Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 1241 – 1252
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doi:10.3906/mat-1901-17

The differential-symbol method of constructing the quasipolynomial solutions of two-point in time problem for nonhomogeneous partial differential equation

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Received: 06.01.2019 • **Accepted/Published Online:** 15.03.2019 • **Final Version:** 29.05.2019

Abstract: The existence of the solution of nonhomogeneous partial differential equations (PDE) of second order in time and finite or infinite order in spatial variable with quasipolynomial right-hand side is proved. This solution satisfies the homogeneous two-point in time conditions. The differential-symbol method for constructing the solution of the problem is proposed. The examples of applying this method for solving some two-point problems for PDE are suggested.

Key words: Characteristic determinant of a problem, two-point in time conditions, differential-symbol method, differential-functional equation

1. Introduction

A well-known fact in the theory of ordinary differential equation (ODE) should be reminded. Let us consider nonhomogeneous ODE

$$L\Big(\frac{d}{dt}\Big)T(t) \equiv \Big(\frac{d^2}{dt^2} + a_1\frac{d}{dt} + a_2\Big)T(t) = f(t), \tag{1.1}$$

where $a_1, a_2 \in \mathbb{C}$, $f(t)$ is quasipolynomial of the form

$$f(t) = Q(t)e^{\alpha t}, \tag{1.2}$$

in which $\alpha \in \mathbb{C}$, $Q(t)$ is polynomial of degree $n \in \mathbb{Z}_+$. If α is a zero of the polynomial $L(\lambda) = \lambda^2 + a_1\lambda + a_2$ of multiplicity $p \in \{0, 1, 2\}$ then the partial solution of equation (1.1) can be found by the method of indefinite coefficients in the form

$$T(t) = R(t)e^{\alpha t}.$$

$R(t)$ is a polynomial of degree $n + p$ with indefinite coefficients. The case $p = 0$ means that α is not zero of $L(\lambda)$.

According to the differential-symbol method (see [7, 13]), the partial solution of equation (1.1) can be found by the formula

$$T(t) = f\Big(\frac{\partial}{\partial\lambda}\Big)\left\{\frac{e^{\lambda t} - T_0(t) - \lambda T_1(t)}{L(\lambda)}\right\}\Bigg|_{\lambda=0}. \tag{1.3}$$

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2010 *AMS Mathematics Subject Classification:* 35G15

In (1.3), $\{T_0(t), T_1(t)\}$ is the normal fundamental system of solutions of equation $L(d/dt)T(t) = 0$. Moreover, the curly brackets of formula (1.3) contain the entire function of parameter λ. In particular, if $f(t)$ has the form (1.2) then partial solution of equation (1.1) is calculated by the following formula

$$T(t) = Q\Big(\frac{\partial}{\partial\lambda}\Big) \left\{ \frac{e^{\lambda t} - T_0(t) - \lambda T_1(t)}{L(\lambda)} \right\}\Bigg|_{\lambda=\alpha}. \tag{1.4}$$

$T(t)$ can be found by differentiation of finite order.

It should be noted that the formula (1.3) determines not only partial solution of equation (1.1) but also the solution of the Cauchy problem for equation (1.1) with homogeneous initial conditions

$$T(0) = T'(0) = 0.$$

In this paper, for nonhomogeneous partial differential equations (PDE) of second order in time and finite or infinite order in spatial variable with quasipolynomial right-hand side, the method of constructing such solution that satisfies homogeneous two-point in time conditions is proposed. We study the case when the coefficient at x of exponential function in right-hand side of equation is zero of the characteristic determinant of the problem.

Problems for PDE with multipoint conditions in time describe many physical, economic, demographic, and other processes. Such problems have a simple interpretation of process observations at different moments of time. An example of such mathematical model is the two-point problem

$$\frac{\partial^2 U}{\partial t^2} - a^2 \frac{\partial^2 U}{\partial x^2} = f(t,x), \;\; t \in \mathbb{R}, \;\; x \in \mathbb{R}, \;\; a \in \mathbb{R}\backslash\{0\}, \tag{1.5}$$

$$U(0,x) = 0, \;\; U(h,x) = 0, \;\;\; x \in \mathbb{R}. \tag{1.6}$$

Problem (1.5), (1.6) describes the oscillation process of the boundless string and involves determining the position of the string at any moment of time t and at an arbitrary point x. Moreover, the string oscillates under the influence of external force $f(t,x)$ and the profiles of the string at two moments of time $t = 0$ and $t = h > 0$ are identical (trivial).

Problems with such conditions for PDE are generalization of Valle-Poussin problems [20, 25, 26].

The first results of studying the n-point problems for hyperbolic PDE were presented in work [21]. Furthermore, the conditions of existence and uniqueness of the solution of multipoint problems for PDE and systems of PDE were studied in bounded domains on the basis of a metric approach (see works [11, 22–24] and bibliography in them) and in unbounded domains using the Fourier transform technique ([3, 4, 27]).

Consequently, the establishment of conditions of the problem solvability for PDE with multipoint in time conditions is significant not only for constructing general theory of boundary-value problems but also for practical problems.

The papers [1, 2, 5, 6, 8, 19] are devoted to the constructing polynomial and quasipolynomial solutions for PDE and boundary-value problems for them.

The application of the differential-symbol method in investigation of problems for PDE with conditions in time variable was considered in [7, 13–15, 17, 18].

Symbol calculus which is inherent for the differential-symbol method can be seen in works [9, 10].

2. Problem statement

Let $a_1(\frac{d}{dx}) = \sum\limits_{j=0}^{\infty} a_{1j}\dfrac{d^j}{dx^j}$, $a_2(\dfrac{d}{dx}) = \sum\limits_{j=0}^{\infty} a_{2j}\dfrac{d^j}{dx^j}$ be differential expressions with complex coefficients a_{10}, a_{20}, a_{11}, $a_{21}, \dots$. Their actions on an infinitely differentiable function $W = W(x)$ with respect to $x \in \mathbb{R}$ are understood as follows:

$$a_k(\frac{d}{dx})W(x) = \sum_{j=0}^{\infty} a_{kj}\frac{d^j W}{dx^j}, \;\; k \in \{1,2\}.$$

In addition, let us consider the differential polynomials $b_{01}(\frac{d}{dx})$, $b_{02}(\frac{d}{dx})$, $b_{11}(\frac{d}{dx})$, and $b_{12}(\frac{d}{dx})$ with complex coefficients. Their symbols are polynomials that $b_{01}(\nu)$, $b_{02}(\nu)$ and $b_{11}(\nu)$, $b_{12}(\nu)$ respectively, do not have the same complex roots.

In present paper, the solvability of problem

$$L\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\Big)U(t,x) \equiv \Big[\frac{\partial^2}{\partial t^2} + 2a_1\Big(\frac{\partial}{\partial x}\Big)\frac{\partial}{\partial t} + a_2\Big(\frac{\partial}{\partial x}\Big)\Big]U(t,x) = f(t,x), \tag{2.1}$$

$$l_k U(t,x) \equiv b_{k1}\Big(\frac{\partial}{\partial x}\Big)U(kh,x) + b_{k2}\Big(\frac{\partial}{\partial x}\Big)\frac{\partial U}{\partial t}(kh,x) = 0, \;\; k \in \{0,1\}, \tag{2.2}$$

where h is a positive number and $f(t,x)$ is a nontrivial quasipolynomial of special form, is studied in domain $(t,x) \in \mathbb{R}^2$.

For nonempty set $P \subseteq \mathbb{C}$, let us introduce the class of quasipolynomials $K_{\mathbb{C},\,P}$, that is the class of functions of the form

$$f(t,x) = \sum_{j=1}^{m}\sum_{k=1}^{N} f_{kj}(t,x)e^{\beta_k t + \alpha_j x}, \;\; m, N \in \mathbb{N}. \tag{2.3}$$

Moreover, $f_{11}(t,x), \dots, f_{Nm}(t,x)$ are nonzero polynomials with complex coefficients, $\beta_1, \dots, \beta_N$ are pairwise different complex numbers, and $\alpha_1, \dots, \alpha_m$ are pairwise different complex numbers that belong to P.

For each quasipolynomial (2.3), replacing t with $\frac{\partial}{\partial \lambda}$ and x with $\frac{\partial}{\partial \nu}$, we obtain the quasipolynomial differential expression $f(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \nu})$, which acts on the entire function $\Gamma(\lambda, \nu)$ of parameters λ and ν by the formula

$$f\Big(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \nu}\Big)\Gamma(\lambda,\nu) \equiv \sum_{j=1}^{m}\sum_{k=1}^{N} f_{kj}\Big(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \nu}\Big)\Gamma(\lambda + \beta_k, \nu + \alpha_j). \tag{2.4}$$

Formula (2.4) contains the derivatives of $\Gamma(\lambda,\nu)$ only finite order.

According to PDE (2.1), we consider homogeneous ODE of the second order

$$\frac{d^2 T}{dt^2} + 2a_1(\nu)\frac{dT}{dt} + a_2(\nu)T = 0, \tag{2.5}$$

which is obtained from (2.1) by substitution of $\frac{\partial}{\partial x}$ with parameter ν (further we consider it as complex parameter).

By $T_0(t,\nu)$ and $T_1(t,\nu)$, we denote the solutions of equation (2.5), which satisfy the conditions

$$\frac{d^k T_j}{dt^k}(0,\nu) = \delta_{kj} = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \qquad k, j \in \{0,1\}.$$

The functions $T_0(t,\nu)$ and $T_1(t,\nu)$ form the fundamental system of solutions of equation (2.5) which is normal at the point $t = 0$. Let us construct the system $\{T_2(t,\nu), T_3(t,\nu)\}$ of solutions of equation (2.5) which for $k, j \in \{0,1\}$ satisfies the conditions:

$$b_{k1}(\nu) T_{2+j}(kh,\nu) + b_{k2}(\nu) \frac{dT_{2+j}}{dt}(kh,\nu) = \delta_{kj}, \qquad k \in \{0,1\}. \tag{2.6}$$

The difference from zero of the characteristic determinant

$$\Delta(\nu) = \begin{vmatrix} b_{01}(\nu) & b_{02}(\nu) \\ \widetilde{b}_{01}(\nu) & \widetilde{b}_{02}(\nu) \end{vmatrix}, \tag{2.7}$$

where

$$\widetilde{b}_{01} = b_{11}(\nu) T_0(h,\nu) + b_{12}(\nu) \frac{dT_0}{dt}(h,\nu), \ \ \widetilde{b}_{02} = b_{11}(\nu) T_1(h,\nu) + b_{12}(\nu) \frac{dT_1}{dt}(h,\nu),$$

is the condition of existence of the functions $T_2(t,\nu)$ and $T_3(t,\nu)$ for fixed $\nu \in \mathbb{C}$.

In the present work, we study the case where the set of zeros M

$$M = \{\nu \in \mathbb{C}\colon \ \Delta(\nu) = 0\} \tag{2.8}$$

and its complement $\overline{M}$ to the space $\mathbb{C}$ are nonempty sets.

Let $D(\nu) = \sqrt{a_1^2(\nu) - a_2(\nu)}$. Then for $\nu \in \mathbb{C}\backslash M$, the functions $T_2(t,\nu)$ and $T_3(t,\nu)$ are quasipolynomials in the variable t of the form

$$T_2(t,\nu) = e^{-a_1(\nu)(t+h)} \frac{\big(b_{11}(\nu) - a_1(\nu) b_{12}(\nu)\big) S(h-t,\nu) + b_{12}(\nu) C(h-t,\nu)}{\Delta(\nu)},$$

$$T_3(t,\nu) = e^{-a_1(\nu)t} \frac{\big(b_{01}(\nu) - a_1(\nu) b_{02}(\nu)\big) S(t,\nu) - b_{02}(\nu) C(t,\nu)}{\Delta(\nu)}.$$

Quasipolynomials $C(t,\nu)$ and $S(t,\nu)$ are defined by the formulas

$$C(t,\nu) = \cosh\big[tD(\nu)\big], \qquad S(t,\nu) = \begin{cases} \dfrac{\sinh\big[tD(\nu)\big]}{D(\nu)}, & D(\nu) \neq 0, \\ t, & D(\nu) = 0. \end{cases}$$

In the paper [16] (theorem 3.2), the existence and uniqueness of solution of problem (2.1), (2.2) in the class $K_{\mathbb{C},\mathbb{C}\backslash M}$ is proved, if the right-hand side of equation (2.1) has the form (2.3); moreoverü $\alpha_1, \ldots, \alpha_m$ do not belong to the set (2.8). Using (2.4), we can find this solution by the formula

$$U(t,x) = f\Big(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\Big)\Big\{F(t,\lambda,\nu)e^{\nu x}\Big\}\Big|_{\lambda=\nu=0} = \sum_{j=1}^{m}\sum_{k=1}^{N} f_{kj}\Big(\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\nu}\Big)\Big\{F(t,\lambda,\nu)e^{\nu x}\Big\}\Big|_{\substack{\lambda=\beta_k, \\ \nu=\alpha_j}}, \tag{2.9}$$

where

$$F\,(t,\lambda,\nu)=\frac{e^{\lambda t}-[b_{01}(\nu)+\lambda b_{02}(\nu)]\,T_2\,(t,\nu)-[b_{11}(\nu)+\lambda b_{12}(\nu)]\,e^{\lambda h}T_3\,(t,\nu)}{L\,(\lambda,\nu)}.$$

It should be noted that the function in curly brackets of formula (2.9) can be represented in the form

$$\widetilde{F}\,(t,x,\lambda,\nu)\equiv F\,(t,\lambda,\nu)\,e^{\nu x}=\frac{e^{\lambda t+\nu x}-T_2\,(t,\nu)\,l_0e^{\lambda t+\nu x}-T_3\,(t,\nu)\,l_1e^{\lambda t+\nu x}}{L\,(\lambda,\nu)}. \tag{2.10}$$

Function (2.10) is analytic function in the domain $\mathbb{R}\times\mathbb{R}\times\mathbb{C}\times\mathbb{C}\backslash M$ and quasipolynomial in variables t and x if $(\lambda,\nu)\in\mathbb{C}\times\mathbb{C}\backslash M$.

The present work is devoted to the case when the right-hand side of equation (2.1) belongs to $K_{\mathbb{C},\,M}$, that is the function $f(t,x)$ has form (2.4) and $\alpha_1,\ldots,\alpha_m\in M$. The existence of a solution of problem (2.1), (2.2) in the class $K_{\mathbb{C},\,M}$ will be proved, and the differential-symbol method of constructing solution of problem (2.1), (2.2) will be proposed. It should be noted that nontrivial solutions of the corresponding homogeneous problem exist in the class $K_{\mathbb{C},\,M}$ [12].

3. Main results

Let $f(t,x)$ belong to $K_{\mathbb{C},\,M}$ and have the form

$$f(t,x)=Q(t,x)e^{\beta t+\alpha x}, \tag{3.1}$$

where $Q(t,x)$ is nontrivial polynomial of variables t and x of degree $n\in\mathbb{Z}_+$ in variable x, α is zero of multiplicity $p\in\mathbb{N}$ of the function $\Delta(\nu)$, β is an arbitrary complex number.

Theorem 3.1 *Let the function $f(t,x)$ of equation (2.1) have the form (3.1), $q\equiv(n+1)p$, and $\widetilde{\Delta}(\nu)\equiv\Delta^{n+1}(\nu)$. Then the solution of problem (2.1), (2.2) exists in the class $K_{\mathbb{C},\,M}$ and it can be found by the following formula:*

$$U(t,x)=\frac{\left(\dfrac{\partial}{\partial\nu}\right)^{q}\rho(t,x,\beta,\nu)\Big|_{\nu=\alpha}}{\widetilde{\Delta}^{(q)}(\alpha)}, \tag{3.2}$$

where

$$\rho(t,x,\beta,\nu)=\widetilde{\Delta}(\nu)\,Q\Big(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\Big)\Big\{\widetilde{F}(t,x,\lambda,\nu)\Big\}\Big|_{\lambda=\beta}, \tag{3.3}$$

$\widetilde{F}(t,x,\lambda,\nu)$ is the function (2.10), $Q\big(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\big)$ is differential polynomial obtained from $Q(t,x)$ by substitution of t and x with $\frac{\partial}{\partial\lambda}$ and $\frac{\partial}{\partial\nu}$, respectively, $\widetilde{\Delta}^{(q)}(\alpha)\equiv\frac{d^q\widetilde{\Delta}}{d\nu^q}\big|_{\nu=\alpha}$.

Proof The function $\widetilde{F}(t,x,\lambda,\nu)$ of form (2.10) has such property: zeros $\lambda_0(\nu)$ and $\lambda_1(\nu)$ of polynomial $L(\lambda,\nu)=\lambda^2+2a_1(\nu)\lambda+a_2(\nu)=(\lambda-\lambda_0(\nu))(\lambda-\lambda_1(\nu))$ are removable and zeros of the determinant $\Delta(\nu)$, which are presented in $T_2(t,\nu)$ and $T_3(t,\nu)$, are the poles. After the action of the differential polynomial $Q\big(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\big)$ on the function $\widetilde{F}(t,x,\lambda,\nu)$, its poles increase their order; however, multiplying by function $\widetilde{\Delta}(\nu)$, we get the entire function

$$\rho(t,x,\lambda,\nu)=\widetilde{\Delta}(\nu)\,Q\Big(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\Big)\{\widetilde{F}(t,x,\lambda,\nu)\}$$

in parameter λ and ν.

In addition, since $\Delta^{(p)}(\alpha) \neq 0$ and

$$\widetilde{\Delta}^{(q)}(\alpha) = \frac{q\,!}{(p\,!)^{n+1}}[\Delta^{(p)}(\alpha)]^{n+1} \neq 0, \tag{3.4}$$

then formula (3.2) defines some quasipolynomial $U(t,x)$ from the class $K_{\mathbb{C},\,M}$.

Let us show that function (2.10) satisfies nonhomogeneous equation

$$L\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\Big)\widetilde{F}(t,x,\lambda,\nu) = e^{\lambda t+\nu x}. \tag{3.5}$$

The action of the differential expression $L(\frac{\partial}{\partial t}, \frac{\partial}{\partial x})$ on function (3.2) should be calculated:

$$L\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\Big)\frac{\Big(\frac{\partial}{\partial \nu}\Big)^q \rho(t,x,\beta,\nu)\Big|_{\nu=\alpha}}{\widetilde{\Delta}^{(q)}(\alpha)} = \frac{\Big(\frac{\partial}{\partial \nu}\Big)^q L\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\Big)\rho(t,x,\beta,\nu)\Big|_{\nu=\alpha}}{\widetilde{\Delta}^{(q)}(\alpha)}$$

$$= \frac{\Big(\frac{\partial}{\partial \nu}\Big)^q \Big[\widetilde{\Delta}(\nu)\, Q\Big(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \nu}\Big) L\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\Big)\Big\{\widetilde{F}(t,x,\lambda,\nu)\Big\}\Big]\Big|_{\substack{\lambda=\beta,\\ \nu=\alpha}}}{\widetilde{\Delta}^{(q)}(\alpha)} = \frac{\Big(\frac{\partial}{\partial \nu}\Big)^q \Big[\widetilde{\Delta}(\nu)\, Q\Big(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \nu}\Big) e^{\lambda t+\nu x}\Big]\Big|_{\substack{\lambda=\beta,\\ \nu=\alpha}}}{\widetilde{\Delta}^{(q)}(\alpha)}$$

$$= \frac{\Big(\frac{\partial}{\partial \nu}\Big)^q \Big[\widetilde{\Delta}(\nu) Q(t,x) e^{\beta t+\nu x}\Big]\Big|_{\nu=\alpha}}{\widetilde{\Delta}^{(q)}(\alpha)} = Q(t,x)e^{\beta t}\frac{\Big(\frac{\partial}{\partial \nu}\Big)^q \Big[\widetilde{\Delta}(\nu) e^{\nu x}\Big]\Big|_{\nu=\alpha}}{\widetilde{\Delta}^{(q)}(\alpha)}$$

$$= Q(t,x)\, e^{\beta t}\, \frac{\sum\limits_{k=0}^{q} C_q^k x^{q-k} e^{\alpha x} \widetilde{\Delta}^{(k)}(\alpha)}{\widetilde{\Delta}^{(q)}(\alpha)} = Q(t,x)e^{\beta t+\alpha x} = f(t,x).$$

In the last line of the formula, the property of the root α of the multiplicity q is used, that is

$$\widetilde{\Delta}(\alpha) = \widetilde{\Delta}'(\alpha) = \ldots = \widetilde{\Delta}^{(q-1)}(\alpha) = 0.$$

Therefore, function (3.2) satisfies equation (2.1).

Let us show that function (3.2) satisfies two-point conditions (2.2). Taking into account (2.6) for $k \in \{0,1\}$, we get the following conditions for function (2.10):

$$l_k\widetilde{F}(t,x,\lambda,\nu) = \frac{l_k e^{\lambda t+\nu x} - \delta_{k0} l_0 e^{\lambda t+\nu x} - \delta_{k1} l_1 e^{\lambda t+\nu x}}{L(\lambda,\nu)} = 0.$$

For $k \in \{0,1\}$, we have

$$l_k U(t,x) = \frac{\Big(\frac{\partial}{\partial \nu}\Big)^q l_k \rho(t,x,\beta,\nu)\Big|_{\nu=\alpha}}{\widetilde{\Delta}^{(q)}(\alpha)} = \frac{\Big(\frac{\partial}{\partial \nu}\Big)^q \Big[\widetilde{\Delta}(\nu)\, Q\Big(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \nu}\Big) l_k \widetilde{F}(t,x,\lambda,\nu)\Big]\Big|_{\substack{\lambda=\beta,\\ \nu=\alpha}}}{\widetilde{\Delta}^{(q)}(\alpha)} = 0.$$

$\square$

Remark 3.2 *After setting $p=0$ in formula (3.2), we get formula (2.9).*

In Theorem 3.1, we consider the case of monomial quasipolynomial $f(t,x)$ of form (3.1) which can be generalized.

Let $f(t,x)$ have form (2.3) and $(m,N) \neq (1,1)$; moreover, the numbers $\alpha_1,\ldots,\alpha_m$ are zeros of function (2.7) of multiplicities $p_1,\ldots,p_m$. Formula (2.3) contains at least two polynomials $f_{kj}(t,x)$. By $n_{kj} \in \mathbb{Z}_+$, we denote the degree of polynomial $f_{kj}(t,x)$ in variable x where $k=\overline{1,N}$, $j=\overline{1,m}$.

Theorem 3.3 *Let the function $f(t,x)$ of equation (2.1) have form (2.3) and $q_{kj} \equiv (n_{kj}+1)p_j$. Then the solution of problem (2.1), (2.2) exists in the class $K_{\mathbb{C},\,M}$ and can be found by the formula*

$$U(t,x) = \sum_{j=1}^{m}\sum_{k=1}^{N} \frac{\left(\dfrac{\partial}{\partial \nu}\right)^{q_{kj}} \rho_{kj}(t,x,\beta_k,\nu)\Big|_{\nu=\alpha_j}}{[\Delta^{n_{kj}+1}]^{(q_{kj})}(\alpha_j)}, \tag{3.6}$$

where

$$\rho_{kj}(t,x,\beta_k,\nu) = \Delta^{n_{kj}+1}(\nu) f_{kj}\Big(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \nu}\Big)\Big\{\widetilde{F}(t,x,\lambda,\nu)\Big\}\Big|_{\lambda=\beta_k},$$

$\widetilde{F}(t,x,\lambda,\nu)$ is function (2.10).

Proof For each monomial summand $f_{kj}(t,x)e^{\beta_k t+\alpha_j x}$ of the function $f(t,x)$ of form (2.3), the solutions $U_{kj}(t,x)$ of problem (2.1), (2.2) can be found by substitution of such value: $\alpha=\alpha_j$, $\beta=\beta_k$, $p=p_j$, $n=n_{kj}$, $q=q_{kj}$, $\rho=\rho_{kj}$, $Q=f_{kj}$ in formulas (3.2) and (3.3). By the principle of superposition of solutions of linear differential equation, the solution of problem (2.1), (2.2) can be found as a sum of mN summands

$$U(t,x) = \sum_{j=1}^{m}\sum_{k=1}^{N} U_{kj}(t,x), \tag{3.7}$$

that is in form (3.6). $\square$

4. Examples of solving the problems

Examples of constructing the solutions of two-point problems for nonhomogeneous PDE with quasipolynomial right-hand side are considered.

Example 4.1 *Let us find solutions of the differential-functional equation*

$$\frac{\partial^2 U(t,x)}{\partial t^2} - \Big(1 + 2\frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2}\Big)U(t,x+2) = e^{-t-x}, \ \ (t,x)\in\mathbb{R}^2, \tag{4.1}$$

which satisfy the two-point conditions

$$\begin{aligned} &\Big(1+\frac{\partial}{\partial x}\Big)U(0,x) + \frac{\partial^2 U}{\partial t \partial x}(0,x) = 0, \\ &\Big(1+\frac{\partial}{\partial x}\Big)U(1,x) + \frac{\partial^2 U}{\partial t \partial x}(1,x) = 0, \ \ x\in\mathbb{R}. \end{aligned} \tag{4.2}$$

∇ Problem (4.1), (4.2) is problem (2.1), (2.2) for which $a_1(\nu)=0,\ a_2(\nu)=-(1+\nu)^2e^{2\nu},\ b_{01}(\nu)=b_{11}(\nu)=1+\nu,$ $b_{02}(\nu)=b_{12}(\nu)=\nu,\ h=1,\ f(t,x)=e^{-t-x}$.

Differential-functional equation (4.1) can be written as differential equation of infinite order in variable x:

$$\Big[\frac{\partial^2}{\partial t^2}-\big(1+\frac{\partial}{\partial x}\big)^2e^{2\frac{\partial}{\partial x}}\Big]U(t,x)=e^{-t-x}.$$

The solutions $T_2(t,\nu)$ and $T_3(t,\nu)$ of the ODE

$$\Big[\frac{d^2}{dt^2}-(1+\nu)^2e^{2\nu}\Big]T(t,\nu)=0$$

have the following form

$$T_2(t,\nu)=\frac{e^{-\nu}\sinh[(1+\nu)e^{\nu}(1-t)]+\nu\cosh[(1+\nu)e^{\nu}(1-t)]}{\Delta(\nu)},$$

$$T_3(t,\nu)=\frac{e^{-\nu}\sinh[(1+\nu)e^{\nu}t]-\nu\cosh[(1+\nu)e^{\nu}t]}{\Delta(\nu)},$$

where

$$\Delta(\nu)=(1+\nu)\big[1-\nu^2e^{2\nu}\big]e^{-\nu}\sinh[(1+\nu)e^{\nu}].$$

The set M has the form

$$M=\{\nu\in\mathbb{C}:\ \Delta(\nu)=0\}.$$

Since $f(t,x)$ has monomial quasipolynomial form and $\alpha=-1$ is a zero of multiplicity $p=2$ of the function $\Delta(\nu)$, Theorem 3.1 can be used.

For quasipolynomial $f(t,x)=e^{-t-x}\in K_{\mathbb{C},M}$, we have $\alpha=-1,\ \beta=-1,\ Q(t,x)=1,\ p=2,\ n=0.$

The function $\widetilde{F}(t,x,\lambda,\nu)$ has the form

$$\widetilde{F}(t,x,\lambda,\nu)=\frac{e^{\lambda t}-(1+\nu+\lambda\nu)[T_2(t,\nu)+e^{\lambda}T_3(t,\nu)]}{\lambda^2-(1+\nu)^2e^{2\nu}}e^{\nu x}.$$

We have

$$\begin{aligned}\rho(t,x,-1,\nu)=\Delta(\nu)\,\widetilde{F}(t,x,-1,\nu)=\Bigg\{&\frac{e^{-t}(1+\nu)\big(1-\nu^2e^{2\nu}\big)e^{-\nu}\sinh[(1+\nu)e^{\nu}]}{1-(1+\nu)^2e^{2\nu}}\\&-\frac{e^{-\nu}\sinh[(1+\nu)e^{\nu}(1-t)]+\nu\cosh[(1+\nu)e^{\nu}(1-t)]}{1-(1+\nu)^2e^{2\nu}}\\&-e^{-1}\frac{e^{-\nu}\sinh[(1+\nu)e^{\nu}t]-\nu\cosh[(1+\nu)e^{\nu}t]}{1-(1+\nu)^2e^{2\nu}}\Bigg\}e^{\nu x}.\end{aligned}$$

According to Theorem 3.1, the solution of problem (4.1), (4.2) can be found by formula (3.2):

$$U(t,x)=\frac{\Big(\dfrac{\partial}{\partial\nu}\Big)^2\rho(t,x,-1,\nu)\Big|_{\nu=-1}}{\Delta''(-1)}.$$

Having calculated

$$\Delta''(-1) = 2(1 - e^{-2}),$$

we obtain

$$U(t,x) = e^{-t-x} - \frac{1}{2(1-e^{-2})}\Big[e^{-3}(t^2+2) - e^{-2}(t^2-2t+3) - x(x+2t-4) + e^{-1}x(x+2t-2)\Big]e^{-x}. \quad (4.3)$$

The found solution (4.3) of problem (4.1), (4.2) in the class $K_{\mathbb{C},M}$ is not unique, because nontrivial solutions of the corresponding problem exist in this class [12]. Such solutions, for example, are the functions $U(t,x) = e^{-x}$, $U(t,x) = (1-t-x)e^{-x}$. $\triangle$

Example 4.2 *In plane $(t,x) \in \mathbb{R}^2$ let us find the solutions of the two-point problem*

$$\begin{gathered}\Big[\frac{\partial^2}{\partial t^2} - 2\frac{\partial^3}{\partial t \partial x^2} + \frac{\partial^4}{\partial x^4}\Big]U(t,x) = t + x^2 + txe^{-x}, \\ \frac{\partial U}{\partial x}(0,x) + \frac{\partial U}{\partial t}(0,x) = 0, \ \ \frac{\partial U}{\partial x}(1,x) + \frac{\partial U}{\partial t}(1,x) = 0.\end{gathered} \quad (4.4)$$

∇ Problem (4.4) is problem (2.1), (2.2), in which $a_1(\nu) = -\nu^2$, $a_2(\nu) = \nu^4$, $b_{01}(\nu) = b_{11}(\nu) = \nu$, $b_{02}(\nu) = b_{12}(\nu) = 1$, $h = 1$, $f(t,x) = t + x^2 + txe^{-x}$.

The solutions $T_2(t,\nu)$ and $T_3(t,\nu)$ of the ODE

$$\Big[\frac{d^2}{dt^2} - 2\nu^2\frac{d}{dt} + \nu^4\Big]T(t,\nu) = 0$$

have the form:

$$T_2(t,\nu) = e^{\nu^2(t+1)}\frac{(\nu^2+\nu)(1-t)+1}{\Delta(\nu)}, \quad T_3(t,\nu) = e^{\nu^2 t}\frac{(\nu^2+\nu)t-1}{\Delta(\nu)},$$

where

$$\Delta(\nu) = \nu^2(\nu+1)^2 e^{\nu^2}.$$

The set M contains only 0 and 1.

For constructing the solution of problem (4.4), Theorem 3.3 can be used, as the function $f(t,x) = t + x^2 + txe^{-x} \in K_{\mathbb{C},M}$ is the sum of two quasipolynomials $f_1(t,x) = Q_1(t,x)e^{\beta_1 t + \alpha_1 x}$ and $f_2(t,x) = Q_2(t,x)e^{\beta_2 t + \alpha_2 x}$, where $\alpha_1 = 0$, $\beta_1 = 0$, $Q_1(t,x) = t + x^2$, $n_1 = 2$, $\alpha_2 = -1$, $\beta_2 = 0$, $Q_2(t,x) = tx$, $n_2 = 1$.

Since numbers $\alpha_1 = 0$ and $\alpha_2 = -1$ are double zeros of $\Delta(\nu)$, then $p_1 = p_2 = 2$.

For problem (4.4), function (2.10) has the form

$$\widetilde{F}(t,x,\lambda,\nu) = \frac{e^{\lambda t} - (\nu+\lambda)\Big[T_2(t,\nu) + e^{\lambda}T_3(t,\nu)\Big]}{(\lambda-\nu^2)^2}e^{\nu x}.$$

The solution $U(t,x)$ of problem (4.4) can be found as the sum of the solution $U_1(t,x)$ of problem (4.4) with $f_1(t,x) = t + x^2$ and the solution $U_2(t,x)$ of problem (4.4) with $f_2(t,x) = txe^{-x}$.

For finding the solution $U_1(t,x)$, function (3.3) is constructed:

$$\rho_1(t,x,\lambda,\nu) = \widetilde{\Delta}(\nu)\,Q_1\Big(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\nu}\Big)\Big\{\widetilde{F}(t,x,\lambda,\nu)\Big\} = \widetilde{\Delta}(\nu)\,\Big(\frac{\partial}{\partial\lambda}+\frac{\partial^2}{\partial\nu^2}\Big)\Big\{\widetilde{F}(t,x,\lambda,\nu)\Big\}$$
$$= \widetilde{\Delta}(\nu)\,\Big\{\widetilde{F}'_{\lambda}(t,x,\lambda,\nu)+\widetilde{F}''_{\nu\nu}(t,x,\lambda,\nu)\Big\},$$

where $\widetilde{\Delta}(\nu) = \Delta^3(\nu)$.

Then

$$\rho_1(t,x,0,\nu) = \widetilde{\Delta}(\nu)\,\Big\{\widetilde{F}'_{\lambda}(t,x,\lambda,\nu)+\widetilde{F}''_{\nu\nu}(t,x,\lambda,\nu)\Big\}\Big|_{\lambda=0}.$$

Since $\widetilde{F}(t,x,\lambda,\nu)$ is an entire function in parameter λ, then we get

$$\widetilde{F}(t,x,0,\nu) = \frac{1-\nu\big[T_2(t,\nu)+T_3(t,\nu)\big]}{\nu^4}e^{\nu x},$$
$$\widetilde{F}'_{\nu}(t,x,\lambda,\nu)\Big|_{\lambda=0} = \frac{\partial}{\partial\nu}\widetilde{F}(t,x,0,\nu), \qquad \widetilde{F}''_{\nu\nu}(t,x,\lambda,\nu)\Big|_{\lambda=0} = \frac{\partial^2}{\partial\nu^2}\widetilde{F}(t,x,0,\nu).$$

The solution $U_1(t,x)$ can be found by formula (3.2), where $q_1 \equiv (n_1+1)p_1 = 6$:

$$U_1(t,x) = \frac{\Big(\dfrac{\partial}{\partial\nu}\Big)^6\rho_1(t,x,0,\nu)\Big|_{\nu=0}}{\widetilde{\Delta}^{(6)}(0)}.$$

Let us calculate the derivative $\widetilde{\Delta}^{(6)}(0)$ by formula (3.4): $\widetilde{\Delta}^{(6)}(0) = \dfrac{6\,!}{(3\,!)^3}\big[\Delta''(0)\big]^3 = 720.$

We get

$$\Big(\frac{\partial}{\partial\nu}\Big)^6\rho_1(t,x,0,\nu)$$
$$= 60\Big(10t^3+6t^2(x-2)^2-2t(2x^3-9x^2+27x+437)+x^4-6x^3+27x^2+874x+3416\Big).$$

Hence,

$$U_1(t,x) = \frac{1}{12}\Big(10t^3+6t^2(x-2)^2-2t(2x^3-9x^2+27x+437)+x^4-6x^3+27x^2+874x+3416\Big). \tag{4.5}$$

For finding $U_2(t,x)$, function (3.3) is constructed:

$$\rho_2(t,x,\lambda,\nu) = \widetilde{\Delta}(\nu)\,\frac{\partial^2}{\partial\lambda\partial\nu}\Big\{\widetilde{F}(t,x,\lambda,\nu)\Big\} = \widetilde{\Delta}(\nu)\,\widetilde{F}''_{\lambda\nu}(t,x,\lambda,\nu),$$

where $\widetilde{\Delta}(\nu) = \Delta^2(\nu)$.

Then $q_2 \equiv (n_2+1)p_2 = 4$, and by formula (3.2), we have

$$U_2(t,x) = \frac{\Big(\dfrac{\partial}{\partial\nu}\Big)^4\rho_2(t,x,0,\nu)\Big|_{\nu=-1}}{\widetilde{\Delta}^{(4)}(-1)}.$$

Let us calculate

$$\widetilde{\Delta}^{(4)}(-1) = \frac{4\,!}{(2\,!)^2}\big[\Delta''(-1)\big]^2 = 6\cdot(2e)^2 = 24e^2,$$

$$\Big(\frac{\partial}{\partial\nu}\Big)^4\rho_2(t,x,0,\nu)\Big|_{\nu=-1} = 4e^{2-x}\Big\{6t(4+x)+12(6+x)+e^t\big(206-6t^2+4t^3-24x+21x^2+x^3 \\ -3t(-20+12x+x^2)\big)-e^{t-1}\big(368+12t^2+8t^3+84x+51x^2+2x^3-6t(14+17x+x^2)\big)\Big\}.$$

We have

$$U_2(t,x) = \frac{1}{6}e^{-x}\Big\{6t(4+x)+12(6+x)+e^t\big(206-6t^2+4t^3-24x+21x^2+x^3 \\ -3t(-20+12x+x^2)\big)-e^{t-1}\big(368+12t^2+8t^3+84x+51x^2+2x^3-6t(14+17x+x^2)\big)\Big\}. \quad (4.6)$$

Therefore, the sum of functions (4.5) and (4.6) is the solution of problem (4.4).

It should be emphasized that the found solution of problem (4.4) is not unique, since nontrivial solutions of corresponding homogeneous problem exist [12] in the class $K_{\mathbb{C},\,M}$, for example, $U(t,x) = x-t$, $U(t,x) = e^{t-x}$. $\triangle$

5. Conclusions

Theorems 3.1 and 3.3 concerning the existence of a solution of nonhomogeneous PDE (2.1) of the second order in time and generally infinite order in spatial variable with quasipolynomial right-hand side are proved. This solution satisfies two-point conditions (2.2). The differential-symbol method of constructing the solution of problem (2.1), (2.2) (formulas (3.2) and (3.7)) is proposed. Besides, we have presented the examples of applying this method to solving some two-point problems, that is for differential-functional equation (infinite order in spatial variable) (example 4.1) and for PDE of fourth order in variable x (example 4.2).

References

[1] Bondarenko B.Bazisnyye Sistemy Polinomial'nykh I Kvazipolinomial'nykh Resheniy Uravneniy v Chastnykh Proizvodnykh. Tashkent, Uzbekistan: Fan, 1987 (in Russian).

[2] Bondarenko B, Filatov A. Kvazipolinominal'nyye Funktsii I Ikh Prilozheniya K Zadacham Teorii Uprugosti. Tashkent, Uzbekistan: Fan, 1978 (in Russian).

[3] Borok VM. Klassy yedinstvennosti resheniya krayevoy zadachi v beskonechnom sloye. Dokl Akad Nauk SSSR+ 1968; 183: 995-998 (in Russian).

[4] Borok V, Perelman M. O klassakh yedinstvennosti, resheniya mnogotochechnoy krayevoy zadachi v beskonechnom sloye. Izv Vuz Mat+ 1973; 8: 29-34 (in Russian).

[5] Hayman W, Shanidze Z. Polynomial solutions of partial differential equations. Methods and Applications of Analysis 1999; 6: 97-108.

[6] Hile G, Stanoyevitch A. Heat polynomial analogous for equations with higher order time derivatives. J Math Anal Appl 2004; 295: 595-610.

[7] Kalenyuk PI, Nytrebych ZM. On an operational method of solving initial-value problems for partial differential equations induced by generalized separation of variables. Journal of Mathematical Science 1999; 97: 3879-3887. https://doi.org/10.1007/BF02364928

[8] Karachik V. Construction of polynomial solutions to some boundary value problems for Poisson's equation. Comput Math Math Phys+ 2011; 51: 1567-1587. https://doi.org/10.1134/S0965542511090120

[9] Kempf A, Jackson DM, Morales AH. New Dirac delta function based methods with applications to perturbative expansions in quantum field theory. J Phys A-Math Theor 2014; 47: arXiv:1404.0747. http://dx.doi.org/10.1088/1751-8113/47/41/415204

[10] Kempf A, Jackson D, Morales A. How to (Parth–) Integrate by Differentiating. J Phys Conf Ser 2015; 626: arXiv:1507.04348.

[11] Kiguradze T. The Vallee-Poussin problem for higher order nonlinear hyperbolic equations. Comput Math Appl 2010; 59: 994-1002. https://doi.org/10.1016/j.camwa.2009.09.009

[12] Malanchuk O, Nytrebych Z. Homogeneous two-point problem for PDE of the second order in time variable and infinite order in spatial variables. Open Math 2017; 15: 101-110. https://doi.org/10.1515/math-2017-0009

[13] Nitrebich Z. An operator method of solving the Cauchy problem for a homogeneous system of partial differential equations. Journal of Mathematical Sciences 1996; 81: 3034-3038. https://doi.org/10.1007/BF02362589

[14] Nytrebych Z, Il'kiv V, Pukach P, Malanchuk O. On nontrivial solutions of homogeneous Dirichlet problem for partial differential equations in a layer. Kragujevac Journal of Mathematics 2018; 42: 193-207.

[15] Nytrebych Z, Malanchuk O. The differential-symbol method of solving the two-point problem with respect to time for a partial differential equation. Journal of Mathematical Sciences 2017; 224: 541-554. https://doi.org/10.1007/s10958-017-3434-0

[16] Nytrebych Z, Malanchuk O. The differential-symbol method of solving the problem two-point in time for a nonhomogeneous partial differential equation. Journal of Mathematical Sciences 2017; 227: 68-80. https://doi.org/10.1007/s10958-017-3574-2

[17] Nytrebych Z, Malanchuk O, Il'kiv V, Pukach P. Homogeneous problem with two-point in time conditions for some equations of mathematical physics. Azerbaijan Journal of Mathematics 2017; 7: 174-190.

[18] Nytrebych Z, Malanchuk O, Il'kiv V, Pukach P. On the solvability of two-point in time problem for PDE. Italian Journal of Pure and Applied Mathematics 2017; 38: 715-726.

[19] Pedersen P. A basis for polynomial solutions for systems of linear constant coefficient PDE's. Adv Math 1996; 117: 157-163. https://doi.org/10.1006/aima.1996.0005

[20] Picone M. Sui Valori Eccezionali di Un Parametro do Cui Dipend un Equazione Differentiale Lineare Ordinaria del Secondo Ordine. Pisa, Italy: Scuola Normale Superiore, 1910 (in Italian).

[21] Ptashnyk B. Zadacha tipu Valle-Pussena dlya giperbolichnikh rivnyan' iz stalimi koyefitsiêntami. Dokl Akad Nauk+ 1966; 10: 1254-1257 (in Ukrainian).

[22] Ptashnyk B. Nekorrektnyye Granichnyye Zadachi Dlya Differentsial'nykh Uravneniy S Chastnymi Proizvodnymi. Kiev, Ukraine: Naukova Dumka, 1984 (in Russian).

[23] Ptashnyk BI, Kmit' IYa, Il'kiv VS, Polishchuk VM. Nelokal'ní Krayoví Zadachí Dlya Rívnyan' Íz Chastinnimi Pokhídnimi. Kiev, Ukraine: Naukova Dumka, 2002 (in Ukrainian).

[24] Ptashnyk BI, Symotyuk MM. Multipoint problem for nonisotropic partial differential equations with constant coefficients. Ukr Math J 2003; 55: 293-310. https://doi.org/10.1023/A:1025468413500

[25] Tamarkin Ya. O Nekotorykh Obshchikh Zadachakh Teorii Obyknovennykh Differentsial'nykh Uravneniy I O Razlozhenii Proizvol'nykh Funktsiy v Ryady. Saint Petersburg, Russia: Tipo-lithography Frolova, 1917 (in Russian).

[26] Vallee-Poussin Ch. Sur l'equation differentielle lineaire du second ordre. Determination d'une integrale par deux valeurs assignees. Journ Math pure appl 1929; 8: 125-144 (in French).

[27] Vilents' IL. Klasi êdiností rozv'yazku zagal'noï krayovoï zadachí v sharí dlya sistem líníynikh diferentsíal'nikh rívnyan' u chastinnikh pokhídnikh. Dokl Akad Nauk+ 1974; 3: 195-197 (in Ukrainian).