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The cissoid of Diocles in the Lorentz–Minkowski plane

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Abstract: This article presents the cissoid of Diocles and the cissoid of two circles with respect to origin in the Lorentz–Minkowski plane.

Key words: Cissoid, cissoid of Diocles, Lorentz–Minkowski plane

1. Introduction

There are some famous problems in the history of mathematics such as angle trisection and squaring a circle. One of them is the problem of doubling the cube which is called Delian problem. A method to solve this problem was invented by Diocles (250 BC) using the cissoid curve, which is nowadays called the cissoid of Diocles. The cissoid of Diocles is the special form of the general cissoid curve built on a circle and a straight line. The cissoid of Diocles which pedal curve is cardioid is very interesting and important curve and has some mechanical applications [4, 5, 7, 11].

In differential geometry, the curves are important. Generally, some special curves as Cassini curves are studied. On the other hand, some curves are introduced in Minkowski space-time plane [10]. These curves are applied in computer-aided geometric design. A locus of a point specified by geometric conditions is drawn by Mathematica [2]. The establishment of a linkage is used for representing a cissoid with continuous motion [9].

In this paper, our aim is to write the cissoid of two curves and the cissoid of Diocles in the Lorentz–Minkowski plane. Additionally, the figures are drawn via MATLAB.

2. Preliminaries

The Lorentz–Minkowski plane is the metric space $E_1^2 = (\mathbb{R}^2, \langle , \rangle)$ where the metric $\langle , \rangle$ is

$$\langle u, v\rangle = u_1 v_1 - u_2 v_2, \quad u = (u_1, u_2), v = (v_1, v_2),$$

which is called the Lorentzian metric.

The Lorentzian metric is a nondegenerate metric of index 1. A vector $u \in E_1^2$ is said space-like if $\langle u, u\rangle > 0$ or $u = 0$, timelike if $\langle u, u\rangle < 0$ and (null) light-like if $\langle u, u\rangle = 0$ and $u \neq 0$ [6, 8].

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Definition 2.1 *A (smooth) curve is a differentiable map* $\alpha : I \subset \mathbb{R} \to E_1^2$, *where* $a < I < b$; $a, b \in \mathbb{R}$. *A curve* α *in* E_1^2 *is called time-like (resp. space-like, light-like) at* t *if* $\alpha'(t)$ *is a time-like (respectively space-like, light-like) vector. The curve* α *is time-like (respectively space-like, light-like) if it is time-like (respectively space-like, light-like) for all* $t \in I$ *[6, 8].*

Definition 2.2 *Let* P *be a point in the Lorentz plane and let* $r > 0$. *The curve*

$$(Q; \langle PQ, PQ \rangle = r^2)$$

has two branches and each of them is called a Lorentzian circle with center P *and radius* r.

If $P = (p_1, p_2)$ *is center, then the equation is*

$$(x_1 - p_1)^2 - (x_2 - p_2)^2 = r^2$$

[1].

3. Cissoid

Diocles (between 250 and 100 BC) utilized the ordinary cissoid (a word from the Greek meaning "ivy") in finding two mean proportionals between given lengths a and b. As early as 1689, J.C. Sturm gave a mechanical device for the construction of the cissoid of Diocles.

Definition 3.1 *Given two curves* $y = f_1(x)$, $y = f_2(x)$ *and the fixed point* O. *Let* Q *and* R *be the intersections of a variable line through* O *with the given curves. The locus of* P *on this secant is such that*

$$OP = OR - OQ = QR$$

is the cissoid of the two curves with respect to O *as shown in Figure 1 [11].*

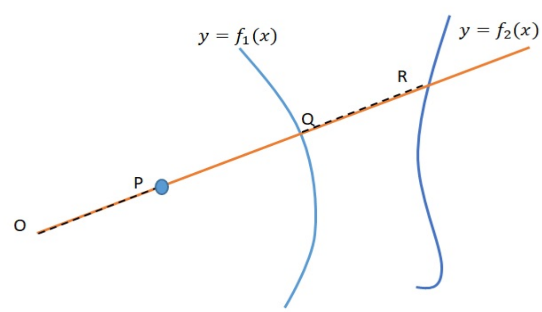

Figure 1. The cissoid of the two curves $y = f_1(x)$, $y = f_2(x)$ with respect to O.

If the two curves are a line and a circle, the ordinary family of cissoids is generated. The following discussion is restricted to this family. Let the two given curves be a fixed circle of radius a, center at K and passing through O, and the line L perpendicular to $OX-$axis at $2(a+b)$ distance from O. The ordinary cissoid is the locus of P on the variable secant through O such that $OP = r = QR$ as shown in Figure 2. $d(A, A') = 2b$, $a = d(O, K)$.

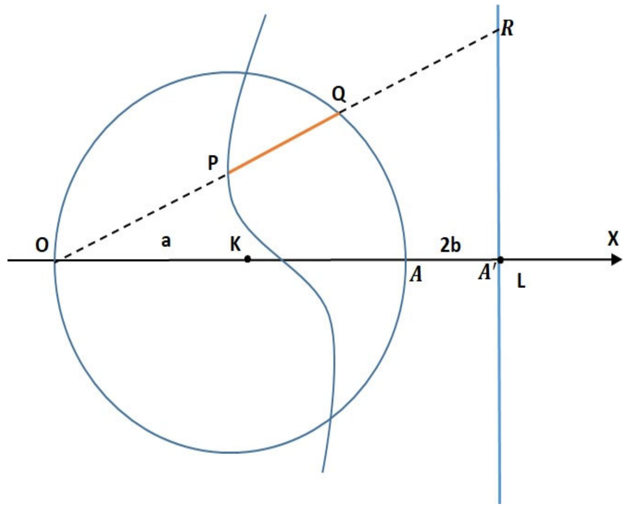

Figure 2. The cissoid of Diocles in general form.

The generation may be effected by the intersection P of the secant OR and the circle of radius a tangent to L at R as this circle rolls upon L.

The curve has a cusp if $b = 0$ (the cissoid of Diocles); a double point if the rolling circle passes between O and K. Its asymptote is the line L.

$$r = 2(a+b)\sec\theta - 2a\cos\theta.$$

$$y^2 = \frac{x^2(2b-x)}{x-2(a+b)}$$

$$\begin{cases} x = \frac{2(b+(a+b)t^2)}{1+t^2} \\ y = \frac{2(bt+(a+b)t^3)}{1+t^2} \end{cases}$$

(If $b = 0$: $r = 2a\sin\theta\tan\theta$; $y^2 = \frac{x^3}{2a-x}$, the cissoid of Diocles) [11].

Diocles takes a circle with radius r and center $(r, 0)$, and a line which tangent to circle at $(2r, 0)$ for the cissoid curve as shown in Figure 3.

4. Cissoid in the Lorentz–Minkowski plane

In this section, we give the cissoid of two circles and the cissoid of Diocles in the Lorentz–Minkowski plane.

Definition 4.1 *Given two curves $y = \alpha_1(x)$, $y = \alpha_2(x)$ and the fixed point O in the Lorentz–Minkowski plane. Let Q and R be the intersections of a variable line through O with the given curves. The locus of P on this secant is such that*

$$d_L(O,P) = d_L(O,R) - d_L(O,Q) = d_L(Q,R)$$

is the cissoid of the two curves with respect to O.

Theorem 4.2 *Let α and β be two different circles with center $(0,0)$ and radius r_1 and r_2, respectively. The cissoid of α and β with respect to origin is the circle with radius $|r_1 - r_2|$ and center $(0,0)$.*

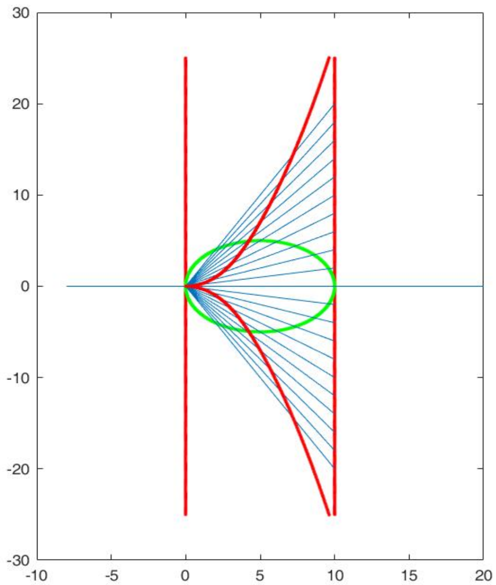

Figure 3. The cissoid of Diocles.

Proof Let α and β be two different circles, d be a straight line, $y = mx$, $|m| < 1$, and α, β and d be in the spacelike.

$$\alpha : x^2 - y^2 = r_1^2$$

and

$$\beta : x^2 - y^2 = r_2^2.$$

Suppose that Q_1 and Q_2 are the intersection points of α and d.

$$\begin{aligned} x^2 - (mx)^2 &= r_1^2 \\ x^2 - m^2x^2 &= r_1^2 \\ (1 - m^2)x^2 &= r_1^2 \\ x_{1,2} &= \mp \frac{r_1}{\sqrt{1 - m^2}}, \\ y_{1,2} &= \mp m \frac{r_1}{\sqrt{1 - m^2}}. \end{aligned}$$

Thus, we have the intersection points as follows

$$\begin{aligned} Q_1 &= (\frac{r_1}{\sqrt{1 - m^2}}, \frac{mr_1}{\sqrt{1 - m^2}}), \\ Q_2 &= (-\frac{r_1}{\sqrt{1 - m^2}}, -\frac{mr_1}{\sqrt{1 - m^2}}). \end{aligned}$$

Similarly, we can calculate the intersection point of β and d. Suppose that R_1 and R_2 are the intersection points of β and d. Thus, we have

$$\begin{aligned} R_1 &= (\frac{r_2}{\sqrt{1-m^2}}, \frac{mr_2}{\sqrt{1-m^2}}), \\ R_2 &= (-\frac{r_2}{\sqrt{1-m^2}}, -\frac{mr_2}{\sqrt{1-m^2}}). \end{aligned}$$

Also, we can calculate the distance between these intersection points.

$$\begin{aligned} k &= d(Q_1, R_1) \\ &= \sqrt{\left|(\frac{r_1-r_2}{\sqrt{1-m^2}})^2 - (\frac{mr_1-mr_2}{\sqrt{1-m^2}})^2\right|} \\ &= \sqrt{\left|\frac{(r_1-r_2)^2 - m^2(r_1-r_2)^2}{1-m^2}\right|} \\ &= \sqrt{\left|\frac{(1-m^2)(r_1-r_2)^2}{1-m^2}\right|} \\ &= \left|r_1-r_2\right|. \end{aligned}$$

In the similar way, we have

$$k' = d(Q_2, R_2) = |r_1 - r_2| = k.$$

Using Pythagorean theorem in the Lorentz plane [3], we can write

$$\begin{aligned} x_m^2 - y_m^2 &= k^2 \\ x_m^2 - m^2 x_m^2 &= k^2 \\ x_m^2 &= \frac{k^2}{1-m^2} \\ x_m &= \frac{k}{\sqrt{1-m^2}}, \\ y_m &= \frac{mk}{\sqrt{1-m^2}}. \end{aligned}$$

Thus, we have the cissoid of α and β as

$$P(m) = (\frac{|r_1-r_2|}{\sqrt{1-m^2}}, \frac{m\,|r_1-r_2|}{\sqrt{1-m^2}}).$$

$P(m)$ is a circle with radius $|r_1 - r_2|$ such that

$$(\frac{k}{\sqrt{1-m^2}})^2 - (\frac{mk}{\sqrt{1-m^2}})^2 = (|r_1-r_2|)^2.$$

This completes the proof of the theorem. □

The cissoid of two circles is illustrated as shown in Figure 4.

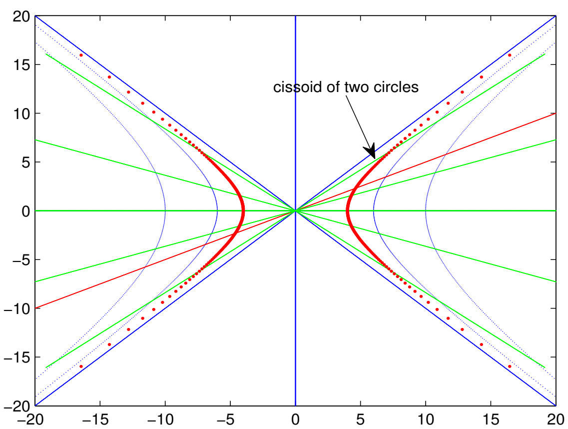

Figure 4. The cissoid of two circles in the Lorentz–Minkowski space-like.

Theorem 4.3 *Let α be a circle and β be a straight line $x = x_0,\ -x_0 < y < x_0$. Then the cissoid of α and β to O is*

$$P(m) = (x - x_0, y - my_0),$$

where $x = \frac{r}{\sqrt{1-m^2}}\,,\ y = mx\,,\ |m| < 1.$

Proof Suppose that

$$\begin{aligned} \alpha &: \quad x^2 - y^2 = r^2, \\ \beta &: \quad x = x_0, \quad -1 < y < 1, \\ d &: \quad y = mx, \quad |m| < 1 \end{aligned}$$

and $Q_{1,2} = d \cap \alpha$ and $R = d \cap \beta$. In this case,

$$Q_{1,2} : \left.\begin{array}{c} x^2 - y^2 = r^2 \\ y = mx \end{array}\right\}$$

we have

$$\begin{aligned} x^2 - m^2x^2 &= r^2 \\ (1 - m^2)x^2 &= r^2 \\ x_{1,2} &= \mp\frac{r}{\sqrt{1-m^2}}, \\ y_{1,2} &= \mp\frac{mr}{\sqrt{1-m^2}}. \end{aligned}$$

And, we have for R,

$$\begin{aligned} R &= d \cap \beta, \\ d &: \ y = mx, \\ \beta &: \ x = x_0, \\ R &= (x_0, mx_0). \end{aligned}$$

To find the cissoid, we have to calculate the distance between $Q_{1,2}$ and R.

$$\begin{aligned} k &= d(Q_1, R) \\ &= \sqrt{\left|(\frac{r}{\sqrt{1-m^2}} - x_0)^2 - (\frac{mr}{\sqrt{1-m^2}} - mx_0)^2\right|} \\ &= r - x_0\sqrt{1-m^2} \end{aligned}$$

and we have

$$k' = d(Q_2, R) = k.$$

Thus, we have for cissoid of α and β,

$$P(m) = (\frac{r}{\sqrt{1-m^2}} - x_0, m(\frac{r}{\sqrt{1-m^2}} - x_0)$$

$$= (x_1 - x_0, y_1 - mx_0), \tag{4.1}$$

where $x_1 = \frac{r}{\sqrt{1-m^2}}, x_2 = \frac{mr}{\sqrt{1-m^2}}$. If we write equation (4.1) as

$$P(m) = (x_1, y_1) - (x_0, 0) - (0, -mx_0),$$

then we can say that the cissoid of d is translation with $-(x_0, 0)$ and a press with $(0, mx_0)$ of the circle. □

The cissoid of a circle and a line according to $x_0 < r$, $x_0 = r$, $x_0 > r$ has three types as shown in Figures 5–7.

Case 1:

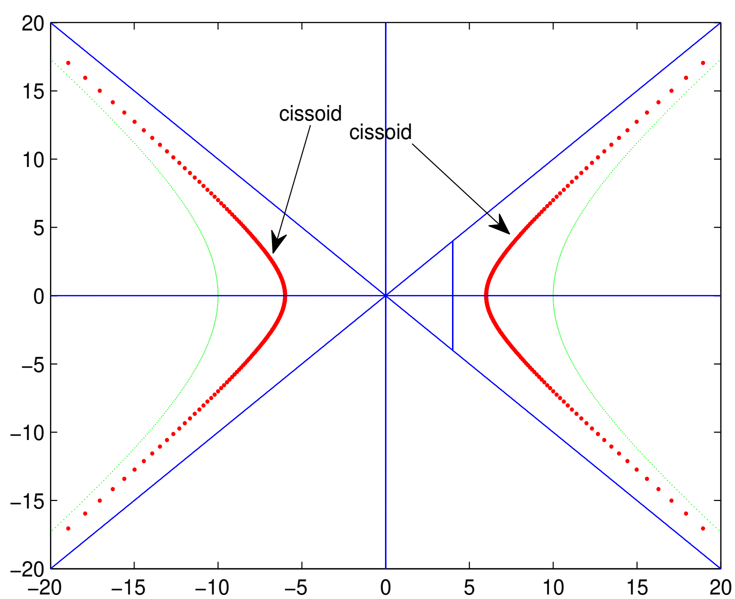

Figure 5. The cissoid of Diocles in the Lorentz–Minkowski space-like for $x_0 < r$.

Case 2:

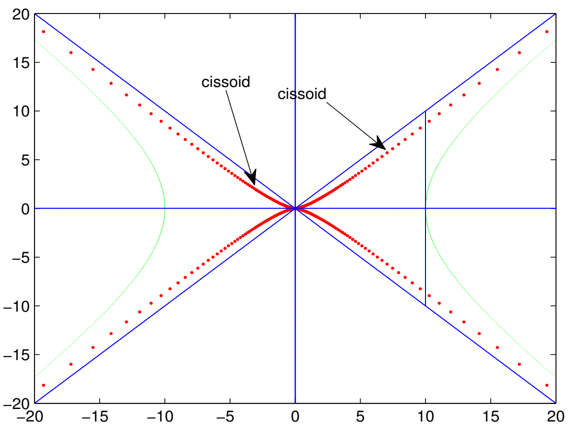

Figure 6. The cissoid of Diocles in the Lorentz–Minkowski space-like for $x_0 = r$.

Case 3:

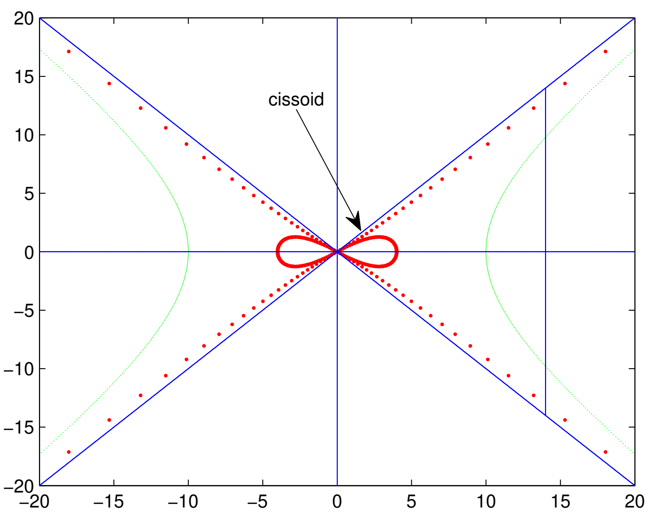

Figure 7. The cissoid of Diocles in the Lorentz–Minkowski space-like for $x_0 > r$.

5. Conclusion

In the Euclidean plane, cissoid of two curves is well known. Every special choosing of two curves gives us a new cissoid curve. The cissoid of Diocles is a special form of general cissoid curve. Diocles takes a circle and a line tangent to the circle. As we know, a circle in the Lorentz–Minkowski plane has a hyperbolic form and it has two parts. Studying in space-like or time-like in the Lorentz–Minkowski plane is sometimes an obligation or sometimes a choice. In this article, we study in space-like of the Lorentz–Minkowski plane and calculate three different types of the cissoid of Diocles in the Lorentz–Minkowski plane, when the curves are taken a Lorentzian circle and a line segment. Furthermore, we show that the cissoid of two Lorentzian circles with different radii, r_1, r_2, and the same center is a circle with radius $|r_1 - r_2|$ and the same center.

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