Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 1253 – 1262
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doi:10.3906/mat-1809-43

A short note on some arithmetical properties of the integer part of αp

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Received: 12.09.2018 • **Accepted/Published Online:** 18.03.2019 • **Final Version:** 29.05.2019

Abstract: Let $\alpha > 0$ be an irrational number. We study some of the arithmetical properties of $\{\lfloor \alpha p \rfloor\}_{p=2}^{\infty}$, where p denotes a prime number and $\lfloor x \rfloor$ denotes the largest integer not exceeding x.

Key words: Arithmetic progressions, Beatty sequences, exponential sums, shifted primes

1. Introduction

For $\alpha > 0$ irrational and β any real number, sequences of the form $\{\lfloor \alpha n + \beta \rfloor\}_{n=1}^{\infty}$ are called Beatty sequences. Beatty sequences may be viewed as generalized arithmetic progressions with irrational moduli. Most of the properties of integers have been extended to Beatty sequences under some restrictions on α (see, for instance, [1, 2, 4, 5, 7, 9]). Following this analogy, it might be of interest to generalize the results involving shifted primes (numbers of the form $p + a$, where p is a prime number and $a \in \mathbb{Z}$ fixed) to $\{\lfloor \alpha p \rfloor\}_{p=2}^{\infty}$. One of the most interesting examples would be the infinitude of primes in $\{\lfloor \alpha p \rfloor\}_{p=2}^{\infty}$; however, as the twin prime conjecture is still open, this problem is still open. Nevertheless, one can confirm the infinitude of primes in $\{\lfloor \alpha p \rfloor\}_{p=2}^{\infty}$ on average (see [10]) .

Thus, our goal in the present paper is to exhibit analogues of some results proved for shifted primes in our setting. For the sake of brevity we have confined ourselves to the case $\beta = 0$. Before we state our results, we introduce irrational numbers of finite type:

An irrational number α is called finite type τ if

$$\tau = \sup \left\{ \beta : \liminf_{\substack{q \to \infty \\ q \in \mathbb{N}}} q^{\beta} ||\alpha q|| = 0 \right\} < \infty,$$

where the notation $\|x\|$ is used to denote the distance from the real number x to the nearest integer. We note here that by Dirichlet's approximation theorem one has $\tau \geqslant 1$. The theorems of Khinchin [11] and of Roth [18, 19] state that $\tau = 1$ for almost all real numbers and for all irrational algebraic numbers, respectively.

1.1. Statement of main results

We first start with a result of Mirsky (see [16]), stating that primes p for which $p - a$ is free of kth powers of primes (k-free) have positive density.

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Theorem 1 *Let $k \geqslant 2$ be an integer. Supposing that $\alpha > 0$ is of type $\tau \geqslant 1$, then*

$$\#\{p \leqslant x : \lfloor \alpha p \rfloor \text{ is } k\text{-free } \} = \frac{1}{\zeta(k)}\pi(x) + O(x^{1-\frac{k-1}{(2\tau+3)k}+\varepsilon}), \tag{1}$$

where $\zeta(k)$ is the value of the Riemann zeta function at $k \geqslant 2$ and $\pi(x)$ is the number of primes not exceeding x.

Taking $k = 2$ and recalling that algebraic irrational numbers are of type 1 (see [18, 19]), then one has the following corollary:

Corollary 1 *Let $\alpha > 0$ be an algebraic irrational number.*

$$\#\{p \leqslant x : \lfloor \alpha p \rfloor \text{ is square-free } \} = \frac{6}{\pi^2}\pi(x) + O(x^{9/10+\varepsilon}),$$

for every $\varepsilon > 0$.

In the same article Mirsky derived an asymptotic formula for the number of representations of a large integer as the sum of a prime number and a k-free number. Below we give an analogue of this asymptotic.

Theorem 2 *Suppose that $\alpha > 0$ is an irrational number of type $\tau \geqslant 1$. Let $R_k(\alpha, n)$ be the number of representations of n as the sum of a k-free integer and a number of the form $\lfloor \alpha p \rfloor$ for some prime p. Then*

$$R_k(\alpha, n) = \frac{1}{\zeta(k)}\pi\left(\frac{n}{\alpha}\right) + O(n^{1-\frac{k-1}{(2\tau+3)k}+\varepsilon}). \tag{2}$$

This theorem indicates that every sufficiently large integer can be expressed as a sum of a square-free number and a number of the form $\lfloor \alpha p \rfloor$.

Secondly, we look at the Titchmarsh divisor problem: For $a \in \mathbb{N}$, Titchmarsh (see [21]) was the first who studied

$$\sum_{p \leqslant x} d(p+a),$$

where $d(n)$ is the number of positive divisors of n. Since then, this problem is called the Titchmarsh divisor problem. Under GRH for all Dirichlet L-functions, he showed that

$$\sum_{p \leqslant x} d(p+a) = c_a x + O\left(\frac{x \log\log x}{\log x}\right) \tag{3}$$

for some $c_a > 0$. He unconditionally proved that the left-hand side of (3) is $\ll x$. Dependence on GRH in his result was first removed by Linnik using his dispersion method (see [13]).

In our setting we obtain only upper and lower bounds with the right order of magnitude.

Theorem 3 *Suppose that $\alpha > 1$ is an irrational of finite type. Then*

$$\sum_{p \leqslant x} d(\lfloor \alpha p \rfloor) \asymp x, \tag{4}$$

where the implied constant depends only on α.

Finally, we prove an Erdős–Kac type result for $\omega(\lfloor \alpha p \rfloor)$. Here $\omega(n)$ is the number of distinct prime divisors of n. The corresponding result for $\omega(p-1)$ was proved by Halberstam (see [8]).

Theorem 4 *Suppose that $\alpha > 0$ is an irrational of finite type. Then*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \leqslant x : \frac{\omega(\lfloor \alpha p \rfloor) - \log\log p}{\sqrt{\log\log p}} < \kappa \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\kappa} e^{-t^2} dt$$

for any $\kappa \in \mathbb{R}$ fixed

1.2. Preliminaries and notation

1.2.1. Notation

Given a real number x, we write $e(x) = e^{2\pi i x}$, $\{x\}$, for the fractional part of x and $\lfloor x \rfloor$ for the greatest integer not exceeding x.

For $N \geqslant 1$ integer, we write $n \sim N$ to mean that n lies in an interval contained in $(N, 2N]$ whenever it is clear from context; otherwise, we shall denote it specifically.

We recall that for functions F and real nonnegative G, the notations $F \ll G$ and $F = O(G)$ are equivalent to the statement that the inequality $|F| \leqslant \alpha G$ holds for some constant $\alpha > 0$. Furthermore, we use $F \asymp G$ to indicate that both $F \gg G$ and $F \ll G$ hold. In a slight departure from convention, we shall frequently use ε and ε' to mean small positive numbers possibly not the same at each occurrence.

1.2.2. Preliminaries

Throughout the paper we shall make use of the following fact: For any $\alpha > 1$,

$$\lfloor \alpha m \rfloor = n \text{ for some } m \in \mathbb{N} \Leftrightarrow 0 < \left\{ \frac{n+1}{\alpha} \right\} \leqslant \frac{1}{\alpha},$$

and thus for any sequence a_n of real numbers, and for any $N > N' \geqslant 1$, one has

$$\sum_{N' < n \leqslant N} a_{\lfloor \alpha n \rfloor} = \sum_{\substack{\alpha N' < m \leqslant \alpha N \\ \left\{ \frac{m+1}{\alpha} \right\} < \frac{1}{\alpha}}} a_m + O\left(\max_{\alpha N' - 1 < n \leqslant \alpha N + 1} |a_n| \right). \tag{5}$$

Lemma 1 (Erdős–Turán inequality) *Let $\{t_k\}_{k=1}^{K}$ be a sequence of real numbers. Suppose that $\mathcal{I} \subset [0,1)$ is an interval. Then*

$$\left| \# \{ k \leqslant K : \{t_k\} \in \mathcal{I} \} - K|\mathcal{I}| \right| \ll \frac{K}{H} + \sum_{h \leqslant H} \frac{1}{h} \left| \sum_{k \leqslant K} e(h t_k) \right|$$

for any $H \geqslant 1$. The constant in the O-term is absolute.

Proof See [3, Theorem 2.1]. □

Lemma 2 *Let α be a real number. Suppose that $H, N \geqslant 1$, and let $\mathcal{L} = \log 2HN$. If*

$$\left| \alpha - \frac{a}{q} \right| \leqslant \frac{1}{q^2}, \quad (a, q) = 1,$$

then

$$\sum_{h\leqslant H}\left|\sum_{n\leqslant N}\Lambda(n)e(\alpha hn)\right|\ll\mathcal{L}^7\{HNq^{-1/2}+HN^{3/4}+DH^{3/5}N^{4/5}+(HN)^{1/2}q^{1/2}\},$$

where

$$D=\max_{m\leqslant HN}\sum_{\substack{d\leqslant H\\ d|m}}1,$$

and $\Lambda(n)$ *is the Von Mangolt function, which is* $\log p$ *if* $n=p^r$ *and zero otherwise.*

Proof See [22, Theorem 1]. □

1.3. Proof of the Main Results

Proof of Theorem 1 The indicator function of k-free integers is $\sum_{d^k|n}\mu(d)$ where μ is the Möbius function, yielding

$$\sum_{\substack{p\leqslant x\\ \lfloor\alpha p\rfloor \text{ is } k\text{-free}}}1=\sum_{p\leqslant x}\sum_{d^k|\lfloor\alpha p\rfloor}\mu(d).$$

Let $2\leqslant y\leqslant(\alpha x)^{1/k}$ be a real number. We swap the summations running over d and p and split the above sum as follows:

$$\sum_{\substack{p\leqslant x\\ \lfloor\alpha p\rfloor \text{ is } k\text{-free}}}1=\sum_{d\leqslant y}\mu(d)\sum_{\substack{p\leqslant x\\ \lfloor\alpha p\rfloor\equiv 0 \pmod{d^k}}}1+\sum_{y<d\ll x^{1/k}}\mu(d)\sum_{\substack{p\leqslant x\\ \lfloor\alpha p\rfloor\equiv 0 \pmod{d^k}}}1. \tag{6}$$

The last sum above on the right is at most

$$\ll\sum_{y<d\ll x^{1/k}}\sum_{\substack{m\ll x\\ m\equiv 0 \pmod{d^k}}}1+x^{1/k}\ll xy^{1-k}.$$

We now turn back to the first sum on the right hand side of (6). It is not hard to see that

$$\lfloor\alpha p\rfloor\equiv 0 \pmod{d^k} \text{ if and only if } \left\{\frac{\alpha p}{d^k}\right\}<\frac{1}{d^k}.$$

Hence, by Lemma 1, we have

$$\sum_{d\leqslant y}\mu(d)\sum_{\substack{p\leqslant x\\ \lfloor\alpha p\rfloor\equiv 0 \pmod{d^k}}}1=\sum_{d\leqslant y}\mu(d)\frac{1}{d^k}\pi(x)+O\left(\frac{xy}{H}\right)+O\left(\sum_{d\leqslant y}\sum_{h\leqslant H}\frac{1}{h}\left|\sum_{p\leqslant x}e\left(\frac{\alpha hp}{d^k}\right)\right|\right) \tag{7}$$

By the standard estimate,

$$\sum_{d\leqslant y}\frac{\mu(d)}{d^k}=\frac{1}{\zeta(k)}+O\left(\frac{1}{y^{k-1}}\right)\qquad(k\geqslant 2),$$

we see that replacing the first term on the right-hand side of (7) by $\frac{\pi(x)}{\zeta(k)}$ introduces an error that is $\ll\frac{\pi(x)}{y^{k-1}}$. To estimate the exponential sum in the O-term above we apply the following lemma.

Lemma 3 *Let $H, D, N \geq 1$. Suppose that $\alpha > 0$ is of type $\tau \geqslant 1$. Then for $k \geqslant 1$, one has*

$$\sum_{h\sim H}\sum_{d\sim D}\left|\sum_{p\leqslant N} e\left(\frac{\alpha ph}{d^k}\right)\right| \ll (HDN)^{\varepsilon}\Big\{D^{1+k/2}H^{1/2}N^{1/2} + DH^{3/5}N^{4/5} + DHN^{3/4} + D^{(k+2\tau+2)/(2\tau+2)}H^{(2\tau+1)/(2\tau+2)}N^{(2\tau+1)/(2\tau+2)}\Big\}. \quad (8)$$

Proof By a standard procedure and partial summation it suffices to show that the same upper (possibly with a different ε) holds for the following sum:

$$S := \sum_{d\sim D}\sum_{h\sim H}\left|\sum_{n\leqslant t}\Lambda(n)e\left(\frac{\alpha hn}{d^k}\right)\right|,$$

uniformly for every $t \leqslant N$.

Let $Q \geqslant 1$ be a parameter to be determined. Suppose $\left|\frac{\alpha}{d^k} - \frac{r}{q}\right| < \frac{1}{qQ}$ for some reduced fraction r/n with $1 \leqslant q \leqslant Q$ yielding $||\alpha q|| < \frac{d^k}{Q}$. Since α is of type $\tau \geqslant 1$, we have $q^{-\tau-\varepsilon} \ll \frac{d^k}{Q}$ and hence $\left(\frac{Q}{d^k}\right)^{1/\tau-\varepsilon} \ll q \leqslant Q$. Assuming that $D^k \ll Q < \infty$ (so that $q \geqslant 1$), it follows by Lemma 2 that

$$S(HDN)^{-\varepsilon} \ll \Big\{HNQ^{-1/2\tau}D^{k/2\tau+1} + DH^{3/5}N^{4/5} + DHN^{3/4} + DH^{1/2}N^{1/2}Q^{1/2}\Big\}.$$

Applying [6, Lemma 2.4] we derive the upper bound in (8). □

Splitting the ranges of d and h of the exponential sum in (7) into intervals of the form $d \sim D$ and $h \sim H$, respectively, applying Lemma 3, and finally summing over H and D, one has

$$\sum_{d\leqslant y}\sum_{h\leqslant H}\frac{1}{h}\left|\sum_{p\leqslant x} e\left(\frac{\alpha hp}{d^k}\right)\right| \ll (xyH)^{\varepsilon}\Big\{y^{1+k/2}x^{1/2} + yx^{4/5} + y^{(k+2\tau+2)/(2\tau+2)}x^{(2\tau+1)/(2\tau+2)}\Big\} \quad (9)$$

for $k \geqslant 1$ and every $y \geqslant 1$. We choose $H = x^{1/5}$ to make the penultimate in (7) small. Using (9), it follows that the left-hand side of (6) differs from $\pi(x)/\zeta(k)$ at most

$$\ll x^{\varepsilon}\Big\{y^{1+k/2}x^{1/2} + yx^{4/5} + y^{(k+2\tau+2)/(2\tau+2)}x^{(2\tau+1)/(2\tau+2)} + xy^{1-k}\Big\}$$

for every $2 \leqslant y \ll x^{1/k}$. Applying [6, Lemma 2.4], we see that only the error term in Theorem 1 survives, thereby proving the claim. □

Proof of Theorem 2 The proof of Theorem 2 is similar to that of Theorem 1; thus, we shall be brief. It is clear that

$$R_k(\alpha, n) = \sum_{\lfloor \alpha p\rfloor \leqslant n}\sum_{d^2|(n-\lfloor \alpha p\rfloor)}\mu(d).$$

Swapping the sums running over p and d, and splitting the range of d depending on whether $d \leqslant y$ or $d > y$, one has

$$R_k(\alpha, n) = \sum_{d \leqslant y} \sum_{\substack{p \leqslant \frac{n}{\alpha} \\ \lfloor \alpha p \rfloor \equiv n_d \mod d^k}} + O\left(\frac{n}{y^{k-1}}\right)$$

where n_d is the reduced residue class of $n \mod d^k$. Here we note that

$$\lfloor \alpha p \rfloor \equiv n_d \pmod{d^k} \text{ if and only if } \left\{\frac{\alpha p - n_d}{d^k}\right\} < \frac{1}{d^k}.$$

Applying Lemma 1 and following the proof of Theorem 1, we arrive at (2). □

Proof of Theorem 3 The lower bound is easy and follows from the observation

$$\sum_{p \leqslant x} d(\lfloor \alpha p \rfloor) \geqslant \sum_{d \leqslant x^{\varepsilon}} \sum_{\substack{p \leqslant x \\ \{\frac{\alpha p}{d}\} < \frac{1}{d}.}} 1 \quad (\text{ for every } \varepsilon > 0).$$

Now choosing $\varepsilon > 0$ sufficiently small and using Lemma 1 and then the upper bound in (9), the lower bound follows.

As for the upper bound, one may follow the strategy in the proof of Theorem 1. However, we shall use Selberg's sieve (we actually borrow the main idea of [17]). To do this, we start with the following standard identity:

$$d(n) = 2 \sum_{\substack{d|n \\ d \leqslant \sqrt{n}}} 1 - \delta(n),$$

where $\delta(n)$ is the characteristic function of squares. It follows that

$$\sum_{p \leqslant x} d(\lfloor \alpha p \rfloor) = 2 \sum_{d \leqslant \sqrt{x}} \sum_{\substack{p \leqslant x \\ \lfloor \alpha p \rfloor \equiv 0 \pmod d}} + O(\sqrt{x}). \tag{10}$$

Let $2 \leqslant z \leqslant x$ be a parameter. Let P_z be the product of primes not exceeding z. From Selberg's sieve (see [20]), we know that there is a sequence $\{\lambda_d\}_{d=1}^{\infty}$ of real numbers satisfying the following properties:

1. $\lambda_1 = 1$,
2. $\lambda_d = 0$ whenever $d > z$,
3. $|\lambda_d| \leqslant 1$,
4. $\sum_{\substack{d_1 | P_z \\ d_2 | P_z}} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]} \ll \frac{1}{\log z}$ where $[d_1, d_2]$ denotes the least common multiple of d_1 and d_2.

Hence, we have

$$\sum_{\substack{p \leqslant x \\ \lfloor \alpha p \rfloor \equiv 0 \pmod d}} \leqslant \sum_{\substack{n \leqslant x \\ \lfloor \alpha n \rfloor \equiv 0 \pmod d}} \left(\sum_{d | (P_z, n)} \lambda_d\right)^2 + O(z).$$

Swapping the sums, one has

$$\sum_{\substack{n\leqslant x\\ \lfloor \alpha n\rfloor\equiv 0 \pmod d}}\left(\sum_{d|(P_z,n)}\lambda_d\right)^2=\sum_{\substack{d_1|P_z\\ d_2|P_z}}\lambda_{d_1}\lambda_{d_2}\sum_{\substack{n\leqslant x\\ \lfloor \alpha n\rfloor\equiv 0 \pmod d\\ d'|n}}$$

where $d'=[d_1,d_2]$. Setting $n=d'k$, it follows that the right-hand side of the above is

$$\sum_{\substack{d_1|P_z\\ d_2|P_z}}\lambda_{d_1}\lambda_{d_2}\sum_{\substack{k\leqslant x/d'\\ \lfloor \alpha d'k\rfloor\equiv 0 \pmod d}}.$$

By the standard procedure (see (5)), the right-hand side of the above is

$$\sum_{\substack{d_1|P_z\\ d_2|P_z}}\lambda_{d_1}\lambda_{d_2}\sum_{\substack{m\leqslant \alpha x\\ m\equiv 0 \pmod d\\ \{\frac{m+1}{\alpha d'}\}<\frac{1}{\alpha d'}}}+O\left(\sum_{\substack{d_1|P_z\\ d_2|P_z}}|\lambda_{d_1}||\lambda_{d_2}|\right) \tag{11}$$

By Lemma 1, we pick up the main term and hence (11) is

$$\frac{x}{d}\sum_{\substack{d_1|P_z\\ d_2|P_z}}\frac{\lambda_{d_1}\lambda_{d_2}}{[d_1,d_2]}+O\left(\frac{x}{dH}\sum_{\substack{d_1|P_z\\ d_2|P_z}}|\lambda_{d_1}||\lambda_{d_2}|\right)+O\left(\sum_{\substack{d_1|P_z\\ d_2|P_z}}|\lambda_{d_1}||\lambda_{d_2}|\right)$$

$$+O\left(\sum_{\substack{d_1|P_z\\ d_2|P_z}}|\lambda_{d_1}||\lambda_{d_2}|\sum_{h\leqslant H}\frac{1}{h}\left|\sum_{n\leqslant \frac{x}{d}}e\left(\frac{\alpha^{-1}hdn}{d'}\right)\right|\right). \tag{12}$$

Using the properties of the sequence λ_r and summing over d, we see that the left-hand side of (10) is

$$\ll \frac{x\log x}{\log z}+O\left(z^2\left(\frac{x\log x}{H}+\sqrt{x}\right)\right)+O\left(z^{\varepsilon}\sum_{\substack{r\leqslant z^2\\ d\leqslant\sqrt{x}}}\sum_{h\leqslant H}\frac{1}{h}\left|\sum_{n\leqslant\frac{x}{d}}e\left(\frac{\alpha^{-1}hdn}{r}\right)\right|\right).$$

Here, for the last part, we use the elementary estimate $d(r)\ll r^{\varepsilon/2}$, together with the observation

$$\sum_{\substack{d_1,d_2\\ [d_1,d_2]=r}}1\leqslant d^2(r).$$

Using the upper bound

$$\sum_{n\leqslant x}e(\alpha n)\ll \min\left\{x,\frac{1}{||\alpha||}\right\}$$

for any $\alpha > 0$, it follows that

$$\sum_{\substack{r\leqslant z^2\\ d\leqslant\sqrt{x}}}\sum_{h\leqslant H}\frac{1}{h}\left|\sum_{n\leqslant\frac{x}{d}}e\left(\frac{\alpha^{-1}hdn}{r}\right)\right| \ll \sum_{r\leqslant z^2}\sum_{h\leqslant H}\frac{1}{h}\sum_{d\leqslant\sqrt{x}}\min\left\{\frac{x}{d},\frac{1}{||\frac{\alpha^{-1}hd}{r}||}\right\}.$$

To estimate the innermost sum we use the following lemma.

Lemma 4 *Let α be a real number. Suppose that $1 < T \leqslant x$. If*

$$\left|\alpha-\frac{a}{q}\right|\leqslant\frac{1}{q^2},\quad (a,q)=1,$$

then

$$\sum_{d\leqslant T}\min\left\{\frac{x}{d},\frac{1}{||\alpha d||}\right\}\ll\left(\frac{x}{q}+T+q\right)\log(2qT).$$

Proof See [23, Lemma 8b]. □

As we have done before, let $Q \geqslant 1$ be a parameter to be determined, and let

$$\left|\alpha^{-1}\frac{h}{r}-\frac{m}{n}\right|\leqslant\frac{1}{nQ},$$

where $n \leqslant Q$ and $(m,n)=1$. By a similar argument used in the proof of Theorem 1, since α^{-1} is of type τ, it follows by Lemma 4 that

$$\sum_{r\leqslant z^2}\sum_{h\leqslant H}\frac{1}{h}\sum_{d\leqslant\sqrt{x}}\min\left\{\frac{x}{d},\frac{1}{||\frac{\alpha^{-1}hd}{r}||}\right\}\ll\sum_{\substack{r\leqslant z^2\\ h\leqslant H}}\frac{1}{h}\left(xhr^{1/\tau-\varepsilon}Q^{-\frac{1}{\tau}+\varepsilon}+\sqrt{x}+Q\right)x^{\varepsilon}.$$

We next pick $H = z^2\log x$, $z = x^{\theta}$, where θ is sufficiently small depending on τ and ε, and we choose Q appropriately. This completes the proof. □

Proof of Theorem 4 We shall make use of [14, Theorem 3] and follow the notations therein.

Let us fix the notation first:

We pick the set $S(x)$ to be the set of primes $\leqslant x$ and $f(n) = \lfloor \alpha n\rfloor$. Let $l \geqslant 2$ be a positive integer. We set

$$\frac{1}{\pi(x)}\#\{p\leqslant x:[\alpha p]\equiv 0 \mod l\}=\lambda_l+e_l(x)$$

where $\lambda_l = \frac{1}{l}$, and by Lemma 1

$$|e_l(x)|\leqslant\frac{1}{H}+\frac{1}{\pi(x)}\sum_{h\leqslant H}\frac{1}{h}\left|\sum_{p\leqslant x}e\left(\frac{\alpha hp}{l}\right)\right| \tag{13}$$

for every $H \geqslant 2$.

Pick $\varepsilon > 0$ sufficiently small. Then it is clear that for $p \leqslant x$, $\lfloor \alpha p \rfloor$ has a uniformly bounded number of prime divisors $\geqslant x^{\varepsilon}$. We next put $y = x^{\frac{1}{\log\log x}}$, and take $\beta = \varepsilon$. By Merten's estimate (see [15]), we obviously have

$$\sum_{\substack{y < l \leqslant x^{\beta} \\ l \text{ prime}}} \frac{1}{l} = \log\log\log x + \log\varepsilon + O\left(\frac{\log\log x}{\log x}\right) = o((\log\log x)^{1/2}).$$

If $\varepsilon > 0$ is sufficiently small (depending on τ), then on choosing H optimally, by (9), it follows that

$$\sum_{y < l \leqslant x^{\beta}} |e_l| \ll x^{2\varepsilon}\left(x^{\frac{4}{5}-1} + x^{\frac{2\tau+1}{2\tau+2}-1}\right) = o((\log\log x)^{1/2}).$$

We have shown conditions (1), (2), and (3) are satisfied in [14, Theorem 3]. Obviously (4) and (5) follow by Merten's estimate and the convergence of the sum of squares of reciprocals of primes. As for condition (5), we need to show that, for every $r \geqslant 1$ and every $u \leqslant r$,

$$\sum_{\substack{(l_1, l_2, \cdots, l_u) \\ l_i \leqslant y \\ i=1,\cdots,u}} |e_{l_1, l_2 \cdots l_u}| = o((\log\log x)^{-r/2}).$$

This estimate is also clear because $y^r = o(x^{\varepsilon})$ together with the upper bound in (9). Thus, the proof is completed. □

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