On a nonnegativity principle with applications to a certain multiterm fractional boundary value problem

Noureddine FERFAR[ID], **Saïd MAZOUZI***[ID]
Laboratory of Applied Mathematics, Badji Mokhtar-Annaba University, Annaba, Algeria

Abstract: The main object of the present paper is to state and prove a general nonnegativity principle in the framework of multiterm fractional differential equations, which we use to investigate some iterative monotone sequences of lower and upper solutions to a certain fractional eigenvalue problem. The obtained results can be easily extended to fractional differential equations of distributed orders since the latter are the natural extension of multiterm fractional differential equations.

Key words: Nonnegativity principle, multiterm fractional differential equations, boundary value problem, lower and upper solutions

1. Introduction

Fractional calculus is now present in different fields of science and technology. We may encounter it in medicine, ecology, seismology, physics, electronics, mechanics, viscoelasticity, etc. We also notice its traces in the classical Abel singular integral equation. We address the reader to references [7, 12, 15]. Furthermore, some recent papers have used the fractional derivative to model love and emotions, diffusion of oxygen to tissues through capillaries, and the modeling of stochastic equations. Overall, there is no powerful method to explicitly solve such equations. We point out that in recent papers some successful attempts in numerical approaches to the solution were provided to solve fractional differential problems and the results were satisfactory. Regarding fractional differential equations of distributed orders, they were first used by Caputo in 1967 to study elastic media as well as to model dielectric induction and diffusion [5, 6]. As far as we are concerned, for the investigation of lower and upper solutions of fractional boundary value problems as well as the maximum principle, we may cite the leading works of [1, 3, 16]. Our main concern in this paper is first the extension of the nonnegativity principle stated in Lemma 3.3 [2] to multiterm fractional differentials, and then its application to the investigation of some iterative monotone sequences of lower and upper solutions to the following multiterm fractional eigenvalue problem:

$$
\begin{cases}
\mathcal{L}u\,(t) = -\lambda q\,(t, u)\,, & t \in J = (a, b)\,, \\
u\,(a) - \alpha u'\,(a) = \gamma_1, \\
u\,(b) + \beta u'\,(b) = \gamma_2,
\end{cases}
\tag{1}
$$

where

$$\mathcal{L}u\left(t\right) = \sum_{i=1}^{m} a_i\left(t\right)^C D_a^{\alpha_i} u\left(t\right) + b_0\left(t\right) u'\left(t\right) - \sum_{i=1}^{p} b_i\left(t\right)^C D_a^{\beta_i} u\left(t\right) + c\left(t\right) u\left(t\right).$$

Throughout this paper we shall use the following notations:

$$\begin{aligned}
\mathcal{L}_1 u\left(t\right) &= \sum_{i=1}^{m} a_i\left(t\right) D^{\alpha_i} u\left(t\right) + c\left(t\right) u\left(t\right), \\
\mathcal{L}_2 u\left(t\right) &= b_0\left(t\right) u'\left(t\right) - \sum_{i=1}^{p} b_i\left(t\right) D^{\beta_i} u\left(t\right), \\
\mathcal{L}u\left(t\right) &= \mathcal{L}_1 u\left(t\right) + \mathcal{L}_2 u\left(t\right), \\
B_1 u &= u\left(a\right) - \alpha u'\left(a\right) \text{ and } B_2 u = u\left(b\right) + \beta u'\left(b\right).
\end{aligned}$$

Here are some problems regarding multiterm fractional differential equations and fractional differential equations of distributed orders investigated in the last few years:

1) Linear case for constant coefficients:

$$\begin{cases} \sum_{i=1}^{m} c_i^C D_0^{\alpha_i} u\left(t\right) = f\left(t\right), \ t \in \left(0, T\right) \\ u^{(k)}\left(0\right) = u_k, \text{ for } k = 0, 1, ..., n-1, \end{cases}$$

where $0 < \alpha_1 < \alpha_2 < ... < \alpha_m$, $c_1, ..., c_m$ are real constant coefficients and $n = [\alpha_m] + 1$, if α_m is not an integer; otherwise, $n = \alpha_m$. This problem was studied first by Hadid [10], then one year later by Hadid and Luchko [11].

2) Linear case for variable coefficients:

$$\begin{cases} \sum_{i=1}^{m} a_i\left(t\right)^C D_0^{\alpha_i} u\left(t\right) = f\left(t\right), \ t \in \left(0, T\right) \\ u^{(k)}\left(0\right) = u_k, \text{ for } k = 0, 1, ..., n-1, \end{cases}$$

where $0 < \alpha_1 < \alpha_2 < ... < \alpha_m$, $c_1, ..., c_m$ are real constant coefficients, and $n = [\alpha_m] + 1$, if α_m is not an integer; otherwise, $n = \alpha_m$. Such a problem was investigated by Miller and Ross [14].

3) Nonlinear case:

$$\begin{cases} ^C D_0^{\alpha} u\left(t\right) = f\left(t, u\left(t\right), ^C D_0^{\alpha_1} u\left(t\right), ^C D_0^{\alpha_2} u\left(t\right), ..., ^C D_0^{\alpha_n} u\left(t\right)\right), \ t \in \left(0, T\right) \\ u^{(k)}\left(0\right) = u_k, \text{ for } k = 0, 1, ..., m, \end{cases}$$

where $m < \alpha \leq m + 1$, $0 < \alpha_1 < \alpha_2 < ... < \alpha_n < \alpha$. This problem was studied by Gejji and Jafari [9].

4) Distributed order case:

$$\begin{cases} \int_0^m \beta\left(r\right)^C D_0^r y\left(t\right) dr = f\left(t\right), \ t \in \left[0, T\right], \\ y^{(k)}\left(a\right) = y_k, \ k = 0, 1, ..., m-1, \ \left(0 < a \leq T\right), \end{cases}$$

studied by Ford and Morgado [8].

5) Application to partial differential equations:

$$
\begin{cases}
{}^{C}D_t^{\alpha} u\left(t, x\right) + \sum_{i=1}^{m} \lambda_i^C D_t^{\alpha_i} u\left(t, x\right) = \operatorname{div} p\left(x\right) \nabla u\left(t, x\right) - q\left(x\right) u\left(t, x\right), \\
\qquad\qquad\qquad\quad \left(t, x\right) \in \left(0, T\right) \times \Omega, \\
\qquad\qquad\quad u_{|t=0} = u_0\left(x\right), \ x \in \bar{\Omega}, \\
\qquad u_{|\partial\Omega} = v\left(t, x\right), \ \left(t, x\right) \in \left[0, T\right] \times \partial\Omega,
\end{cases}
$$

Ω being an open and bounded subset of $\mathbb{R}^n$ and $0 < \alpha_m < ... < \alpha_1 < \alpha \leq 1$, $\lambda_i \geq 0$, for $i = 1, ..., m$, with $m \in \mathbb{N}^*$. This problem was investigated by Luchko [13].

6) Investigation of monotone lower and upper solutions of the following boundary value problem (BVP, for short):

$$
\begin{cases}
{}^{C}D_0^{\delta} u\left(t\right) + g\left(t\right) u'\left(t\right) + h\left(t\right) u\left(t\right) = -\lambda k\left(t, u\left(t\right)\right), \ t \in \left(0, 1\right), 1 < \delta \leq 2, \\
u\left(0\right) - \alpha u'\left(0\right) = 0, \ u\left(1\right) + \beta u'\left(1\right) = 0, \ \alpha, \beta \geq 0,
\end{cases}
$$

by Al-Refai [2].

The paper is organized as follows. We next present some preliminaries, and then the following section is devoted to the generalization of the nonnegativity principle to multiterm fractional differential equations. Next, we apply the obtained nonnegativity principle in the comparison between any two lower and upper solutions to a certain multiterm fractional boundary value problem. Finally, we give some concluding remarks.

2. Preliminaries

We recall the notion of the (left-sided) fractional Riemann–Liouville integral of order α of a function $f : [a, b] \to \mathbb{R}$ as well as its (left-sided) fractional derivative of order α in the sense of Caputo [7, 12].

Definition 1 *Let α be a positive constant. We define the left-sided Riemann–Liouville fractional integral of order α of a function $f : [a, b] \to \mathbb{R}$ by the following formula:*

$$
J_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} f(s) ds, \quad t > a,
$$

where $\Gamma(\alpha)$ is the Gamma function, given by $\Gamma\left(\alpha\right) = \int_0^{\infty} e^{-t} t^{\alpha - 1} dt$.

Let $n = \alpha$ if α is an integer number and $n = [\alpha] + 1$ if α is not ($[\alpha]$ being the integral part of α). We define the left-sided Caputo fractional derivative of order α of f by

$$
\begin{aligned}
{}^{C}D_a^{\alpha} f(t) &= J_a^{n-\alpha} f^{(n)}(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - s)^{n - \alpha - 1} f^{(n)}(s) ds, \\
&\quad \text{if } \alpha \text{ is non integer, and} \\
{}^{C}D_a^{\alpha} f(t) &= f^{(n)}(t), \text{ for every } t > a, \text{ if } \alpha \text{ is integer.}
\end{aligned}
$$

We recall the following useful relations:

1) ${}^{C}D_a^{\alpha} J_a^{\alpha} f(t) = f(t)$ (under the continuity of f or $f \in \mathcal{L}^{\infty}$),

2) $J_a^\alpha {}^C D_a^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k$ (under the assumption of $f \in AC^n([a,b])$).

Throughout this paper we shall use the notation of the fractional derivative in the sense of Caputo of order α of f by $D^\alpha f(t)$ instead of ${}^C D_a^\alpha f(t)$.

In the sequel we need the following assumptions:

1) $a_i(t), b_j(t) \in C([a,b];\mathbb{R}^+)$, for $i = 1, ..., m$, $j = 1, ..., p$, with $\sum_{i=1}^m a_i(t) > 0$, for every $t \in [a,b]$;

2) $b_0(t), c(t) \in C([a,b];\mathbb{R})$;

3) $q(t,x) \in C^1([a,b] \times \mathbb{R}; \mathbb{R})$;

4) $\alpha_1, ..., \alpha_m \in (1,2]$ with $\alpha_i \neq \alpha_j$, whenever $i \neq j$, and $\beta_1, ..., \beta_p \in (0,1)$ with $\beta_i \neq \beta_j$, whenever $i \neq j$;

5) α and β are nonnegative real numbers such that $\alpha \geq \frac{b-a}{\alpha_0 - 1}$, where $\alpha_0 = \min_{1 \leq i \leq m} \alpha_i$;

6) λ, γ_1, γ_2 are given real numbers.

3. Nonnegativity principle and existence of eigenfunctions

We begin our investigation by stating and proving the following result proved in [2] just for the closed interval $[0,1]$. We have:

Proposition 2 *If $f \in C^2([a,b])$ has a local minimum at a point $t_0 \in J = (a,b)$, then for every $1 < \gamma < 2$, we have*

$$D^\gamma f(t_0) \geq \frac{(t_0 - a)^{-\gamma}}{\Gamma(2-\gamma)} \left[(\gamma - 1)\{f(a) - f(t_0)\} - f'(a)(t_0 - a) \right]. \tag{2}$$

Proof Let us set $g(t) = f(t) - f(t_0)$, $t \in [a,b]$. Then the function g satisfies the following:

$$g(t) \geq 0, \ g(t_0) = g'(t_0) = 0 \text{ and } g''(t_0) \geq 0,$$

so that

$$D^\gamma f(t_0) = D^\gamma g(t_0) = \frac{1}{\Gamma(2-\gamma)} \int_a^{t_0} (t_0 - s)^{1-\gamma} g''(s)\, ds.$$

Integrating the right-hand side twice by parts, we obtain

$$
\begin{aligned}
\int_a^{t_0} (t_0 - s)^{1-\gamma} g''(s)\, ds &= \lim_{s \to t_0} (t_0 - s)^{1-\gamma} g'(s) - (t_0 - a)^{1-\gamma} g'(a) \\
&\quad + (1-\gamma) \int_a^{t_0} (t_0 - s)^{-\gamma} g'(s)\, ds \\
&= -(t_0 - a)^{1-\gamma} g'(a) \\
&\quad + (1-\gamma) \left\{ \lim_{s \to t_0} (t_0 - s)^{-\gamma} g(s) - (t_0 - a)^{-\gamma} g(a) \right\} \\
&\quad - \gamma(1-\gamma) \int_a^{t_0} (t_0 - s)^{-1-\gamma} g(s)\, ds.
\end{aligned}
$$

Since $g'(a) = f'(a)$, $g''(t) = f''(t)$, $-\gamma(1-\gamma)\int_a^{t_0}(t_0-s)^{-1-\gamma}g(s)\,ds \geq 0$, and

$$\lim_{s \to t_0}(t_0-s)^{1-\gamma}g'(s) = \lim_{s \to t_0}(t_0-s)^{-\gamma}g(s) = 0,$$

then we get the desired inequality (2) at once. □

Corollary 3 *Under the same assumptions of the proposition, if $f'(a) \leq 0$, then*

$$D^\gamma f(t_0) \geq 0, \text{ for every } 1 < \gamma < 2.$$

We need the following lemma:

Lemma 4 *Let $f \in C^1([a,b])$ attain its minimum at $t_0 \in J = (a,b)$. Then for every $0 < \gamma < 1$, we have*

$$D^\gamma f(t_0) \leq \frac{(t_0-a)^{-\gamma}}{\Gamma(1-\gamma)}[f(t_0)-f(a)] \leq 0. \tag{3}$$

Proof Let us set $g(t) = f(t) - f(t_0)$, $t \in [a,b]$. Then the function g satisfies the following:

$$g(t) \geq 0, \ g(t_0) = g'(t_0) = 0,$$

so that

$$D^\gamma f(t_0) = D^\gamma g(t_0) = \frac{1}{\Gamma(1-\gamma)}\int_a^{t_0}(t_0-s)^{-\gamma}g'(s)\,ds.$$

Integrating the right-hand side twice by parts, we obtain

$$\int_a^{t_0}(t_0-s)^{-\gamma}g'(s)\,ds = \lim_{s \to t_0}(t_0-s)^{-\gamma}g(s) - (t_0-a)^{-\gamma}g(a)$$

$$-\gamma\int_a^{t_0}(t_0-s)^{-\gamma-1}g(s)\,ds.$$

Next, due to the fact that $g'(a) = f'(a)$, $-\gamma\int_a^{t_0}(t_0-s)^{-1-\gamma}g(s)\,ds \geq 0$, and

$$\lim_{s \to t_0}(t_0-s)^{-\gamma}g(s) = 0,$$

we get

$$\Gamma(1-\gamma)D^\gamma f(t_0) = \Gamma(1-\gamma)D^\gamma g(t_0) = \int_a^{t_0}(t_0-s)^{-\gamma}g'(s)\,ds$$

$$= -(t_0-a)^{-\gamma}g(a) - \gamma\int_a^{t_0}(t_0-s)^{-\gamma-1}g(s)\,ds$$

$$\leq -(t_0-a)^{-\gamma}g(a) = (t_0-a)^{-\gamma}[f(t_0)-f(a)] \leq 0,$$

and the inequality follows. □

We assume throughout this paper that all the above given hypotheses from 1) to 6) are satisfied. Next, we shall state and prove a general version of the nonnegativity principle given in Lemma 3.3 [2]. We have:

Lemma 5 [Nonnegativity principle] *Let $\tilde{b}(t,x,y)$ be a negative continuous function in $J \times \mathbb{R} \times \mathbb{R}$. Then every function $\varphi \in C^2(J;\mathbb{R})$ satisfying the following problem,*

$$\sum_{i=1}^{m} a_i(t) D^{\alpha_i} \varphi(t) + \mathcal{L}_2 \varphi(t) + \tilde{b}(t, \varphi(t), \varphi'(t)) \varphi(t) \leq 0, \ t \in J = (a,b),$$

$$B_i \varphi \ \geq \ 0, \ for \ i = 1, 2,$$

is necessarily nonnegative on the closed interval $[a, b]$.

Proof Suppose to the contrary that $\varphi(t)$ is negative in $[a, b]$. Then φ attains its negative minimum at some point $t_0 \in [a, b]$; that is, $\min\{\varphi(t), \ t \in [a, b]\} = \varphi(t_0) < 0$.

Let $a < t_0 < b$. Then $\varphi'(t_0) = 0$. Consider the following cases:

- If $\varphi'(a) \leq 0$, then we conclude by Corollary 3 that $D^{\alpha_i} \varphi(t_0) \geq 0$, for $i = 1, ..., m$, and by Lemma 4 that $D^{\beta_i} \varphi(t_0) \leq 0$, for $i = 1, ..., p$. Therefore,

$$\sum_{i=1}^{m} a_i(t_0) D^{\alpha_i} \varphi(t_0) + \mathcal{L}_2 \varphi(t_0) + \tilde{b}(t_0, \varphi(t_0), 0) \varphi(t_0) > 0,$$

  which is a contradiction.

- If $\varphi'(a) > 0$, then the assumption $\alpha \geq \frac{b-a}{\alpha_0 - 1}$ and the first boundary condition imply that

$$\varphi(a) \geq \alpha \varphi'(a) \geq \frac{b-a}{\alpha_0 - 1} \varphi'(a),$$

  from which we get

$$(\alpha_0 - 1) \varphi(a) \geq (b - a) \varphi'(a).$$

Hence, for each $i = 1, ..., n$, one has

$$\begin{aligned}
D^{\alpha_i} \varphi(t_0) \ &\geq \ \frac{(t_0 - a)^{-\alpha_i}}{\Gamma(2 - \alpha_i)} [(\alpha_i - 1)\{\varphi(a) - \varphi(t_0)\} - \varphi'(a)(t_0 - a)] \\
&\geq \ \frac{(t_0 - a)^{-\alpha_i}}{\Gamma(2 - \alpha_i)} [(\alpha_0 - 1)\{\varphi(a) - \varphi(t_0)\} - \varphi'(a)(t_0 - a)] \\
&\geq \ \frac{(t_0 - a)^{-\alpha_i}}{\Gamma(2 - \alpha_i)} [(\alpha_0 - 1)\varphi(a) - (\alpha_0 - 1)\varphi(t_0) - \varphi'(a)(t_0 - a)] \\
&\geq \ \frac{(t_0 - a)^{-\alpha_i}}{\Gamma(2 - \alpha_i)} [(b - a)\varphi'(a) - (\alpha_0 - 1)\varphi(t_0) - \varphi'(a)(t_0 - a)] \\
&\geq \ \frac{(t_0 - a)^{-\alpha_i}}{\Gamma(2 - \alpha_i)} [(b - t_0)\varphi'(a) - (\alpha_0 - 1)\varphi(t_0)] \\
&\geq \ 0.
\end{aligned}$$

Therefore,

$$\sum_{i=1}^{m} a_i(t_0) D^{\alpha_i} \varphi(t_0) + 0 + \tilde{b}(t_0, \varphi(t_0), 0) \varphi(t_0) > 0,$$

which is a contradiction.

Now, if $t_0 = a$, then $\varphi'(a^+) > 0$; therefore, $B_1 \varphi < 0$, which is again a contradiction. Finally, if $t_0 = b$, then $\varphi'(b^-) < 0$; therefore, $B_2 \varphi < 0$, giving once more another contradiction. We conclude that $\varphi(t) \geq 0$ for every $t \in [a, b]$. □

Next, we introduce the notions of lower and upper solutions to problem (1). We have:

Definition 6 *A function* $u \in C^2([a, b]; \mathbb{R})$ *is called a lower (resp. an upper) solution to problem (1) if it satisfies the following:*

$$\begin{cases} \mathcal{L}u(t) + \lambda q(t, u(t)) \geq 0, \ \text{for every } t \in J, \\ B_i u \leq \gamma_i, \ \text{for } i = 1, 2, \end{cases}$$

(resp.

$$\begin{cases} \mathcal{L}u(t) + \lambda q(t, u(t)) \leq 0, \ \text{for every } t \in J, \\ B_i u \geq \gamma_i, \ \text{for } i = 1, 2). \end{cases}$$

Two distinct solutions $\varphi_1(t)$ *and* $\varphi_2(t)$ *to problem (1) are said to be ordered if they satisfy either the inequality* $\varphi_1(t) \leq \varphi_2(t)$ *or* $\varphi_2(t) \leq \varphi_1(t)$, *for every* $t \in [a, b]$.

The following theorem gives the uniqueness of the solution whenever it exists as well as the order character of any lower and upper solutions.

Theorem 7 *Under the above assumptions, if* $c(t) + \lambda \frac{\partial q}{\partial x}(t, x) < 0$, *for every* $(t, x) \in J \times \mathbb{R}$, *then any two lower and upper solutions to problem (1) are ordered; moreover, the solution if it exists must be unique.*

Proof Let $\varphi(t)$ and $\psi(t)$ be respectively a lower and an upper solution to problem (1). Then

$$\begin{cases} \mathcal{L}\varphi(t) + \lambda q(t, \varphi(t)) \geq 0, \ t \in J, \\ B_i \varphi \leq \gamma_i, \ \text{for } i = 1, 2, \end{cases}$$

and

$$\begin{cases} \mathcal{L}\psi(t) + \lambda q(t, \psi(t)) \leq 0, \ t \in J, \\ B_i \psi \geq \gamma_i, \ \text{for } i = 1, 2. \end{cases}$$

It follows that

$$\begin{cases} \mathcal{L}(\psi - \varphi)(t) + \lambda \{q(t, \psi(t)) - q(t, \varphi(t))\} \leq 0, \ t \in J, \\ B_i(\psi - \varphi) \geq 0, \ \text{for } i = 1, 2. \end{cases}$$

We note that by virtue of the mean value theorem we have

$$q(t, \psi(t)) - q(t, \varphi(t)) = \frac{\partial q(t, \theta(t))}{\partial x}(\psi(t) - \varphi(t)), \ t \in J,$$

for some $\theta(t)$ between $\varphi(t)$ and $\psi(t)$. Setting $z(t) = \psi(t) - \varphi(t)$ and

$$\tilde{b}(t, z(t), z'(t)) = c(t) + \lambda \frac{\partial q}{\partial x}(t, \theta(t)) \quad (< 0),$$

we obtain

$$\sum_{i=1}^{m} a_i(t) D^{\alpha_i} z(t) + \mathcal{L}_2(z)(t) + \tilde{b}(t, z(t), z'(t)) z(t) \leq 0, \ t \in J,$$

$$B_i z \geq 0, \ \text{for } i = 1, 2.$$

Applying the nonnegativity principle established earlier we conclude that $z(t) = \psi(t) - \varphi(t) \geq 0$ for every $t \in [a, b]$. Hence, $\varphi(t) \leq \psi(t)$ for every $t \in [a, b]$.

Regarding the uniqueness of the solution we suppose that problem (1) has two solutions $\varphi_1(t)$ and $\varphi_2(t)$, and then setting $\varphi(t) = \varphi_2(t) - \varphi_1(t)$, we get at once

$$\begin{cases} \mathcal{L}(\varphi_2 - \varphi_1)(t) + \lambda\{q(t, \varphi_2(t)) - q(t, \varphi_1(t))\} \\ = \sum_{i=1}^{m} a_i(t) D^{\alpha_i} \varphi(t) + \mathcal{L}_2(\varphi)(t) + \left\{c(t) + \lambda\frac{\partial q}{\partial x}\left(t, \tilde{\theta}(t)\right)\right\}\varphi(t) = 0, \ t \in J, \\ B_i\varphi = 0, \ \text{for } i = 1, 2, \end{cases}$$

for some function $\tilde{\theta}(t)$ between $\varphi_1(t)$ and $\varphi_2(t)$ according to the mean value theorem. We deduce from the nonnegativity principle that $\varphi(t) \geq 0$ for every $t \in [a, b]$; that is, $\varphi_2(t) \geq \varphi_1(t)$, for every $t \in [a, b]$. Employing the same reasoning, we get $\varphi_2(t) \leq \varphi_1(t)$ for every $t \in [a, b]$. Therefore, $\varphi_1(t) = \varphi_2(t)$ for every $t \in [a, b]$, which proves the uniqueness of the solution as claimed. $\square$

An immediate corollary of the previous theorem is the following:

Corollary 8 *Assume that $q(t, 0) = 0$, for every $t \in [a, b]$ and let*

$$\Lambda = \sup_{(t,x)\in[a,b]\times\mathbb{R}} \left\{ c(t) / \left|\frac{\partial q}{\partial x}(t, x)\right| \right\}.$$

Then the necessary condition for which the following eigenvalue problem,

$$\begin{cases} \mathcal{L}u(t) + \lambda q(t, u(t)) = 0, \ t \in J, \\ B_i u = 0, \ \text{for } i = 1, 2, \end{cases} \tag{4}$$

has a nontrivial solution is:

1) for every constant $\lambda \leq \Lambda$, if $\sup_{(t,x)\in[a,b]\times\mathbb{R}} \frac{\partial q}{\partial x}(t, x) < 0$,

2) for every constant $\lambda \geq -\Lambda$, if $\inf_{(t,x)\in[a,b]\times\mathbb{R}} \frac{\partial q}{\partial x}(t, x) > 0$.

Proof

1. Suppose that $\sup_{(t,x)\in[a,b]\times\mathbb{R}} \frac{\partial q}{\partial x}(t, x) < 0$ and $\lambda > \Lambda$. Then $c(t) + \lambda\frac{\partial q}{\partial x}(t, x) < 0$, for every $(t, x) \in [a, b] \times \mathbb{R}$. It follows from the above theorem that the solution of problem (4) is unique, and since $u = 0$ is a solution, then it is a unique one. Thus, problem (4) has no eigenfunction at all; therefore, the condition $\lambda \leq \Lambda$ is necessary for the existence of an eigenfunction of problem (4).

2. Under the assumption $\inf_{(t,x)\in[a,b]\times\mathbb{R}} \frac{\partial q}{\partial x}(t, x) > 0$, if we assume that $\lambda < -\Lambda$, then $c(t) + \lambda\frac{\partial q}{\partial x}(t, x) < 0$ for every $(t, x) \in [a, b] \times \mathbb{R}$. We conclude as before that the condition $\lambda \geq -\Lambda$ is necessary for the existence of an eigenfunction of problem (4).

□

Another important consequence of the previous theorem is the following:

Corollary 9 *If* $q(t,u) = r(t)u$ *with* $r(t) \in C([a,b];\mathbb{R}^*)$ *and* $\lambda_0 = \sup\limits_{t \in [a,b]} \{c(t)/|r(t)|\}$, *then the necessary condition for which the linear eigenvalue problem*

$$\begin{cases} \mathcal{L}u(t) + \lambda r(t)u(t) = 0, \ t \in J, \\ \quad B_i u = 0, \ for \ i = 1, 2 \end{cases} \tag{5}$$

has nontrivial solutions is:

1) for every constant $\lambda \leq \lambda_0$, *if* $\sup\limits_{t \in [a,b]} r(t) < 0$,

2) for every constant $\lambda \geq -\lambda_0$, *if* $\inf\limits_{t \in [a,b]} r(t) > 0$.

Proof It is straightforward. □

4. Existence result via lower and upper solutions

In what follows we will derive two monotone sequences $\{\varphi_n\}_{n \geq 1}$ and $\{\psi_n\}_{n \geq 1}$ of lower and upper solutions to problem (1) that converge pointwisely to some functions φ and ψ such that $\varphi(t) \leq \psi(t)$ for every $t \in [a,b]$.

We have the following theorem:

Theorem 10 *Let* φ_0 *and* ψ_0 *be respectively a lower and an upper solution to problem (1) such that* $\varphi_0 \leq \psi_0$. *Assume that there is a negative constant* γ *such that*

$$\gamma < c(t) + \lambda \frac{\partial q}{\partial x}(t,x), \ (t,x) \in [a,b] \times \mathbb{R}. \tag{6}$$

Let $\{\varphi_n\}_{n \geq 1}$ *and* $\{\psi_n\}_{n \geq 1}$ *respectively satisfy the following iterative problems:*

$$\begin{cases} \mathcal{L}\varphi_n(t) + (\gamma - c(t))\varphi_n(t) \\ = (\gamma - c(t))\varphi_{n-1}(t) - \lambda q(t, \varphi_{n-1}(t)), \ t \in J, \\ B_i\varphi_{n-1} \leq B_i\varphi_n \leq \gamma_i, \ for \ i = 1, 2, \ and \ n = 1, 2, ... \end{cases} \tag{7}$$

and

$$\begin{cases} \mathcal{L}\psi_n(t) + (\gamma - c(t))\psi_n(t) \\ = (\gamma - c(t))\psi_{n-1}(t) - \lambda q(t, \psi_{n-1}(t)), \ t \in J, \\ B_i\psi_{n-1} \geq B_i\psi_n \geq \gamma_i, \ for \ i = 1, 2, and \ n = 1, 2, ... \end{cases} \tag{8}$$

Then:

1. *The sequence* $\{\varphi_n\}_{n \geq 1}$ *is nondecreasing and each term of it is a lower solution to problem (1).*

2. *The sequence* $\{\psi_n\}_{n \geq 1}$ *is nonincreasing and each term of it is an upper solution to problem (1).*

3. $\varphi_n \leq \psi_n$, *for every* $n \geq 1$.

Proof 1. Since φ_0 is a lower solution to problem (1), then

$$\begin{cases} \mathcal{L}\varphi_0\left(t\right) + \lambda q\left(t, \varphi_0\left(t\right)\right) \geq 0, \ t \in J, \\ \qquad B_i\varphi_0 \leq \gamma_i, \text{ for } i = 1, 2. \end{cases} \tag{9}$$

Let φ_1 be a solution to (7) corresponding to $n = 1$. Then

$$\begin{cases} \mathcal{L}\varphi_1\left(t\right) + \left(\gamma - c\left(t\right)\right)\varphi_1\left(t\right) = \left(\gamma - c\left(t\right)\right)\varphi_0\left(t\right) - \lambda q\left(t, \varphi_0\left(t\right)\right), \ t \in J, \\ \qquad B_i\varphi_0 \leq B_i\varphi_1 \leq \gamma_i, \text{ for } i = 1, 2. \end{cases}$$

It follows that

$$\lambda q\left(t, \varphi_0\left(t\right)\right) = -\mathcal{L}\varphi_1\left(t\right) - \left(\gamma - c\left(t\right)\right)\left(\varphi_1\left(t\right) - \varphi_0\left(t\right)\right),$$

and so combining with (9) we get

$$\begin{cases} \mathcal{L}\left(\varphi_1\left(t\right) - \varphi_0\left(t\right)\right) + \left(\gamma - c\left(t\right)\right)\left(\varphi_1\left(t\right) - \varphi_0\left(t\right)\right) \leq 0, \ t \in J, \\ \qquad 0 \leq B_i\left(\varphi_1 - \varphi_0\right), \text{ for } i = 1, 2. \end{cases} \tag{10}$$

Putting $z\left(t\right) = \varphi_1\left(t\right) - \varphi_0\left(t\right)$, problem (10) becomes

$$\begin{cases} \sum_{i=1}^{m} a_i\left(t\right) D^{\alpha_i} z\left(t\right) + \mathcal{L}_2 z\left(t\right) + \gamma z\left(t\right) \leq 0, \ t \in J, \\ \qquad B_i z \geq 0, \text{ for } i = 1, 2. \end{cases} \tag{11}$$

We conclude by the nonnegativity principle that $z\left(t\right) \geq 0$, for every $t \in [a, b]$; that is, $\varphi_0\left(t\right) \leq \varphi_1\left(t\right)$, for every $t \in J$, and by induction we infer that $\varphi_{n-1}\left(t\right) \leq \varphi_n\left(t\right)$, for $n = 1, 2, \ldots$, for every $t \in [a, b]$.

On the other hand, we have

$$\begin{cases} \mathcal{L}\varphi_n\left(t\right) + \lambda q\left(t, \varphi_n\left(t\right)\right) = \left(\gamma - c\left(t\right)\right)\left(\varphi_{n-1}\left(t\right) - \varphi_n\left(t\right)\right) \\ \qquad + \lambda\left\{q\left(t, \varphi_n\left(t\right)\right) - q\left(t, \varphi_{n-1}\left(t\right)\right)\right\}, \ t \in J, \\ = \left\{c\left(t\right) - \gamma + \lambda\frac{\partial q}{\partial x}\left(t, \tilde{\varphi}_n\left(t\right)\right)\right\}\left(\varphi_n\left(t\right) - \varphi_{n-1}\left(t\right)\right) \geq 0 \\ \qquad B_i\varphi_n \leq \gamma_i, \text{ for } i = 1, 2, \text{ and } n = 1, 2, \ldots \end{cases}$$

for some $\tilde{\varphi}_n\left(t\right)$ between $\varphi_{n-1}\left(t\right)$ and $\varphi_n\left(t\right)$ according to the mean value theorem. This shows that $\varphi_n\left(t\right)$ is a lower solution to problem (1).

2. Using the same reasoning we find $\psi_n\left(t\right) \leq \psi_{n-1}\left(t\right)$, for $n = 1, 2, \ldots$, for every $t \in [a, b]$, and $\psi_n\left(t\right)$ is an upper solution to problem (1).

3. Since φ_1 (respectively ψ_1) satisfies (7) (respectively (8)), then by subtraction we get

$$\begin{cases} \mathcal{L}\left(\psi_1\left(t\right) - \varphi_1\left(t\right)\right) + \left(\gamma - c\left(t\right)\right)\left(\psi_1\left(t\right) - \varphi_1\left(t\right)\right) \\ = \left(\gamma - c\left(t\right)\right)\left(\psi_0\left(t\right) - \varphi_0\left(t\right)\right) - \lambda\left\{q\left(t, \psi_0\left(t\right)\right) - q\left(t, \varphi_0\left(t\right)\right)\right\}, \\ = \left(\gamma - c\left(t\right) - \lambda\frac{\partial q}{\partial x}\left(t, \tilde{\varphi}_0\left(t\right)\right)\right)\left(\psi_0\left(t\right) - \varphi_0\left(t\right)\right) \leq 0 \\ \qquad B_i\left(\psi_1 - \varphi_1\right) \geq 0, \text{ for } i = 1, 2. \end{cases}$$

We deduce once again by the nonnegativity principle that $\psi_1 \geq \varphi_1$, and by induction we conclude that

$$\varphi_n\left(t\right) \leq \psi_n\left(t\right), \text{ for every } t \in [a, b] \text{ and for } n = 1, 2, \ldots$$

$\square$

We have the following result regarding the convergence of the sequences of the above lower and upper solutions:

Corollary 11 *The above sequences of lower and upper solutions* $\{\varphi_n(t)\}_{n\geq 0}$ *and* $\{\psi_n(t)\}_{n\geq 0}$ *converge pointwisely to some functions* φ *and* ψ, *respectively, so that*

$$\varphi_0(t) \leq \varphi(t) \leq \psi(t) \leq \psi_0(t), \ t \in [a, b].$$

Moreover, if the functions φ *and* ψ *are continuous on* $[a, b]$, *then the above convergence is uniform.*

Proof

1) We have from the preceding theorem

$$\begin{aligned} \varphi_0(t) &\leq& \varphi_1(t) \leq \ldots \leq \varphi_{n-1}(t) \leq \varphi_n(t) \\ &\leq& \psi_n(t) \leq \psi_{n-1}(t) \leq \ldots \leq \psi_1(t) \leq \psi_0(t). \end{aligned}$$

Hence, $\{\varphi_n(t)\}_{n\geq 0}$ is a nondecreasing (resp. $\{\psi_n(t)\}_{n\geq 0}$ is a nonincreasing) sequence, which is bounded from above (resp. from below) by $\psi_0(t)$ (resp. by $\varphi_0(t)$). They must be pointwisely convergent.

2) Assume that the limits $\varphi(t)$ and $\psi(t)$ are continuous on $[a, b]$. Then by virtue of Dini's theorem (see [4]) the given monotone sequences $\{\varphi_n(t)\}_{n\geq 0}$ and $\{\psi_n(t)\}_{n\geq 0}$ converge uniformly on $[a, b]$ to the functions $\varphi(t)$ and $\psi(t)$, respectively.

$\square$

The next important consequence of the preceding theorem is:

Corollary 12 *Under the assumption*

$$\gamma < c(t) + \lambda \frac{\partial q}{\partial x}(t, x) < 0, (t, x) \in [a, b] \times \mathbb{R}, \tag{12}$$

if $B_1\varphi_n = B_1\psi_n = \gamma_1$ *and* $B_2\varphi_n = B_2\psi_n = \gamma_2$, *for* $n = 1, 2, \ldots$, *in (7) and (8), and if the limits* $lim_{n\to\infty}\varphi_n(t)$ *and* $lim_{n\to\infty}\psi_n(t)$ *are continuous functions on* $[a, b]$, *then they are equal and the common limit* $u(t)$ *is the unique solution to problem (1).*

Proof It follows from the previous theorem that if the limits $\varphi(t) := lim_{n\to\infty}\varphi_n(t)$ and $\psi(t) := lim_{n\to\infty}\psi_n(t)$ are continuous on $[a, b]$, then $\varphi_n \rightrightarrows \varphi$ (uniformly on $[a, b]$) and $\psi_n \rightrightarrows \psi(t)$ (uniformly on $[a, b]$). Taking the limits in (7) and (8), as $n \to \infty$, we realize that both $\varphi(t)$ and $\psi(t)$ are solutions to the same problem (1), and we conclude by Theorem 7 that $\varphi \equiv \psi$, showing that problem (1) has a unique solution. $\square$

5. Concluding remarks

Thanks to the new nonnegativity principle stated in Lemma 5 we have proved that under appropriate assumptions any two lower and upper solutions to problem (1) are ordered, and whenever a solution exists it must be unique. On the other hand, the eigenvalue problems (4) and (5) have nontrivial solutions under some handy conditions. Next, using inequality (6), we derived two monotone sequences of lower and upper solutions to problem (1) that converge pointwisely to some lower and upper solutions to the same problem, respectively. It turns out that if the limits are continuous, then under condition (12) a unique solution to problem (1) is obtained as a uniform limit of either the earlier defined sequences of lower or upper solutions.

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