Multiplication alteration by two-cocycles for bialgebras with weak antipode

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Abstract: In this paper we introduce the theory of multiplication alteration by two-cocycles for bialgebras with weak antipode. Moreover, by the connection between two-cocycles and invertible skew pairings, we show that a special case of the double cross product of these bialgebras can be obtained as a deformation of a bialgebra with weak antipode.

Key words: Bialgebras, weak antipode, 2-cocycle, skew pairing

1. Introduction

Let R be a commutative ring with a unit and denote the tensor product over R by $\otimes$. In [17], we can find one of the first interesting examples of multiplication alteration by a 2-cocycle for R-algebras. In this case, Sweedler proved that if U is an associative unitary R-algebra with a commutative subalgebra A and $\sigma = \sum a_i \otimes b_i \otimes c_i \in A \otimes A \otimes A$ is an Amistur 2-cocycle, then U admits a new associative an unitary product defined by $u \cdot v = \sum a_i\, ub_i\, vc_i$ for all $u, v \in U$. Moreover, if U is central separable, U with the new product is still central separable and is isomorphic to the Rosenberg–Zelinsky central separable algebra obtained from the 2-cocycle σ^{-1} (see [15]). Later, in [3], Doi discovered a new construction to modify the algebra structure of a bialgebra A over a field $\mathbb{F}$ using an invertible 2-cocycle σ in A. In this case, if $\sigma : A \otimes A \to \mathbb{F}$ is the 2-cocycle, the new product on A is defined by

$$a * b = \sum \sigma(a_1 \otimes b_1) a_2 b_2 \sigma^{-1}(a_3 \otimes b_3)$$

for $a, b \in A$. With this new algebra structure and the original coalgebra structure, A is a new bialgebra denoted by A^σ, and if A is a Hopf algebra with antipode λ_A, so is A^σ whit antipode given by

$$\lambda_{A^\sigma}(a) = \sum \sigma(a_1 \otimes \lambda_A(a_2)) \lambda_A(a_3) \sigma^{-1}(\lambda_A(a_4) \otimes a_5)$$

for $a \in A$.

A particular case of alterations of products by 2-cocycles are provided by invertible skew pairings on bialgebras. If A and H are bialgebras and $\tau : A \otimes H \to \mathbb{F}$ is an invertible skew pairing, in [4], Doi and Takeuchi defined a new biagebra $A \bowtie_\tau H$ in the following way: The morphism $\omega : A \otimes H \otimes A \otimes H \to \mathbb{F}$

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defined by $\omega((a \otimes g) \otimes (b \otimes h)) = \varepsilon_A(a)\varepsilon_H(h)\tau(b \otimes g)$, for $a, b \in A$ and $g, h \in H$, is a 2-cocycle in $A \otimes H$ and $A \bowtie_\tau H = (A \otimes H)^\omega$. The construction of $A \bowtie_\tau H$ is an example of Majid's double crossproduct $A \bowtie H$ ([11], [12]) where the left H-module structure of A, denoted by φ_A, and the right A-module structure of H, denoted by ϕ_H, are defined by

$$\varphi_A(h \otimes a) = \sum \tau(a_1 \otimes h_1)a_2\tau^{-1}(a_3 \otimes h_2), \quad \phi_H(h \otimes a) = \sum \tau(a_1 \otimes h_1)h_2\tau^{-1}(a_2 \otimes h_3)$$

for $a \in A$ and $h \in H$.

The main motivation of this paper is to introduce a theory of alteration multiplication, in the sense of [3], for bialgebras with weak antipode in monoidal categories. This kind of bialgebra was introduced by Li in [8] in order to construct some singular solutions of the quantum Yang–Baxter equation (see also [10]). It is necessary to highlight that the name chosen by Li was weak Hopf algebra, but there is another notion with the same name which is well established in the literature (see [2]), and in order to avoid confusion (they are different notions and none is contained in the other), we will use the name of bialgebra with weak antipode to refer to our structure. Anyway, we are convinced that the construction can be carried out for classical weak Hopf algebras, which will be the goal of a future work.

Throughout this paper, $\mathcal{C}$ denotes a strict symmetric monoidal category with tensor product $\otimes$, unit object K, and natural isomorphism of symmetry c. For each object M in $\mathcal{C}$, we denote the identity morphism by $id_M : M \to M$ and, for simplicity of notation, given objects M, N, and P in $\mathcal{C}$ and a morphism $f : M \to N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$. There is no loss of generality in assuming the strict character for $\mathcal{C}$ because it is well known that we can construct a strict monoidal category $\mathcal{C}^{st}$ which is tensor equivalent to $\mathcal{C}$ (see [7] for the details). As a consequence, the results proved in this paper hold for every nonstrict symmetric monoidal category.

An outline of the paper is as follows. After this introduction, in Section 2, we consider the notion of 2-cocycle and prove some properties related with bialgebras with weak antipode. The main result recovers Doi's construction on Hopf algebras [3] by showing that the deformation of a bialgebra with weak antipode by a 2-cocycle is also a bialgebra with weak antipode (Theorem 2.6). In Section 3, we introduce the notion of skew pairing for bialgebras with weak antipode (inspired in the definition of weak Hopf pair given by Li in [9]) and we prove that, as in the Hopf algebra setting, if there exists a skew pairing for two bialgebras with anti-comultiplicative weak antipode A and H, we can define a new bialgebra with anti-comultiplicative weak antipode $A \bowtie_\tau H$ such that $A \bowtie_\tau H = (A \otimes H)^\omega$ for some 2-cocycle ω induced by τ (Proposition 3.7). Finally, we explain the reasons why, unlike what happens for Hopf algebras, our construction is not applicable in order to give a description of the Drinfeld's double (Remark 3.8).

2. Product alterations by two-cocycles for bialgebras with weak antipode

In this section, we prove that, as in the Hopf algebra case (see [4]), 2-cocycles provide a way of altering the product of a bialgebra with weak antipode to produce another bialgebra with weak antipode, but firstly, and for completeness and consistency, we will remind some useful concepts.

An algebra in $\mathcal{C}$ is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in $\mathcal{C}$ and $\eta_A : K \to A$ (unit), $\mu_A : A \otimes A \to A$ (product) are morphisms in $\mathcal{C}$ such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. If A, B are algebras in $\mathcal{C}$, so is $A \otimes B$, where $\eta_{A \otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$.

A coalgebra in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in $\mathcal{C}$ and $\varepsilon_D : D \to K$ (counit), $\delta_D : D \to D \otimes D$ (coproduct) are morphisms in $\mathcal{C}$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If D, E are coalgebras in $\mathcal{C}$, so is $D \otimes E$, where $\varepsilon_{D \otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D \otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

If A is an algebra, B is a coalgebra and $f : B \to A$, $g : B \to A$ are morphisms, we define the convolution product by $f * g = \mu_A \circ (f \otimes g) \circ \delta_B$. We will say that $f : B \to A$ is convolution invertible if there exists $f^{-1} : B \to A$ such that $f * f^{-1} = \varepsilon_B \otimes \eta_A = f^{-1} * f$.

A bialgebra H is an algebra (H, η_H, μ_H) and a coalgebra $(H, \varepsilon_H, \delta_H)$ such that η_H and μ_H are morphisms of coalgebras (equivalently, ε_H and δ_H are morphisms of algebras), i.e., $\delta_H \circ \eta_H = \eta_H \circ \eta_H$ and $\delta_H \circ \mu_H = \mu_{H \otimes H} \circ (\delta_H \otimes \delta_H)$. Moreover, if there exists a morphism $T_H : H \to H$ (called weak antipode) such that

$$id_H * T * id_H = id_H, \tag{1}$$

$$T * id_H * T = T, \tag{2}$$

we will say that H is a bialgebra with weak antipode. The morphism T is not unique, and using (1), it is easy to see that $T \circ \eta_H = \eta_H$ and $\varepsilon_H \circ T = \varepsilon_H$. Moreover, T is antimultiplicative if $T \circ \mu_H = \mu_H \circ c_{H,H} \circ (T \otimes T)$, and anti-comultiplicative if $\delta_H \circ T = (T \otimes T) \circ c_{H,H} \circ \delta_H$. Finally, we will define the morphisms target and source as $\Pi_H^L = id_H * T$ and $\Pi_H^R = T * id_H$, respectively.

Examples 2.1 *The most natural examples of bialgebras with weak antipode coming by considering S a finite Clifford monoid. Then the semigroup algebra kS is a finite dimensional bialgebra with antimultiplicative and anti-comultiplicative weak antipode, and so is $(kS)^*$ (see [8]). Moreover the tensor product $kS \otimes (kS)^*$ is also a finite dimensional bialgebra with antimultiplicative and anti-comultiplicative weak antipode. Note that none of these bialgebras are Hopf algebras unless S is a group. Other more sophisticated examples are the weak quantized enveloping algebras of semisimple Lie algebras, generalized Kac–Moody algebras and superalgebras (see [1] and [6] for details).*

Two more concrete examples can be given from those obtained in [1]. In this article, the authors consider the example of finite dimensional Hopf algebra given by Sweedler in [16] and, modifying slightly the relations between the generators, they obtain two new bialgebras with weak antipode that are not Hopf algebras. It can be proven that the weak antipode of these bialgebras is antimultiplicative, which makes their dual bialgebras with anti-comultiplicative weak antipode.

Now we recall the notion of 2-cocycle.

Definition 2.2 Let H be a bialgebra, and let $\sigma : H \otimes H \to K$ be a convolution invertible morphism. We say that σ is a 2-cocycle if the equality

$$\partial^1(\sigma) * \partial^3(\sigma) = \partial^4(\sigma) * \partial^2(\sigma) \tag{3}$$

holds, where $\partial^1(\sigma) = \varepsilon_H \otimes \sigma$, $\partial^2(\sigma) = \sigma \circ (\mu_H \otimes H)$, $\partial^3(\sigma) = \sigma \circ (H \otimes \mu_H)$ and $\partial^4(\sigma) = \sigma \otimes \varepsilon_H$. Equivalently, a convolution invertible morphism $\sigma : H \otimes H \to K$ is a 2-cocycle if

$$\sigma \circ (H \otimes ((\sigma \otimes \mu_H) \circ \delta_{H \otimes H})) = \sigma \circ (((\sigma \otimes \mu_H) \circ \delta_{H \otimes H}) \otimes H). \tag{4}$$

The 2-cocycle σ is called normal if further

$$\sigma \circ (\eta_H \otimes H) = \varepsilon_H = \sigma \circ (H \otimes \eta_H). \tag{5}$$

It is not difficult to show that, if σ is a 2-cocycle, then $\chi = (\sigma^{-1} \circ (\eta_H \otimes \eta_H)) \otimes \sigma$ is a normal 2-cocycle with convolution inverse $\chi^{-1} = (\sigma \circ (\eta_H \otimes \eta_H)) \otimes \sigma^{-1}$. As a consequence, in the following, we assume all 2-cocycles are normal. Moreover, if σ is normal so is σ^{-1}, and the following equalities hold:

$$\partial^3(\sigma) * \partial^2(\sigma^{-1}) = \partial^1(\sigma^{-1}) * \partial^4(\sigma), \tag{6}$$

$$\partial^2(\sigma) * \partial^3(\sigma^{-1}) = \partial^4(\sigma^{-1}) * \partial^1(\sigma), \tag{7}$$

i.e.

$$(\sigma \otimes \sigma^{-1}) \circ (H \otimes \mu_{H \otimes H} \otimes H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (\delta_H \otimes \delta_H \otimes \delta_H) = (\sigma \otimes \sigma^{-1}) \circ (H \otimes (c_{H,H} \circ \delta_H) \otimes H), \tag{8}$$

$$((\sigma \circ (\mu_H \otimes H)) \otimes (\sigma^{-1} \circ (H \otimes \mu_H))) \circ \delta_{H \otimes H \otimes H} = (\sigma^{-1} \otimes \sigma) \circ (H \otimes \delta_H \otimes H). \tag{9}$$

Proposition 2.3 *Let H be a bialgebra with anti-comultiplicative weak antipode T and let $\sigma : H \otimes H \to K$ be a convolution invertible morphism with inverse σ^{-1}. Then, for $i = L, R$:*

(i) The following conditions are equivalent:

$$\sigma \circ (\Pi_H^i \otimes H) = \varepsilon_H \otimes \varepsilon_H. \tag{10}$$

$$\sigma^{-1} \circ (\Pi_H^i \otimes H) = \varepsilon_H \otimes \varepsilon_H. \tag{11}$$

(ii) The following conditions are equivalent:

$$\sigma \circ (H \otimes \Pi_H^i) = \varepsilon_H \otimes \varepsilon_H. \tag{12}$$

$$\sigma^{-1} \circ (H \otimes \Pi_H^i) = \varepsilon_H \otimes \varepsilon_H. \tag{13}$$

Proof We begin by showing the if part of (i). The only if part is similar and we leave the details to the reader. First of all, note that

$$\delta_H \circ \Pi_H^L = (\mu_H \otimes \Pi_H^L) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes T) \circ \delta_H. \tag{14}$$

Indeed,

$$\delta_H \circ \Pi_H^L = \mu_{H \otimes H} \circ (\delta_H \otimes (\delta_H \circ T)) \circ \delta_H \, (H \text{ bialgebra})$$

$$= \mu_{H \otimes H} \circ (\delta_H \otimes ((T \otimes T) \circ c_{H,H} \circ \delta_H)) \circ \delta_H \, (T \text{ anti-comultiplicative})$$

$$= (\mu_H \otimes \Pi_H^L) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes T) \circ \delta_H \, (\text{naturality of } c).$$

Then, if we assume that $\sigma \circ (\Pi_H^L \otimes H) = \varepsilon_H \otimes \varepsilon_H$,

$$\varepsilon_H \otimes \varepsilon_H = (\sigma^{-1} * \sigma) \circ (\Pi_H^L \otimes H)(\sigma \text{ convolution invertible})$$

$$= (\sigma^{-1} \otimes \sigma) \circ (H \otimes c_{H,H} \otimes H) \circ (((\mu_H \otimes \Pi_H^L) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes T) \circ \delta_H) \otimes \delta_H) \text{ (by (14))}$$

$$= \sigma^{-1} \circ (\Pi_H^L \otimes H) \text{ (by (10))}.$$

On the other hand, by using

$$\delta_H \circ \Pi_H^R = (\Pi_H^R \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (T \otimes \delta_H) \circ \delta_H, \tag{15}$$

we get the corresponding equality involving Π_H^R.

The proof for (ii) follows a similar pattern.

$\square$

Proposition 2.4 *Let H be a bialgebra with anti-comultiplicative weak antipode T and let $\sigma : H \otimes H \to K$ be a 2-cocycle. Define $f : H \to K$ as $f = \sigma \circ (H \otimes T) \circ \delta_H$. If the equalities*

$$\sigma \circ (\Pi_H^i \otimes H) = \varepsilon_H \otimes \varepsilon_H = \sigma \circ (H \otimes \Pi_H^i),$$

$i = L, R$ hold, then f is convolution invertible with inverse $f^{-1} = \sigma^{-1} \circ (T \otimes H) \circ \delta_H$.

Proof

Indeed,

$$f * f^{-1}$$

$$= (\sigma \otimes \sigma^{-1}) \circ (H \otimes (c_{H,H} \circ \delta_H \circ T) \otimes H) \circ (\delta_H \otimes H) \circ \delta_H \ (T \text{ anti-comultiplicative})$$

$$= (\partial^1(\sigma^{-1}) * \partial^4(\sigma)) \circ (((H \otimes T) \circ \delta_H) \otimes H) \circ \delta_H \text{ (by naturality of } c \text{ and counit properties)}$$

$$= (\partial^3(\sigma) * \partial^2(\sigma^{-1})) \circ (((H \otimes T) \circ \delta_H) \otimes H) \circ \delta_H \ (\text{by (6)})$$

$$= (\sigma \otimes \sigma^{-1}) \circ (H \otimes \mu_{H \otimes H} \otimes H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H) \circ (((\delta_H \otimes ((T \otimes T) \circ c_{H,H} \circ \delta_H)) \otimes \delta_H) \circ (\delta_H \otimes H) \circ \delta_H$$

$(T \text{ anti-comultiplicative})$

$$= ((\sigma \circ (H \otimes \Pi_H^R)) \otimes (\sigma^{-1} \circ (\Pi_H^L \otimes H))) \circ \delta_{H \otimes H} \circ \delta_H \ (\text{naturality of } c \text{ and coassociativity})$$

$$= (\varepsilon_H \otimes \varepsilon_H \otimes \varepsilon_H \otimes \varepsilon_H) \circ \delta_{H \otimes H} \circ \delta_H \ (\text{by (11) and (12)})$$

$$= \varepsilon_H \ (\text{counit properties}).$$

On the other hand,

$$f^{-1} * f$$

$$= (\partial^4(\sigma^{-1}) * \partial^1(\sigma)) \circ (T \otimes H \otimes T) \circ (\delta_H \otimes H) \circ \delta_H \ (\text{by coassociativity, naturality of } c \text{ and counit properties})$$

$$= (\partial^2(\sigma) * \partial^3(\sigma^{-1})) \circ (T \otimes H \otimes T) \circ (\delta_H \otimes H) \circ \delta_H \ (\text{by (7)})$$

$$= (\sigma \otimes \sigma^{-1}) \circ (\mu_H \otimes c_{H,H} \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H)$$

$$\circ(((T \otimes T) \circ c_{H,H} \circ \delta_H) \otimes \delta_H \otimes ((T \otimes T) \circ c_{H,H} \circ \delta_H)) \circ (\delta_H \otimes H) \circ \delta_H \quad (T \text{ anti-comultiplicative})$$

$$= (((\sigma \circ (\Pi_H^R \otimes T)) \otimes (\sigma^{-1} \circ (T \otimes \Pi_H^L))) \circ (H \otimes c_{H,H} \otimes H) \circ ((c_{H,H} \circ \delta_H) \otimes (c_{H,H} \circ \delta_H)) \circ \delta_H \quad (\text{coassociativity}$$
and naturality of c)

$$= (\varepsilon_H \otimes \varepsilon_H \otimes \varepsilon_H \otimes \varepsilon_H) \circ (H \otimes c_{H,H} \otimes H) \circ ((c_{H,H} \circ \delta_H) \otimes (c_{H,H} \circ \delta_H)) \circ \delta_H \quad (\text{by (13) and (10)})$$

$$= \varepsilon_H \quad (\text{counit properties}),$$

and the proof is complete.

$\square$

Proposition 2.5 *Let H be a bialgebra and let σ be a 2-cocycle. Define the product μ_{H^σ} as*

$$\mu_{H^\sigma} = (\sigma \otimes \mu_H \otimes \sigma^{-1}) \circ (H \otimes H \otimes \delta_{H \otimes H}) \circ \delta_{H \otimes H}.$$

Then $H^\sigma = (H, \eta_{H^\sigma} = \eta_H, \mu_{H^\sigma}, \varepsilon_{H^\sigma} = \varepsilon_H, \delta_{H^\sigma} = \delta_H)$ is a bialgebra.

Proof [3], Theorem 1.6. $\square$

The following theorem is the main result of this section. We will show that, under suitable conditions, H^σ is also a bialgebra with weak antipode.

Theorem 2.6 *Let H be a bialgebra with anti-comultiplicative weak antipode T and let σ be a 2-cocycle such that (10) and (12) hold, $i = L, R$. Then H^σ is a bialgebra with anti-comultiplicative weak antipode*

$$T_{H^\sigma} = (f \otimes T \otimes f^{-1}) \circ (H \otimes \delta_H) \circ \delta_H,$$

and the corresponding equalities of Proposition 2.3 hold for H^σ.

Proof By Proposition 2.5, H^σ is a bialgebra. Now we compute the target and source morphisms:

$$\Pi_{H^\sigma}^L$$

$$= \mu_{H^\sigma} \circ (H \otimes T_{H^\sigma}) \circ \delta_H$$

$$= \mu_{H^\sigma} \circ (H \otimes \sigma \otimes H) \circ (\delta_H \otimes (c_{H,H} \circ \delta_H \circ T) \otimes f^{-1}) \circ (H \otimes \delta_H) \circ \delta_H \quad (T \text{ anti-comultiplicative})$$

$$= (\sigma \otimes \mu_H) \circ (\delta_{H \otimes H} \otimes (\sigma^{-1} * \sigma)) \circ (\delta_{H \otimes H} \otimes f^{-1}) \circ (H \otimes T \otimes H) \circ (\delta_H \otimes H) \circ \delta_H \quad (\text{coassociativity and naturality}$$
of c)

$$= (\sigma \otimes \mu_H) \circ (\delta_{H \otimes H} \otimes f^{-1}) \circ (H \otimes T \otimes H) \circ (\delta_H \otimes H) \circ \delta_H \quad (\sigma \text{ convolution invertible})$$

$$= (\sigma \otimes \mu_H) \circ (\delta_{H \otimes H} \otimes \sigma^{-1}) \circ (H \otimes (c_{H,H} \circ \delta_H \circ T) \otimes H) \circ (\delta_H \otimes H) \circ \delta_H \quad (T \text{ anti-comultiplicative})$$

$$= ((\partial^1(\sigma^{-1}) * \partial^4(\sigma)) \otimes H) \circ (H \otimes H \otimes (c_{H,H} \circ (\mu_H \otimes H))) \circ (\delta_{H \otimes H} \otimes H) \circ (H \otimes T \otimes H) \circ (\delta_H \otimes H) \circ \delta_H$$
(coassociativity and naturality of c)

$$= ((\partial^3(\sigma) * \partial^2(\sigma^{-1})) \otimes H) \circ (H \otimes H \otimes (c_{H,H} \circ (\mu_H \otimes H))) \circ (\delta_{H \otimes H} \otimes H) \circ (H \otimes T \otimes H) \circ (\delta_H \otimes H) \circ \delta_H$$
(by (6))

$$= (((\sigma \otimes \sigma^{-1}) \circ (H \otimes \mu_{H \otimes H} \otimes H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H)$$

$$\circ (\delta_H \otimes ((T \otimes T) \circ c_{H,H} \circ \delta_H) \otimes \delta_H)) \otimes H) \circ (H \otimes H \otimes (c_{H,H} \circ (\mu_H \otimes H))) \circ (H \otimes c_{H,H} \otimes T \otimes H)$$

$$\circ (\delta_H \otimes (c_{H,H} \circ \delta_H) \otimes H) \circ (\delta_H \otimes H) \circ \delta_H \quad (T \text{ anti-comultiplicative})$$

$$= ((((\sigma \circ (H \otimes \Pi_H^R)) \otimes \sigma^{-1}) \circ (H \otimes c_{H,H} \otimes H) \circ (H \otimes \mu_H \otimes \delta_H)) \otimes H) \circ (\delta_H \otimes T \otimes c_{H,H})$$

$$\circ (H \otimes (c_{H,H} \circ (\Pi_H^L \otimes H) \circ \delta_H) \otimes H) \circ (\delta_H \otimes H) \circ \delta_H \quad (\text{coassociativity and naturality of } c)$$

$$= (\sigma^{-1} \otimes \Pi_H^L) \circ (\mu_H \otimes c_{H,H}) \circ (H \otimes (c_{H,H} \circ (H \otimes T) \circ \delta_H) \otimes H) \circ (\delta_H \otimes H) \circ \delta_H \quad (\text{by (12)}).$$

Now, using (10), we have that

$$\sigma \circ (\Pi_{H^\sigma}^L \otimes H) = (\sigma \circ (\Pi_H^L \otimes H) \circ \delta_H) \otimes \varepsilon_H = \varepsilon_H \otimes \varepsilon_H,$$

and by (11), $\sigma \circ (H \otimes \Pi_{H^\sigma}^L) = \varepsilon_H \otimes \varepsilon_H$.

In a similar way,

$$\Pi_{H^\sigma}^R = (\Pi_H^R \otimes \sigma) \circ (c_{H,H} \otimes \mu_H) \circ (H \otimes (c_{H,H} \circ (T \otimes H) \circ \delta_H) \otimes H) \circ (H \otimes \delta_H) \circ \delta_H,$$

and $\sigma \circ (\Pi_{H^\sigma}^R \otimes H) = \varepsilon_H \otimes \varepsilon_H = \sigma \circ (H \otimes \Pi_H^R)$.

Now we get that T_{H^σ} is a weak antipode for H^σ. First of all, using the equalities (14), (15), (10), and (12), it is not difficult to see that the following equalities hold:

$$\mu_{H^\sigma} \circ (\Pi_H^L \otimes H) = (\sigma \otimes \mu_H) \circ \delta_{H \otimes H} \circ (\Pi_H^L \otimes H). \tag{16}$$

$$\mu_{H^\sigma} \circ (H \otimes \Pi_H^L) = (\sigma \otimes \mu_H) \circ \delta_{H \otimes H} \circ (H \otimes \Pi_H^L). \tag{17}$$

$$\mu_{H^\sigma} \circ (\Pi_H^R \otimes H) = (\mu_H \otimes \sigma^{-1}) \circ \delta_{H \otimes H} \circ (\Pi_H^R \otimes H). \tag{18}$$

$$\mu_{H^\sigma} \circ (H \otimes \Pi_H^R) = (\mu_H \otimes \sigma^{-1}) \circ \delta_{H \otimes H} \circ (H \otimes \Pi_H^R). \tag{19}$$

Then

$$id_H * T_{H^\sigma} * id_H$$

$$= \Pi_{H^\sigma}^L * id_H$$

$$= (\sigma^{-1} \otimes (\mu_{H^\sigma} \circ (\Pi_H^L \otimes H))) \circ (H \otimes c_{H,H} \otimes H) \circ (((\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes T) \circ \delta_H) \otimes \delta_H) \circ \delta_H$$

(coassociativity and computations for $\Pi_{H^\sigma}^L$)

$$= (\sigma^{-1} \otimes ((\sigma \otimes \mu_H) \circ \delta_{H \otimes H} \circ (\Pi_H^L \otimes H))) \circ (H \otimes c_{H,H} \otimes H)$$

$$\circ (((\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes T) \circ \delta_H) \otimes \delta_H) \circ \delta_H \quad (\text{by (16)})$$

$$= ((\sigma^{-1} * \sigma) \otimes \mu_H) \circ \delta_{H \otimes H} \circ (\Pi_H^L \otimes H) \circ \delta_H \quad (\text{by (14)})$$

$$= \Pi_H^L * id_H \quad (\sigma \text{ convolution invertible})$$

$= id_H$ (by (1)).

On the other hand,

$T_{H^\sigma} * id_H * T_{H^\sigma}$

$= \mu_{H^\sigma} \circ (\mu_{H^\sigma} \otimes T_{H^\sigma}) \circ (H \otimes \sigma^{-1} \otimes \delta_H) \circ ((c_{H,H} \circ \delta_H \circ T) \otimes T \otimes T) \circ (f \otimes \delta_H \otimes H) \circ (\delta_H \otimes H) \circ \delta_H$ (T anti-comultiplicative)

$= \mu_{H^\sigma} \circ (\mu_H \otimes \sigma^{-1} \otimes T_{H^\sigma}) \circ ((\sigma * \sigma^{-1}) \otimes \delta_{H \otimes H} \otimes H) \circ (H \otimes c_{H,H} \otimes \delta_H) \circ ((\delta_H \circ T) \otimes H \otimes H)$

$\circ (f \otimes \delta_H \otimes H) \circ (\delta_H \otimes H) \circ \delta_H$ (coassociativity)

$= \mu_{H^\sigma} \circ (\mu_H \otimes \sigma^{-1} \otimes T_{H^\sigma}) \circ (\delta_{H \otimes H} \otimes H) \circ (f \otimes T \otimes \delta_H) \circ (\delta_H \otimes H) \circ \delta_H$ (σ convolution invertible)

$= \mu_{H^\sigma} \circ (\mu_H \otimes \sigma^{-1} \otimes T_{H^\sigma}) \circ (H \otimes c_{H,H} \otimes H \otimes H) \circ (f \otimes ((T \otimes T) \circ c_{H,H} \circ \delta_H) \otimes \delta_H \otimes H) \circ (\delta_H \otimes \delta_H) \circ \delta_H$ (T anti-comultiplicative)

$= \mu_{H^\sigma} \circ (\Pi_H^R \otimes H) \circ (H \otimes \sigma^{-1} \otimes T_{H^\sigma}) \circ (f \otimes (c_{H,H} \circ (T \otimes H) \circ \delta_H) \otimes \delta_H) \circ (\delta_H \otimes H) \circ \delta_H$ (naturality of c)

$= (\mu_H \otimes \sigma^{-1}) \circ \delta_{H \otimes H} \circ (\Pi_H^R \otimes H) \circ (H \otimes \sigma^{-1} \otimes T_{H^\sigma}) \circ (f \otimes (c_{H,H} \circ (T \otimes H) \circ \delta_H) \otimes \delta_H) \circ (\delta_H \otimes H) \circ \delta_H$ (by (18))

$= (\mu_H \otimes \sigma^{-1}) \circ \delta_{H \otimes H} \circ (\Pi_H^R \otimes \sigma^{-1} \otimes \sigma \otimes H) \circ (f \otimes ((H \otimes T) \circ c_{H,H} \circ \delta_H) \otimes \delta_H \otimes T \otimes ((T \otimes f^{-1}) \circ \delta_H))$

$\circ (\delta_H \otimes H \otimes \delta_H) \circ (H \otimes \delta_H) \circ \delta_H$ (coassociativity)

$= (\mu_H \otimes \sigma^{-1}) \circ \delta_{H \otimes H} \circ (\Pi_H^R \otimes (\partial^4(\sigma^{-1}) * \partial^1(\sigma)) \otimes ((T \otimes f^{-1}) \circ \delta_H)) \circ (((H \otimes T) \circ c_{H,H} \circ \delta_H) \otimes H \otimes ((T \otimes H) \circ \delta_H))$

$\circ (((f \otimes H) \circ \delta_H) \otimes \delta_H) \circ \delta_H$ (definition of ∂ morphisms)

$= (\mu_H \otimes \sigma^{-1}) \circ \delta_{H \otimes H} \circ (\Pi_H^R \otimes (\partial^2(\sigma) * \partial^3(\sigma^{-1})) \otimes ((T \otimes f^{-1}) \circ \delta_H)) \circ (((H \otimes T) \circ c_{H,H} \circ \delta_H) \otimes H \otimes ((T \otimes H) \circ \delta_H))$

$\circ (((f \otimes H) \circ \delta_H) \otimes \delta_H) \circ \delta_H$ (by (7))

$= (\mu_H \otimes \sigma^{-1}) \circ \delta_{H \otimes H} \circ (\Pi_H^R \otimes ((\sigma \otimes \sigma^{-1}) \circ (\mu_H \otimes c_{H,H} \otimes \mu_H) \circ (H \otimes c_{H,H} \otimes c_{H,H} \otimes H)) \otimes H)$

$\circ (H \otimes (\delta_H \circ T) \otimes \delta_H \otimes ((T \otimes T) \circ c_{H,H} \circ \delta_H) \otimes T) \circ (f \otimes (c_{H,H} \circ \delta_H) \otimes \delta_H \otimes ((H \otimes f^{-1}) \circ \delta_H))$

$\circ (\delta_H \otimes \delta_H) \circ \delta_H$ (definition of ∂ morphisms, T anti-comultiplicative)

$= (\mu_H \otimes \sigma^{-1}) \circ \delta_{H \otimes H} \circ (H \otimes \sigma \otimes H) \circ (H \otimes \mu_H \otimes (\sigma^{-1} \circ (H \otimes \Pi_H^L)) \otimes T \otimes H)$

$\circ (H \otimes H \otimes c_{H,H} \otimes H \otimes H \otimes H) \circ (H \otimes \delta_H \otimes H \otimes \delta_H \otimes H)$

$\circ (f \otimes (c_{H,H} \circ (T \otimes \Pi_H^R) \circ \delta_H) \otimes \delta_H \otimes ((T \otimes f^{-1}) \circ \delta_H)) \circ (\delta_H \otimes \delta_H) \circ \delta_H$ (naturality of c)

$= (\mu_H \otimes \sigma^{-1}) \circ \delta_{H \otimes H} \circ (H \otimes (\sigma \circ (\mu_H \otimes H)) \otimes H) \circ (f \otimes (c_{H,H} \circ (T \otimes \Pi_H^R) \circ \delta_H)$

$\otimes H \otimes ((T \otimes T) \circ \delta_H) \otimes f^{-1}) \circ (\delta_H \otimes H \otimes \delta_H) \circ (H \otimes \delta_H) \circ \delta_H$ (by (13))

$= (\mu_H \otimes \sigma^{-1}) \circ \delta_{H \otimes H} \circ (H \otimes \sigma \otimes H) \circ (f \otimes ((\Pi_H^R \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (T \otimes \delta_H) \circ \delta_H)$

$$\otimes (c_{H,H} \circ \delta_H \circ T) \otimes f^{-1}) \circ (\delta_H \otimes \delta_H) \circ \delta_H \quad (T \text{ anti-comultiplicative})$$

$$= (\mu_H \otimes \sigma^{-1}) \circ \delta_{H \otimes H} \circ (H \otimes \sigma \otimes H) \circ (f \otimes (\delta_H \circ \Pi_H^R) \otimes (c_{H,H} \circ \delta_H \circ T) \otimes f^{-1}) \circ (\delta_H \otimes \delta_H) \circ \delta_H \quad (\text{by (15)})$$

$$= (\mu_H \otimes (\sigma^{-1} * \sigma)) \circ \delta_{H \otimes H} \circ (f \otimes \Pi_H^R \otimes T \otimes f^{-1}) \circ (\delta_H \otimes \delta_H) \circ \delta_H \quad (\text{ naturality of } c)$$

$$= \mu_H \circ (\Pi_H^R \otimes T) \circ \delta_H \circ (f \otimes H \otimes f^{-1}) \circ (H \otimes \delta_H) \circ \delta_H \quad (\sigma \text{ convolution invertible})$$

$$= (T * id_H * T) \circ (f \otimes H \otimes f^{-1}) \circ (H \otimes \delta_H) \circ \delta_H$$

$$= T_{H^\sigma} \quad (\text{by (2)}).$$

Finally, T_{H^σ} is anti-comultiplicative because so is T and by Proposition 2.4. Indeed,

$$\delta_H \circ T_{H^\sigma}$$

$$= c_{H,H} \circ (T \otimes T) \circ \delta_H \circ (f \otimes H \otimes f^{-1}) \circ (H \otimes \delta_H) \circ \delta_H$$

$$= c_{H,H} \circ (T \otimes (f^{-1} * f) \otimes T) \circ (f \otimes \delta_H \otimes H \otimes f^{-1}) \circ (\delta_H \otimes \delta_H) \circ \delta_H$$

$$= c_{H,H} \circ (T_{H^\sigma} \otimes T_{H^\sigma}) \circ \delta_H,$$

and the proof is complete.

$\square$

Remark 2.7 Note that, by Proposition 2.4, $T = T_{H^\sigma} \circ (f^{-1} \otimes H \otimes f) \circ (H \otimes \delta_H) \circ \delta_H$. As a consequence, it is easy to see that T is bijective if and only if so is T_{H^σ}.

3. Two-cocycles, skew pairings, and double crossproducts

It is a well known fact that a class of 2-cocycles is provided by invertible skew pairings on bialgebras. In this section, we will show that when we consider bialgebras with weak antipode the bialgebra built with this method is also a bialgebra with weak antipode.

First of all, we introduce our notion of skew pairing for bialgebras with weak antipode. The definition is inspired in the one given by Li in [9] by the name of weak Hopf pair. Although it may seem that we have removed the conditions involving the unit morphisms, we will show that they can be obtained under the hypothesis of considering convolution invertible skew pairings, and this assumption will be essential to obtain the results of this section.

Definition 3.1 Let A and H be bialgebras with weak antipodes T_A and T_H, respectively. A pairing between A and H over K is a morphism $\tau : A \otimes H \to K$ such that the equalities

(a1) $\tau \circ (\mu_A \otimes H) = (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ (A \otimes A \otimes \delta_H)$,

(a2) $\tau \circ (A \otimes \mu_H) = (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ (\delta_A \otimes H \otimes H)$,

hold.

A skew pairing between A and H is a pairing between A^{cop} and H, i. e. a morphism $\tau : A \otimes H \to K$ satisfying (a1) and

(a2') $\tau \circ (A \otimes \mu_H) = (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ ((c_{A,A} \circ \delta_A) \otimes H \otimes H)$.

Proposition 3.2 *Let A and H be bialgebras with weak antipodes T_A and T_H, respectively. Let $\tau : A \otimes H \to K$ be a skew pairing. Then the following assertions are equivalent:*

(i) τ is convolution invertible.

(ii) $\tau \circ (\Pi_A^i \otimes H) = \varepsilon_A \otimes \varepsilon_H$, $i = L, K$.

Moreover, in this case $\tau^{-1} = \tau \circ (T_A \otimes H)$ is the convolution inverse of τ.

Proof

(i) $\Rightarrow$ (ii). Assume that τ is convolution invertible with inverse τ^{-1}. Then, by (1) for A and (a1),

$$\tau = \tau \circ ((id_A * T_A * id_A) \otimes H) = (\tau \circ (\Pi_A^L \otimes H)) * \tau.$$

Then

$$\varepsilon_A \otimes \varepsilon_H = \tau * \tau^{-1} = (\tau \circ (\Pi_A^L \otimes H)) * \tau * \tau^{-1} = \tau \circ (\Pi_A^L \otimes H),$$

and in a similar way, but using that $id_A * T_A * id_A = id_A * \Pi_A^R$, we get that $\tau \circ (\Pi_A^R \otimes H) = \varepsilon_A \otimes \varepsilon_H$. Now consider the morphism $\tau^{-1} = \tau \circ (T_A \otimes H)$. Then, by (a1),

$$\tau * \tau^{-1} = \tau \circ (\Pi_A^L \otimes H) = \varepsilon_A \otimes \varepsilon_H,$$

$$\tau^{-1} * \tau = \tau \circ (\Pi_A^R \otimes H) = \varepsilon_A \otimes \varepsilon_H,$$

and τ^{-1} is the convolution inverse of τ.

(ii) $\Rightarrow$ (i). Define $\tau^{-1} = \tau \circ (T_A \otimes H)$. By (a1), τ^{-1} is the convolution inverse of τ.

$\square$

Proposition 3.3 *Let A and H be bialgebras with anti-comultiplicative weak antipodes T_A and T_H, respectively. Let $\tau : A \otimes H \to K$ be a convolution invertible skew pairing. Then, for $i = L, K$,*

$$\tau \circ (A \otimes \Pi_H^i) = \varepsilon_A \otimes \varepsilon_H. \tag{20}$$

As a consequence, $\tau = \tau \circ (T_A \otimes T_H) = \tau^{-1} \circ (A \otimes T_H)$ and $\tau^{-1} \circ (A \otimes \Pi_H^i) = \varepsilon_A \otimes \varepsilon_H$, $i = L, K$.

Proof

By Proposition 3.2, $\tau^{-1} = \tau \circ (T_A \otimes H)$. Then

τ^{-1}

$= \tau \circ (T_A \otimes (id_H * T_H * id_H))$ (by (1))

$= (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ ((c_{A,A} \circ \delta_A \circ T_A) \otimes ((\Pi_H^L \otimes H) \circ \delta_H))$ (by (a2))

$= (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ (((T_A \otimes T_A) \circ \delta_A) \otimes ((\Pi_H^L \otimes H) \circ \delta_H))$ (T anti-comultiplicative)

$= ((\tau^{-1} \circ (A \otimes \Pi_H^L)) * \tau^{-1})$ (by Proposition 3.2).

Then,

$$\tau^{-1} \circ (A \otimes \Pi_H^L) = (\tau^{-1} \circ (A \otimes \Pi_H^L)) * \tau^{-1} * \tau = \tau^{-1} * \tau = \varepsilon_A \otimes \varepsilon_H.$$

Now, using (14),

$$\varepsilon_A \otimes \varepsilon_H$$

$$= (\tau * \tau^{-1}) \circ (A \otimes \Pi_H^L)$$

$$= (\tau \otimes \tau^{-1}) \circ (A \otimes c_{A,H} \otimes H) \circ (\delta_A \otimes ((\mu_H \otimes \Pi_H^L) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes T) \circ \delta_H))$$

$$= \tau \circ (A \otimes \Pi_H^L).$$

In a similar way we get that $\tau^{-1} \circ (A \otimes \Pi_H^R) = \varepsilon_A \otimes \varepsilon_H$, and using (15), we obtain that $\varepsilon_A \otimes \varepsilon_H = \tau \circ (A \otimes \Pi_H^R)$.

On the other hand,

$$\tau^{-1} * (\tau \circ (T_A \otimes T_H))$$

$$= (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ (((T_A \otimes T_A) \circ \delta_A) \otimes ((H \otimes T_H) \circ \delta_H))$$

$$= (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ ((c_{A,A} \circ \delta_A \circ T_A) \otimes ((H \otimes T_H) \circ \delta_H))$$

$$= \tau \circ (T_A \otimes \Pi_H^L)$$

$$= \varepsilon_A \otimes \varepsilon_H,$$

and with similar computations we obtain that $(\tau \circ (T_A \otimes T_H)) * \tau^{-1} = \varepsilon_A \otimes \varepsilon_H$. As a consequence, $\tau = \tau \circ (T_A \otimes T_H) = \tau^{-1} \circ (A \otimes T_H)$. Finally, by the definition of τ^{-1}, it is obvious that $\tau^{-1} \circ (A \otimes \Pi_H^i) = \varepsilon_A \otimes \varepsilon_H$, $i = L, K$.

$\square$

Remark 3.4 As a consequence of Propositions 3.2 and 3.3, we have that if the skew pairing τ is invertible, the equalities

$$\tau \circ (\eta_A \otimes H) = \varepsilon_H = \tau^{-1} \circ (\eta_A \otimes H), \tag{21}$$

$$\tau \circ (A \otimes \eta_H) = \varepsilon_A = \tau^{-1} \circ (A \otimes \eta_H), \tag{22}$$

hold.

In the Hopf algebra setting, (21) implies that τ is convolution invertible with inverse $\tau^{-1} = \tau \circ (T_A \otimes H)$, and therefore it also implies (22), because by (a1),

$$\tau * \tau^{-1} = \tau \circ (\Pi_A^L \otimes H) = \tau \circ (\eta_A \otimes H) \circ (\varepsilon_A \otimes H) = \varepsilon_A \otimes \varepsilon_H$$

and

$$\tau^{-1} * \tau = \tau \circ (\Pi_A^R \otimes H) = \tau \circ (\eta_A \otimes H) \circ (\varepsilon_A \otimes H) = \varepsilon_A \otimes \varepsilon_H.$$

Note that the above proof does not work in the bialgebra with weak antipode setting because in this case $\Pi_A^L \neq \eta_A \circ \varepsilon_A \neq \Pi_A^R$.

Proposition 3.5 *Let A and H be bialgebras with anti-comultiplicative weak antipodes T_A and T_H, respectively. Moreover, assume that T_H is bijective. Let $\tau : A \otimes H \to K$ be a convolution invertible skew pairing. Then,*

$$\tau^{-1} = \tau \circ (A \otimes T_H^{-1}). \tag{23}$$

As a consequence, for $i = L, K$,

$$\tau^{-1} \circ (\Pi_A^i \otimes H) = \varepsilon_A \otimes \varepsilon_H. \tag{24}$$

Proof Indeed,

$$\tau * \tau^{-1}$$

$$= (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ ((c_{A,A} \circ \delta_A) \otimes (c_{H,H} \circ (H \otimes T_H^{-1}) \circ \delta_H)) \text{ (by naturality)}$$

$$= \tau \circ (A \otimes (\mu_H \circ c_{H,H} \circ (H \otimes T_H^{-1}) \circ \delta_H)) \text{ (by (a2'))}$$

$$= \tau \circ (A \otimes (\Pi_H^L \circ T_H^{-1})) \text{ (T_H anti-comultiplicative)}$$

$$= \varepsilon_A \otimes \varepsilon_H \text{ (by 20)},$$

and

$$\tau^{-1} * \tau$$

$$= (\tau \otimes \tau) \circ (A \otimes c_{A,H} \otimes H) \circ ((c_{A,A} \circ \delta_A) \otimes (c_{H,H} \circ (T_H^{-1} \otimes H) \circ \delta_H))$$

$$= \tau \circ (A \otimes (\mu_H \circ c_{H,H} \circ (T_H^{-1} \otimes H) \circ \delta_H))$$

$$= \tau \circ (A \otimes (\Pi_H^R \circ T_H^{-1}))$$

$$= \varepsilon_A \otimes \varepsilon_H.$$

Finally, by Proposition 3.2 we get that $\tau^{-1} \circ (\Pi_A^i \otimes H) = \varepsilon_A \otimes \varepsilon_H$, $i = L, K$.

$\square$

Proposition 3.6 *Let A, H be bialgebras with anti-comultiplicative weak antipodes T_A and T_H, respectively. Then $A \otimes H = (A \otimes H, \eta_{A \otimes H}, \mu_{A \otimes H}, \varepsilon_{A \otimes H}, \delta_{A \otimes H})$ is a bialgebra with anti-comultiplicative weak antipode $T_{A \otimes H} = T_A \otimes T_H$.*

Moreover, let $\tau : A \otimes H \to K$ be a convolution invertible skew pairing. The morphism $\sigma = \varepsilon_A \otimes (\tau \circ c_{H,A}) \otimes \varepsilon_H$ is a normal 2-cocycle with convolution inverse $\sigma^{-1} = \varepsilon_A \otimes (\tau^{-1} \circ c_{H,A}) \otimes \varepsilon_H$ and satisfies the conditions (10) and (12).

Proof Straightforward. $\square$

By [4], if A and H are bialgebras and $\tau : A \otimes H \to K$ is a convolution invertible skew pairing, we can form a new bialgebra $A \bowtie_\tau H$ built on $A \otimes H$ with tensor product coproduct and unit, and algebra structure given by

$$\mu_{A \bowtie_\tau H} = (\mu_A \otimes \mu_H) \circ (A \otimes \tau \otimes A \otimes H \otimes \tau^{-1} \otimes H) \circ (A \otimes \delta_{A \otimes H} \otimes A \otimes H \otimes H) \circ (A \otimes \delta_{A \otimes H} \otimes H) \circ (A \otimes c_{H,A} \otimes H).$$

Moreover, if A and H are Hopf algebras with antipodes T_A and T_H, respectively, $A \bowtie_\tau H$ is a Hopf algebra with antipode

$$T_{A \bowtie_\tau H} = (\tau^{-1} \otimes T_A \otimes T_H \otimes \tau) \circ (A \otimes H \otimes \delta_{A \otimes H}) \circ \delta_{A \otimes H}.$$

Following [4], the Hopf algebra $A \bowtie_\tau H$ can be explained as a deformation of $A \otimes H$ by the 2-cocycle associated to a skew pairing. Now we extend this result to the bialgebra with weak antipode setting.

Proposition 3.7 *Let A, H be bialgebras with anti-comultiplicative weak antipodes T_A and T_H, respectively. Let $\tau : A \otimes H \to K$ be a convolution invertible skew pairing. Then*

$$A \bowtie_\tau H = (A \otimes H, \eta_{A \otimes H}, \mu_{A \bowtie_\tau H}, \varepsilon_{A \otimes H}, \delta_{A \otimes H}, T_{A \bowtie_\tau H})$$

has a structure of bialgebra with anti-comultiplicative weak antipode.

Proof We only need to see that $A \bowtie_\tau H$ is a deformation of $A \otimes H$ by the 2-cocycle σ defined in Proposition 3.6. Indeed,

$\mu_{(A \otimes H)^\sigma}$

$= (\varepsilon_A \otimes (\tau \circ c_{H,A}) \otimes \varepsilon_H \otimes \mu_{A \otimes H} \otimes \varepsilon_A \otimes (\tau^{-1} \circ c_{H,A}) \otimes \varepsilon_H) \circ (A \otimes H \otimes A \otimes H \otimes \delta_{A \otimes H \otimes A \otimes H}) \circ \delta_{A \otimes H \otimes A \otimes H}$

$= (\mu_A \otimes \mu_H) \circ (A \otimes \tau \otimes A \otimes H \otimes \tau^{-1} \otimes H) \circ (A \otimes \delta_{A \otimes H} \otimes A \otimes H \otimes H) \circ (A \otimes \delta_{A \otimes H} \otimes H) \circ (A \otimes c_{H,A} \otimes H)$

$= \mu_{A \bowtie_\tau H},$

and

$T_{(A \otimes H)^\sigma}$

$= (\varepsilon_A \otimes (\tau \circ c_{H,A} \circ (H \otimes T_A)) \otimes (\varepsilon_H \circ T_H) \otimes T_A \otimes T_H$

$\qquad \otimes (\varepsilon_A \circ T_A) \otimes (\tau^{-1} \circ c_{H,A} \circ (T_H \otimes A)) \otimes \varepsilon_H) \circ (\delta_{A \otimes H} \otimes A \otimes H \otimes \delta_{A \otimes H}) \circ (A \otimes H \otimes \delta_{A \otimes H}) \circ \delta_{A \otimes H}$

$= (\tau^{-1} \otimes T_A \otimes T_H \otimes \tau) \circ (A \otimes H \otimes \delta_{A \otimes H}) \circ \delta_{A \otimes H}$

$= T_{A \bowtie_\tau H}.$

$\square$

Remark 3.8 The Hopf algebra $A \bowtie_\tau H$ is a is a special case of the Majid–Radford double crossproduct $A \bowtie H$ (see [13], [14]), one of whose most celebrated examples is the Drinfeld double [5] (roughly speaking, a double crossproduct involving H and $(H^*)^{cop}$, where H^* is the dual of H). In the Hopf algebra world, if $A = H^{*cop} \otimes H$, the Drinfeld double $D(H)$ can be obtained as A^σ, where σ is defined by

$$\sigma((x \otimes a), (y \otimes b)) = x(1)y(a)\varepsilon(b),$$

for $x, y \in H^*$ and $a, b \in H$. Taking this into account, by Theorem 1.10 of [8], if H is a finite bialgebra with weak antipode so is its dual, it seems natural to ask if our result can be applied in order to describe the Drinfeld double of a bialgebra with weak antipode. Unfortunately, as we will explain next, the answer is negative and the description of the Drinfeld double of bialgebras with weak antipode in terms of 2-cocycles remains open.

Indeed, assume that H is a finite bialgebra with weak antipode T. Consider the evaluation map $b : H \otimes H^* \to K$. It is not difficult to see that b is a skew pairing and the associated 2-cocycle will be $\sigma = \varepsilon_{H^*} \otimes b \otimes \varepsilon_H$. However, if we assume that b is convolution invertible, by Proposition 3.2 we have that

$$b \circ (\Pi_H^L \otimes H^*) = \varepsilon_H \otimes \varepsilon_{H^*} = b \circ (\Pi_H^R \otimes H^*).$$

Then, by the suitable compositions, we get that $\Pi_H^L = \eta_H \otimes \varepsilon_H = \Pi_H^R$ and H is a classical Hopf algebra.

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