, by identifying $L^2(M)$ with its dual, one comes to a Gel'fand triple

$$\mathcal{S}(M) \subset L^2(M) \subset \mathcal{S}^*(M), \tag{2.9}$$

which we refer to as the Gel'fand triple associated with M .

Throughout this paper, we denote by \mathcal{L} the set of all continuous linear operators from $\mathcal{S}(M)$ to $\mathcal{S}^*(M)$, that is $\mathcal{L} = \mathcal{L}[\mathcal{S}(M), \mathcal{S}^*(M)]$.

Definition 2.2 [21] *For an operator $T \in \mathcal{L}$, its 2D-Fock transform is the function \widetilde{T} on $\Gamma \times \Gamma$ given by*

$$\widetilde{T}(\sigma, \tau) = \langle\langle T(Z_\sigma), Z_\tau \rangle\rangle, \quad (\sigma, \tau) \in \Gamma \times \Gamma, \tag{2.10}$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ is the canonical bilinear form on $\mathcal{S}^*(M) \times \mathcal{S}(M)$.

For two operators $T_1, T_2 \in \mathcal{L}$, their usual product $T_1 T_2$ may not make sense. However, one can introduce a product of other type for them.

Definition 2.3 *Let $T_1, T_2 \in \mathcal{L}$, then their convolution $T_1 * T_2$ is defined as*

$$\widetilde{T_1 * T_2}(\sigma, \tau) = \widetilde{T_1}(\sigma, \tau) \widetilde{T_2}(\sigma, \tau), \quad (\sigma, \tau) \in \Gamma \times \Gamma. \tag{2.11}$$

It can be verified that $(\mathcal{L}, *)$ forms a commutative algebra with an involution and a unit.

Lemma 2.3 [3] Let $(T_n)_{n \geq 1} \subset \mathcal{L}$ be such

- (1) $\widetilde{T}_n(\sigma, \tau) \rightarrow \widetilde{T}(\sigma, \tau)$ for all $\sigma, \tau \in \Gamma$;
- (2) there exist constants $C \geq 0$ and $p \geq 0$ such that

$$\sup_{n \geq 1} |\widetilde{T}_n(\sigma, \tau)| \leq C \lambda_\sigma^p \lambda_\tau^p, \quad (\sigma, \tau) \in \Gamma \times \Gamma.$$

Then there exists a unique $T \in \mathcal{L}$ such that sequence $(T_n)_{n \geq 1}$ converges strongly to T .

Lemma 2.4 [3] Let (E, \mathcal{E}, ν) be a measure space and $T(\cdot) : E \rightarrow \mathcal{L}$ satisfies

- (1) for any $\sigma, \tau \in \Gamma$, the function $\widetilde{T}(\cdot)(\sigma, \tau) : E \rightarrow \mathbb{C}$ is measurable;
- (2) there exist $p \geq 0$ and a nonnegative function $C(\omega) \in L^1(E, \nu)$, such that for ν -a.e., $\omega \in E$ it holds that

$$|\widetilde{T}(\omega)(\sigma, \tau)| \leq C(\omega) \lambda_\sigma^p \lambda_\tau^p \quad \sigma, \tau \in \Gamma. \tag{2.12}$$

Then the strongly Bochner integrable $\int_E T(\omega) d\nu(\omega)$ exist and

$$\int_E \widetilde{T(\omega)}(\sigma, \tau) d\nu(\omega) = \int_E \widetilde{T}(\omega)(\sigma, \tau) d\nu(\omega), \quad \sigma, \tau \in \Gamma. \tag{2.13}$$

In particular, $\int_E T(\omega) d\nu(\omega) \in \mathcal{L}$.

Lemma 2.5 [3] Let $0 \leq l < +\infty$, $T(\cdot) : [0, l] \rightarrow \mathcal{L}$ is a \mathcal{L} -valued function, then $T(\cdot)$ is continuous if and only if:

- (1) $\widetilde{T}(\cdot)(\sigma, \tau) : [0, l] \rightarrow \mathbb{C}$ is continuous for any $\sigma, \tau \in \Gamma$;
- (2) there exist $p \geq 0$ and $C \geq 0$ such that

$$|\widetilde{T}(\cdot)(\sigma, \tau)| \leq C \lambda_\sigma^p \lambda_\tau^p$$

for each $\sigma, \tau \in \Gamma$ and $t \in [0, l]$.

3. Existence and uniqueness of solution

In this section, we will establish the existence and uniqueness of solutions to equation (1.2) and also give an explicit express of the solution. Throughout this section, $l > 0$ is given.

Theorem 3.1 Let the kernel $\{K(t, s) | 0 \leq t, s \leq l\}$ and the free term $\{J(t) | 0 \leq t \leq l\}$ of equation (1.2) satisfies

- (1) for any $\sigma, \tau \in \Gamma$, the function

$$\widetilde{K}(\cdot, \cdot)(\sigma, \tau) : [0, l] \times [0, l] \rightarrow \mathbb{C}$$

is continuous, and

$$\widetilde{J(\cdot)}(\sigma, \tau) : [0, l] \rightarrow \mathbb{C}$$

is measurable;

(2) there exist constants $p \geq 0$, $M \geq 1$, and a positive function $c(t) \in L^1[0, l]$ such that

$$|\widetilde{J(t)}(\sigma, \tau)| \leq c(t)\lambda_\sigma^p \lambda_\tau^p, \quad t \in [0, l],$$

for any $\sigma, \tau \in \Gamma$ and

$$|\widetilde{K(t, s)}(\sigma, \tau)| \leq M, \quad 0 \leq s \leq t \leq l.$$

Then there exists a unique generalized \mathcal{L} -valued solution $T : [0, l] \rightarrow \mathcal{L}$ to equation (1.2). Moreover, the solution T is given by

$$T(t) = J(t) + \int_0^t \left[\sum_{n=1}^\infty K_n(t, s) \right] * J(s) ds, \quad 0 \leq t \leq l, \tag{3.1}$$

where K_n is defined by

$$\begin{aligned} K_1(t, s) &= K(t, s), \\ K_{n+1}(t, s) &= \int_s^t K(t, u) * K_n(u, s) du, \quad n \geq 1. \end{aligned} \tag{3.2}$$

where the series $\sum_{n=1}^\infty K_n(t, s)$ converges in \mathcal{L} strongly.

Proof (Existence) According to Lemma 2.4, for all $n \geq 1$ and $0 \leq s \leq t \leq l$, the Bochner integral $\int_s^t K(t, u) * K_n(u, s) du$ exists and belongs to \mathcal{L} .

Furthermore, by induction we can prove that

$$|\widetilde{K_n(t, s)}(\sigma, \tau)| \leq \frac{(t-s)^{n-1}}{(n-1)!} M^n. \tag{3.3}$$

for any $\sigma, \tau \in \Gamma$, $0 \leq s \leq t \leq l$ and $n \geq 1$.

In fact, (3.3) is obviously true for $n = 1$. If the inequality (3.3) is proven for some $n \geq 1$. Then

$$\begin{aligned} &|\widetilde{K_{n+1}(t, s)}(\sigma, \tau)| \\ &= \left| \int_s^t \widetilde{K(t, u)}(\sigma, \tau) \widetilde{K_n(u, s)}(\sigma, \tau) du \right| \\ &\leq \int_s^t M \frac{(u-s)^{n-1}}{(n-1)!} M^n du \\ &= \frac{t-s}{n!} M^{n+1}. \end{aligned}$$

Now we consider the series $\sum_{n=1}^\infty K_n(t, s)$. By (3.3), it is easy to see that for $0 \leq s \leq t \leq l$, we have

$$\sum_{n=1}^\infty |\widetilde{K_n(t, s)}(\sigma, \tau)| \leq \sum_{n=1}^\infty \frac{(t-s)^{n-1}}{(n-1)!} M^n \leq M e^{(t-s)M}.$$

and for any $n \geq 0$, the function

$$(t, s) \rightarrow \widetilde{K}_n(t, s)(\sigma, \tau)$$

is continuous. Then the sequence

$$\left(\sum_{i=1}^n K_i(t, s)\right)_{n \geq 1} \subset \mathcal{L}$$

satisfies the conditions of Lemma 2.3, so it converges to some $A(s, t) \in \mathcal{L}$. We denoted $A(s, t) = \sum_{n=1}^{\infty} K_n(t, s)$.

Obviously, for any $n \geq 1, 0 \leq t < l$, the Bochner integral

$$\int_0^t \left[\sum_{i=1}^n K_i(t, s)\right] * J(s) ds$$

exists and belongs to \mathcal{L} . And

$$\widetilde{A}(s, t)(\sigma, \tau) = \sum_{n=1}^{\infty} \widetilde{K}_n(t, s)(\sigma, \tau)$$

for any $\sigma, \tau \in \Gamma, 0 \leq s \leq t \leq l$, which implies

$$|\widetilde{A}(s, t)(\sigma, \tau)| \leq \sum_{n=1}^{\infty} |\widetilde{K}_n(t, s)(\sigma, \tau)| \leq M e^{(t-s)M}.$$

On the other hand, by the uniform convergence of $\sum_{n=1}^{\infty} \widetilde{K}_n(t, s)(\sigma, \tau)$, the function

$$(t, s) \rightarrow \widetilde{A}_n(t, s)(\sigma, \tau)$$

is continuous. Which means that the Bochner integral $\int_0^t A(t, s) * J(s) ds$ also exists and belongs to \mathcal{L} .

Now let

$$T(t) = J(t) + \int_0^t A(t, s) * J(s) ds, \quad 0 \leq t \leq l$$

and

$$T_n(t) = J(t) + \int_0^t \left[\sum_{i=1}^n K_i(t, s)\right] * J(s) ds, \quad 0 \leq t \leq l, \quad n \geq 1.$$

For any $\sigma, \tau \in \Gamma$ and $0 \leq t \leq l$, by the dominated convergence theorem, we have

$$\widetilde{T}_n(t)(\sigma, \tau) \rightarrow \widetilde{T}(t)(\sigma, \tau).$$

On the other hand, for $0 \leq t \leq l$, we have

$$\begin{aligned}
 & J(t) + \int_0^t K(t, s) * T_n(s) ds \\
 &= J(t) + \int_0^t K(t, s) * \{J(s) + \int_0^s [\sum_{i=1}^n K_i(s, u)] * J(u) du\} ds \\
 &= J(t) + \int_0^t K(t, s) * J(s) ds + \int_0^t \int_0^s K(t, s) * [\sum_{i=1}^n K_i(s, u)] * J(u) du ds \\
 &= J(t) + \int_0^t K(t, s) * J(s) ds + \int_0^t \{ \int_0^s K(t, s) * [\sum_{i=1}^n K_i(s, u)] ds \} * J(u) du \\
 &= J(t) + \int_0^t K(t, s) * J(s) ds + \int_0^t [\sum_{i=2}^{n+1} K_i(s, u)] * J(u) du \\
 &= J(t) + \int_0^t [\sum_{i=1}^{n+1} K_i(t, s)] * J(s) ds \\
 &= T_{n+1}(t).
 \end{aligned}$$

Therefore

$$T_{n+1}(t) = J(t) + \int_0^t K(t, s) * T_n(s) ds.$$

Thus,

$$\widetilde{T}_{n+1}(t)(\sigma, \tau) = \widetilde{J}(t)(\sigma, \tau) + \int_0^t \widetilde{K}(t, s)(\sigma, \tau) \cdot \widetilde{T}_n(s)(\sigma, \tau) ds.$$

By taking limit, we have

$$\widetilde{T}(t)(\sigma, \tau) = \widetilde{J}(t)(\sigma, \tau) + \int_0^t \widetilde{K}(t, s)(\sigma, \tau) \widetilde{T}(s)(\sigma, \tau) ds,$$

which implies that

$$T(t) = J(t) + \int_0^t K(t, s) * T(s) ds.$$

This means that T is a solution of equation (1.2).

(Uniqueness) Let $S : [0, l] \rightarrow \mathcal{L}$ be another solution of (1.2). Then for any $\sigma, \tau \in \Gamma$, we have

$$\begin{aligned}
 & |\widetilde{T}(t)(\sigma, \tau) - \widetilde{S}(t)(\sigma, \tau)| \\
 &= | \int_0^t \widetilde{K}(t, s)(\sigma, \tau) [\widetilde{T}(t)(\sigma, \tau) - \widetilde{S}(t)(\sigma, \tau)] ds | \\
 &\leq M \int_0^t |\widetilde{T}(t)(\sigma, \tau) - \widetilde{S}(t)(\sigma, \tau)| ds
 \end{aligned}$$

by Gronwall lemma

$$|\widetilde{T}(t)(\sigma, \tau) - \widetilde{S}(t)(\sigma, \tau)| = 0, \quad 0 \leq t \leq l,$$

which implies that $T = S$. □

4. Regularity of solution

In this section, we prove regularity solution of equation (1.2), we first study the continuity of solutions.

Theorem 4.1 *Let the kernel $\{K(t, s)|0 \leq t, s \leq l\}$ and the free term $\{J(t)|t \in [0, l]\}$ of (1.2) be such that (1) for any $\sigma, \tau \in \Gamma$, the function*

$$\widetilde{K}(\cdot, \cdot)(\sigma, \tau) : [0, l] \times [0, l] \rightarrow \mathbb{C}$$

and

$$\widetilde{J}(\cdot)(\sigma, \tau) : [0, l] \rightarrow \mathbb{C}$$

are continuous;

(2) there exist constants $p \geq 0, M \geq 1$, and a positive function $c(t) \in L^1[0, l]$ such that

$$|\widetilde{J}(t)(\sigma, \tau)| \leq c(t)\lambda_\sigma^p \lambda_\tau^p, \quad t \in [0, l],$$

for any $\sigma, \tau \in \Gamma$ and

$$|\widetilde{K}(t, s)(\sigma, \tau)| \leq M, \quad 0 \leq s \leq t \leq l.$$

Then, the solution T is continuous as a map from $[0, l]$ to \mathcal{L} .

Proof Let $p \geq 0, M \geq 1, c > 0$, for any $\sigma, \tau \in \Gamma$, we have

$$|\widetilde{K}(t, s)(\sigma, \tau)| \leq M,$$

and

$$|\widetilde{J}(t)(\sigma, \tau)| \leq c\lambda_\sigma^p \lambda_\tau^p, \quad t \in [0, l],$$

then for any $\sigma, \tau \in \Gamma$, we have

$$\begin{aligned} & |\widetilde{T}(t)(\sigma, \tau)| \\ & \leq |\widetilde{J}(t)(\sigma, \tau)| + \int_0^t |\widetilde{K}(t, s)(\sigma, \tau)| \cdot |\widetilde{T}(s)(\sigma, \tau)| ds \\ & \leq c\lambda_\sigma^p \lambda_\tau^p + M \int_0^t |\widetilde{T}(s)(\sigma, \tau)| ds \end{aligned}$$

$0 \leq t \leq l$, by Gronwall inequality we get

$$|\widetilde{T}(t)(\sigma, \tau)| \leq ce^{Mt} \lambda_\sigma^p \lambda_\tau^p, \quad t \in [0, L].$$

On the other hand, for any $\sigma, \tau \in \Gamma$, by (3.1) we get

$$\widetilde{T}(t)(\sigma, \tau) = \widetilde{J}(t)(\sigma, \tau) + \int_0^t \left[\sum_{n=1}^\infty \widetilde{K}_n(t, s)(\sigma, \tau) \right] \cdot \widetilde{J}(s)(\sigma, \tau) ds,$$

$0 \leq t \leq l$. By induction on $n \geq 1$ we see that the function

$$(t, s) \rightarrow \widetilde{K}_n(t, s)(\sigma, \tau), \quad 0 \leq s \leq t \leq l$$

are continuous. Since the series $\sum_{n=1}^{\infty} \widetilde{K}_n(t, s)(\sigma, \tau)$ is uniformly convergent on the set $0 \leq s \leq t \leq l$, it follows that the function

$$(t, s) \rightarrow \sum_{n=1}^{\infty} \widetilde{K}_n(t, s)(\sigma, \tau)$$

is continuous. Noting that the function $\widetilde{J}(\cdot)(\sigma, \tau) : [0, l] \rightarrow \mathbb{C}$ is also continuous, by Lemma 2.5, we come to the conclusion. \square

The next theorem tells us under what conditions the solutions would take values in $\mathcal{L}[\mathcal{S}(M), \mathcal{S}(M)]$.

Theorem 4.2 *Let $\{T(t)|0 \leq t \leq l\}$ be a solution of (1.2). Suppose that the kernel $\{K(t, s)|0 \leq t, s \leq l\}$ and the free term $\{J(t)|t \in [0, l]\}$ of (1.2) satisfy the conditions: for any $p \geq 0$, there exist $q \geq 0$ and $M \geq 0$ such that*

$$|\widetilde{K}(t, s)(\sigma, \tau)| \leq M$$

and

$$|\widetilde{J}(t)(\sigma, \tau)| \leq \rho(t)\lambda_{\sigma}^{p+q}\lambda_{\tau}^{-q}$$

for any $\sigma, \tau \in \Gamma, 0 \leq t \leq l$. Then the solution $\{T(t)|0 \leq t \leq l\}$ lies in $\mathcal{L}[\mathcal{S}(M), \mathcal{S}(M)]$.

Proof The conditions imply that

$$\begin{aligned} |\widetilde{T}(t)(\sigma, \tau)| &= |\widetilde{J}(t)(\sigma, \tau) + \int_0^t \widetilde{K}(t, s)(\sigma, \tau) \cdot \widetilde{T}(s)(\sigma, \tau) ds| \\ &\leq |\widetilde{J}(t)(\sigma, \tau)| + \int_0^t |\widetilde{K}(t, s)(\sigma, \tau)| \cdot |\widetilde{T}(s)(\sigma, \tau)| ds \\ &\leq c\lambda_{\sigma}^{p+q}\lambda_{\tau}^{-p} + M \int_0^t |\widetilde{T}(t)(\sigma, \tau)| ds \end{aligned}$$

for any $\sigma, \tau \in \Gamma, 0 \leq t \leq l$. By Gronwall inequality, we get

$$|\widetilde{T}(t)(\sigma, \tau)| \leq ce^{Mt}\lambda_{\sigma}^{p+q}\lambda_{\tau}^{-p}, \quad t \in [0, l].$$

Hence, by a characterization theorem for operators in $\mathcal{L}[\mathcal{S}(M), \mathcal{S}(M)]$, we come to the conclusion. \square

5. Continuous dependence of solutions on free terms

In this section, we prove that the solution T of equation (1.2) depends continuously on the free term J .

Let

$$C([0, l], \mathcal{L}) = \{S|S : [0, l] \rightarrow \mathcal{L} \text{ is continuous}\}.$$

For $\sigma, \tau \in \Gamma$, define a seminorm $\|\cdot\|_{\sigma, \tau}$ on $C([0, l], \mathcal{L})$ as follows:

$$\|\cdot\|_{\sigma, \tau} = \sup_{0 \leq t \leq l} |\widetilde{S}(t)(\sigma, \tau)|, \quad S \in C([0, l], \mathcal{L}).$$

We equip $C([0, l], \mathcal{L})$ with the topology generated by seminorms $\{\|\cdot\|_{\sigma, \tau} \mid \sigma, \tau \in \Gamma\}$. Then $C([0, l], \mathcal{L})$ becomes a Hausdorff topological vector space. For $p \geq 0$, let $C_p([0, l], \mathcal{L}) = \{S \mid S \in C([0, l], \mathcal{L}), \text{ and there exists } \alpha, \text{ such that}$

$$\sup_{0 \leq t \leq l} |\widetilde{S}(t)(\sigma, \tau)| \leq \alpha \lambda_\sigma^p \lambda_\tau^p, \quad \forall \sigma, \tau \in \Gamma.$$

Then, $C_p([0, l], \mathcal{L})$ is a subspace of $C([0, l], \mathcal{L})$.

In the sequel, we assume that $p \geq 0$, and the kernel $K(t, s)$ of (1.2) satisfies that:

- (1) $\widetilde{K}(\cdot, \cdot)(\sigma, \tau) : [0, l] \times [0, l] \rightarrow \mathbb{C}$ is continuous for all $\sigma, \tau \in \Gamma$;
- (2) $\sup_{0 \leq t \leq l} |\widetilde{K}(t, s)(\sigma, \tau)| \leq M$, for all $\sigma, \tau \in \Gamma$.

Theorem 5.1 *Let $T \in C_p([0, l], \mathcal{L})$. Define*

$$S(t) = \int_0^t K(t, s) * T(s) ds, \quad 0 \leq t \leq l.$$

Then $S \in C_p([0, l], \mathcal{L})$.

Proof For $t \in [0, l]$, the integral $\int_0^t K(t, s) * T(s) ds$ exists. Hence, $S(t)$ is well defined. Obviously, for all $\sigma, \tau \in \Gamma$, the function $\widetilde{S}(\cdot)(\sigma, \tau)$ is continuous. Let $\alpha > 0$ be such

$$\sup_{0 \leq t \leq l} |\widetilde{T}(t)(\sigma, \tau)| \leq \alpha \lambda_\sigma^p \lambda_\tau^p, \quad \sigma, \tau \in \Gamma.$$

Then for all $\sigma, \tau \in \Gamma$, we have

$$\begin{aligned} |\widetilde{S}(t)(\sigma, \tau)| &= \left| \int_0^t \widetilde{K}(t, s)(\sigma, \tau) \widetilde{T}(s)(\sigma, \tau) ds \right| \\ &\leq M e^{Mt}, \end{aligned}$$

Therefore, $S \in C_p([0, l], \mathcal{L})$. □

Theorem 5.2 *For each $J \in C_p([0, l], \mathcal{L})$, let T_J be the unique solution of equation (1.2) corresponding to J , then $T_J \in C_p([0, l], \mathcal{L})$.*

Proof Let $J \in C_p([0, l], \mathcal{L})$. Then by Theorem 3.1 and Theorem 4.1, we know that (1.2) has a unique solution $T_J \in C([0, l], \mathcal{L})$. Now let $c > 0$ be such that

$$\sup_{0 \leq t \leq l} |\widetilde{J}(t)(\sigma, \tau)| \leq c \lambda_\sigma^p \lambda_\tau^p, \quad \forall \sigma, \tau \in \Gamma.$$

Then

$$\begin{aligned} &|\widetilde{T}(t)(\sigma, \tau)| \\ &\leq |\widetilde{J}(t)(\sigma, \tau)| + \left| \int_0^t \widetilde{K}(t, s)(\sigma, \tau) \widetilde{T}(s)(\sigma, \tau) ds \right| \\ &\leq c \lambda_\sigma^p \lambda_\tau^p + M \int_0^t |\widetilde{T}(s)(\sigma, \tau)| ds \end{aligned}$$

By Gronwall inequality,

$$|\widetilde{T}(t)(\sigma, \tau)| \leq \alpha \lambda_\sigma^p \lambda_\tau^p, \quad \forall \sigma, \tau \in \Gamma, 0 \leq t \leq l.$$

where $\alpha = ce^{Ml}$, hence $T \in C_p([0, l], \mathcal{L})$. □

Theorem 5.3 For each $J \in C_p([0, l], \mathcal{L})$, Let $T_J \in C_p([0, l], \mathcal{L})$ be the unique solution of equation (1.2) corresponding to J . Define a map $\Theta : C_p([0, l], \mathcal{L}) \rightarrow C_p([0, l], \mathcal{L})$ as,

$$\Theta(J) = T_J, \quad J \in C_p([0, l], \mathcal{L}).$$

Then Θ is a topological homeomorphism, that is, Θ is a linear bijection, and both Θ and Θ^{-1} are continuous.

Proof The uniqueness of the solution to equation (1.2) implies that Θ is linear.

In fact, $\forall \sigma, \tau \in \Gamma$ and $J \in C_p([0, l], \mathcal{L})$, we have

$$\begin{aligned} |\widetilde{T_J}(t)(\sigma, \tau)| &\leq |\widetilde{J}(t)(\sigma, \tau)| + \int_0^t |K(t, s)(\sigma, \tau)| |\widetilde{T_J}(s)(\sigma, \tau)| ds \\ &\leq \|J\|_{\sigma, \tau} + M \int_0^t |\widetilde{T_J}(t)(\sigma, \tau)| ds \end{aligned}$$

$0 \leq t \leq l$. Hence,

$$|\widetilde{T_J}(t)(\sigma, \tau)| \leq \|J(t)\|_{\sigma, \tau} e^{Mt},$$

which implies that

$$\|T_J\|_{(\sigma, \tau)} \leq e^{Ml} \|J(t)\|_{\sigma, \tau},$$

i.e.

$$\|\Theta(J)\|_{(\sigma, \tau)} \leq e^{Ml} \|J(t)\|_{\sigma, \tau}.$$

Therefore, Θ is a continuous injection.

Now we prove that Θ is a surjective. For any $T \in C_p([0, l], \mathcal{L})$, according to Theorem 5.2, we can find a $S \in C_p([0, l], \mathcal{L})$ such that

$$S(t) = \int_0^t K(t, s) * T(s) ds, \quad 0 \leq t \leq l.$$

Put $J = T - S$, then $J \in C_p([0, l], \mathcal{L})$ and

$$T(t) = J(t) + \int_0^t K(t, s) * T(s) ds, \quad 0 \leq t \leq l.$$

which means that $T = \Theta(J)$. Therefore, Θ is surjective.

So far, we have shown that Θ is a continuous linear bijection; thus, Θ^{-1} exists. It remains to prove that Θ^{-1} is continuous.

In fact, for $\forall \sigma, \tau \in \Gamma$ and $T \in C_p([0, l], \mathcal{L})$, let $J = \Theta^{-1}(T)$, then

$$J(t) = T(t) - \int_0^t K(t, s) * T(s) ds, \quad 0 \leq t \leq l.$$

Hence,

$$\begin{aligned} & |\widetilde{J}(t)(\sigma, \tau)| \\ & \leq |\widetilde{T}(t)(\sigma, \tau)| + \int_0^t |\widetilde{K}(t, s)(\sigma, \tau)| \cdot |\widetilde{T}(t)(\sigma, \tau)| ds \\ & \leq \|T\|_{\sigma, \tau} + tM\|T\|_{\sigma, \tau} \end{aligned}$$

$0 \leq t \leq l$, which implies that

$$\|J\|_{\sigma, \tau} \leq (1 + Ml)\|T\|_{\sigma, \tau}.$$

Therefore, Θ^{-1} is continuous. □

Remark If we define a map $A : C_p([0, l], \mathcal{L}) \rightarrow C_p([0, l], \mathcal{L})$ by

$$(AT)(t) = \int_0^t K(t, s) * T(s) ds, \quad 0 \leq t \leq l.$$

$T \in C_p([0, l], \mathcal{L})$, then we have $(I - A)^{-1} = \Theta$. Thus, A is a continuous linear map from $C_p([0, l], \mathcal{L})$ to itself. It is interesting to compare the map A with the classical Volterra integral operator.

Acknowledgments

This work is supported by National Natural Science Foundation of China (Grant No. 11861057, 11761047) and the Hongliu Foundation of First-class Disciplines of Lanzhou University of Technology, China.

References

- [1] Bellman R, Cooke KL. Differential-Difference Equations. New York, NY, USA: Academic, 1963.
- [2] Bellman R, Kalaba R, Lockett J. Numerical Inversion of the Laplace Transform. New York, NY, USA: American Elsevier, 1966.
- [3] Chen JS. Convergence theorems for operators sequences on functionals of discrete-time normal martingales. Journal of Function Spaces 2018; Article ID 8430975. doi: 10.1155/2018/8430975
- [4] Cochran WG, Lee JS, Potthoff J. Stochastic Volterra equation with singular kernels. Stochastic Processes and Their Applications 1995; 56: 337-349. doi: 10.1016/0304-4149(94)00072-2
- [5] Émery M. A discrete approach to the chaotic representation property. In: Séminaire de Probabilités, XXXV. Lecture Notes in Mathematics; Berlin, Germany: Springer 2001, pp. 123-138.
- [6] Friedman A. On integral equations of Volterra type. Journal d'Analyse Mathématique 1963; 11 (1): 381-413. doi: 10.1007/BF02789991
- [7] Friedman A, Shinbrot M. Volterra integral equations in Banach space. Transactions of the American Mathematical Society 1967; 126: 131-179. doi: 10.1090/S0002-9947-1967-0206754-7
- [8] Gel'fand IM, Vilenkin NY. Generalized Functions, vol. 4. Applications of Harmonic Analysis. New York, NY, USA: Academic Press, 1964.
- [9] Hida T. Analysis of Brownian Functionals. CarletonMath, Lectures 13. Ottawa, Carleton University, 1975.
- [10] Hida T, Kuo HH, Potthoff J, Streit L. White Noise: An Infinite Dimensional Calculus. Dordrecht, The Netherlands: Kluwer Academic, 1993.

- [11] Huang ZY, Yan JA. Introduction to Infinite Dimensional Stochastic Analysis. Dordrecht, The Netherlands: Kluwer Academic, 1999.
- [12] Motwani R, Raghavan P. Randomized Algorithms. New York, NY, USA: Cambridge University Press, 1995.
- [13] ksendal B, Zhang TS. The general linear stochastic Volterra equations with anticipating coefficients. In: Proceedings Gregynog Conference on Stochastic Analysis, 1996.
- [14] Obata N. An analytic characterization of symbols of operators on white noise functionals. Journal of the Mathematical Society of Japan 1993; 45: 421-445. doi: 10.2969/jmsj/04530421
- [15] Privault N. Stochastic analysis of Bernoulli processes. Probability Surveys 2008; 5: 435-483. doi: 10.1214/08-PS139
- [16] Rudnick J, Gaspari G. Elements of the Random Walk. Cambridge, UK: Cambridge University Press, 2004.
- [17] Tricomi FG. Integral Equations. New York, NY, USA: Interscience, 1957.
- [18] Wang CS, Chen JS. Characterization theorems for generalized functionals of discrete-time normal martingale. Journal of Function Spaces 2015; Article ID 714745. doi: 10.1155/2015/714745
- [19] Wang CS, Lu YC, Chai H F. An alternative approach to Privault's discrete-time chaotic calculus. Journal of Mathematical Analysis and Applications 2011; 373: 643-654. doi: 10.1016/j.jmaa.2010.08.021
- [20] Wang CS, Zhang JH. Wick analysis for Bernoulli noise functionals. Journal of Function Spaces Volume 2014; Article ID 727341. doi: 10.1155/2014/727341
- [21] Wang CS, Chen JS. A Characterization of Operators on Functionals of Discrete-Time Normal Martingales. Stochastic Analysis and Applications 2017; 35 (2): 305-316. doi: 10.1080/07362994.2016.1248779