A new general subclass of analytic bi-univalent functions

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Abstract: In a very recent work, Şeker [Seker B. On a new subclass of bi-univalent functions defined by using Salagean operator. Turkish Journal of Mathematics 2018; 42: 2891–2896] defined two subclasses of analytic bi-univalent functions by means of Sălăgean differential operator and he obtained the initial Taylor–Maclaurin coefficient estimates for functions belonging to these classes. The main purpose of this paper is to improve the results obtained by Şeker in the aforementioned study. For this purpose, we define a general subclass of bi-univalent functions.

Key words: Analytic functions, univalent functions, bi-univalent functions, coefficient bounds, Sălăgean differential operator

1. Introduction

Let $\mathcal{A}$ be the family of analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. Also let $\mathcal{S} := \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{U}\}$.

For a function $f \in \mathcal{A}$ defined by (1.1), the differential operator D^n $(n \in \mathbb{N} = \{1, 2, \ldots\})$ introduced by Sălăgean [4] as follows:

$$
\begin{aligned}
D^0 f(z) &= f(z), \\
D^1 f(z) &= z f'(z) =: D f(z), \\
D^n f(z) &= D(D^{n-1} f(z)) \qquad (n \in \mathbb{N}).
\end{aligned}
$$

So from the above equalities, we easily obtain that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

with $D^n f(0) = 0$.

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Theorem 1.1 *[2] (Koebe One-Quarter Theorem) The range of every function of class $\mathcal{S}$ contains the disk of radius $\left\{w : |w| < \frac{1}{4}\right\}$.*

Thus by Theorem 1.1, every function $f \in \mathcal{A}$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}) \qquad \text{and} \qquad f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f); \; r_0(f) \geq \frac{1}{4}\right).$$

For the inverse function f^{-1}, we have:

$$g(w) := f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

Definition 1 *(see [3]) If both the function f and its inverse function f^{-1} are univalent in $\mathbb{U}$, then the function f is called bi-univalent. We say that f is in the class Σ for such functions.*

Very recently, Şeker [5] introduced the two subclasses of the bi-univalent function class Σ in Definitions 2 and 3 by using the Sălăgean differential operator and obtained nonsharp bounds on $|a_2|$ and $|a_3|$ for functions $f \in \Sigma$ in each of these subclasses.

Definition 2 *(see [5]) A function $f(z)$ given by (1.1) is said to be in the class*

$$\mathcal{H}_{\Sigma}^{m,n}(\alpha) \qquad (m, n \in \mathbb{N}_0, \; m > n, \; 0 < \alpha \leq 1)$$

if the following conditions are satisfied:

$$f \in \Sigma \qquad \text{and} \qquad \left|\arg\left(\frac{D^m f(z)}{D^n f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \mathbb{U}) \tag{1.3}$$

and

$$\left|\arg\left(\frac{D^m g(w)}{D^n g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \mathbb{U}), \tag{1.4}$$

where the function g is given by (1.2).

Theorem 1.2 *(see [5]) Let the function $f(z)$ given by (1.1) be in the class*

$$\mathcal{H}_{\Sigma}^{m,n}(\alpha) \qquad (m, n \in \mathbb{N}_0, \; m > n, \; 0 < \alpha \leq 1).$$

Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha\left(3^m - 3^n\right) + \left(2^m - 2^n\right)^2 - \alpha\left(2^{2m} - 2^{2n}\right)}} \tag{1.5}$$

and

$$|a_3| \leq \frac{2\alpha}{3^m - 3^n} + \frac{4\alpha^2}{\left(2^m - 2^n\right)^2}. \tag{1.6}$$

Definition 3 (*see [5]*) *A function* $f(z)$ *given by* (1.1) *is said to be in the class*

$$\mathcal{H}_{\Sigma}^{m,n}(\beta) \qquad (m, n \in \mathbb{N}_0, \, m > n, \, 0 \leq \beta < 1)$$

if the following conditions are satisfied:

$$f \in \Sigma \qquad and \qquad \Re\left(\frac{D^m f(z)}{D^n f(z)}\right) > \beta \qquad (z \in \mathbb{U}) \tag{1.7}$$

and

$$\Re\left(\frac{D^m g(w)}{D^n g(w)}\right) > \beta \qquad (w \in \mathbb{U}), \tag{1.8}$$

where the function g *is given by* (1.2)*.*

Theorem 1.3 (*see [5]*) *Let the function* $f(z)$ *given by* (1.1) *be in the class*

$$\mathcal{H}_{\Sigma}^{m,n}(\beta) \qquad (m, n \in \mathbb{N}_0, \, m > n, \, 0 \leq \beta < 1).$$

Then

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{(3^m - 3^n) - 2^n(2^m - 2^n)}} \tag{1.9}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(2^m - 2^n)^2} + \frac{2(1-\beta)}{3^m - 3^n}. \tag{1.10}$$

Now we introduce the a new subclass of $\mathcal{A}$ which generalizes Definitions 2 and 3.

Definition 4 *Let the functions* $h, p : \mathbb{U} \to \mathbb{C}$ *satisfy the conditions*

$$\min\{\Re(h(z)), \Re(p(z))\} > 0 \quad (z \in \mathbb{U}) \quad and \quad h(0) = p(0) = 1.$$

For a function $f \in \mathcal{A}$ *defined by* (1.1)*, we say that*

$$f \in \mathcal{B}_{\Sigma}^{m,n,\gamma}(h, p) \qquad (m, n \in \mathbb{N}_0, \, m > n, \, \gamma \in \mathbb{C}\backslash\{0\})$$

if the following conditions are satisfied:

$$f \in \Sigma \qquad and \qquad 1 + \frac{1}{\gamma}\left(\frac{D^m f(z)}{D^n f(z)} - 1\right) \in h(\mathbb{U}) \quad (z \in \mathbb{U}) \tag{1.11}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{D^m g(w)}{D^n g(w)} - 1\right) \in p(\mathbb{U}) \quad (w \in \mathbb{U}), \tag{1.12}$$

where the function g *is defined by* (1.2)*.*

Remark 1 *If we choose the functions* h *and* p *as*

$$h\left(z\right) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad and \quad p\left(z\right) = \left(\frac{1-z}{1+z}\right)^{\alpha} \qquad \left(0 < \alpha \leq 1,\, z \in \mathbb{U}\right),$$

then it is clear that the functions h *and* p *satisfy the conditions in Definition* 4. *Therefore, we get a new subclass*

$$\mathcal{B}_{\Sigma}^{m,n,\gamma}\left(\left(\frac{1+z}{1-z}\right)^{\alpha}, \left(\frac{1-z}{1+z}\right)^{\alpha}\right) = \mathcal{B}_{\Sigma}^{m,n,\gamma}\left(\alpha\right) \qquad \left(0 < \alpha \leq 1,\, z \in \mathbb{U}\right)$$

consists of functions $f \in \Sigma$ *satisfying*

$$\left|\arg\left(1 + \frac{1}{\gamma}\left(\frac{D^m f\left(z\right)}{D^n f\left(z\right)} - 1\right)\right)\right| < \frac{\alpha\pi}{2} \qquad \left(0 < \alpha \leq 1,\, z \in \mathbb{U}\right)$$

and

$$\left|\arg\left(1 + \frac{1}{\gamma}\left(\frac{D^m g(w)}{D^n g(w)} - 1\right)\right)\right| < \frac{\alpha\pi}{2} \qquad \left(0 < \alpha \leq 1,\, w \in \mathbb{U}\right)$$

where the function g *is defined by* (1.2).

Remark 2 *If we choose the functions* h *and* p *as*

$$h\left(z\right) = \frac{1 + (1 - 2\beta)\, z}{1 - z} \quad and \quad p\left(z\right) = \frac{1 - (1 - 2\beta)\, z}{1 + z} \qquad \left(0 \leq \beta < 1,\, z \in \mathbb{U}\right),$$

it is evident that the functions h *and* p *satisfy the conditions in Definition* 4. *Therefore, we get a new subclass*

$$\mathcal{B}_{\Sigma}^{m,n,\gamma}\left(\frac{1 + (1 - 2\beta)\, z}{1 - z}, \frac{1 - (1 - 2\beta)\, z}{1 + z}\right) = \mathcal{B}_{\Sigma}^{m,n,\gamma}\left(\beta\right) \qquad \left(0 \leq \beta < 1,\, z \in \mathbb{U}\right)$$

consisting of functions $f \in \Sigma$ *satisfying*

$$\Re\left(1 + \frac{1}{\gamma}\left(\frac{D^m f\left(z\right)}{D^n f\left(z\right)} - 1\right)\right) > \beta \qquad \left(0 \leq \beta < 1,\, z \in \mathbb{U}\right)$$

and

$$\Re\left(1 + \frac{1}{\gamma}\left(\frac{D^m g(w)}{D^n g(w)} - 1\right)\right) > \beta \qquad \left(0 \leq \beta < 1,\, w \in \mathbb{U}\right),$$

where the function g *is defined by* (1.2).

Remark 3 *For the special choices of the parameters* m, n, *and* γ *in Remarks* 1 *and* 2, *we get the following known classes:*

(i) Letting $\gamma = 1$, *we get the classes*

$$\mathcal{B}_{\Sigma}^{m,n,1}\left(\alpha\right) = \mathcal{H}_{\Sigma}^{m,n}\left(\alpha\right) \qquad and \qquad \mathcal{B}_{\Sigma}^{m,n,1}\left(\beta\right) = \mathcal{H}_{\Sigma}^{m,n}\left(\beta\right)$$

defined in Definitions 2 *and* 3, *respectively.*

(ii) *For* $m = 1$*,* $n = 0$*, and* $\gamma = 1$*, we get the class*

$$\mathcal{B}_{\Sigma}^{1,0,1}(\alpha) = \mathcal{S}_{\Sigma}^{*}[\alpha] \qquad and \qquad \mathcal{B}_{\Sigma}^{1,0,1}(\beta) = \mathcal{S}_{\Sigma}^{*}(\beta)$$

of strongly bi-starlike functions of order α $(0 < \alpha \leq 1)$ *and bi-starlike functions of order* β $(0 \leq \beta < 1)$*, respectively.*

(iii) *For* $m = 2$*,* $n = 1$ *and* $\gamma = 1$*, we get the class*

$$\mathcal{B}_{\Sigma}^{2,1,1}(\beta) = \mathcal{C}_{\Sigma}(\beta)$$

of bi-convex functions of order β*. The classes* $\mathcal{S}_{\Sigma}^{*}[\alpha]$*,* $\mathcal{S}_{\Sigma}^{*}(\beta)$*, and* $\mathcal{C}_{\Sigma}(\beta)$ *are introduced and studied by Brannan and Taha [1].*

In the light of the work by Şeker [5], we investigate the coefficient problem for functions $f \in \mathcal{A}$ given by (1.1) belonging to the bi-univalent function class $\mathcal{B}_{\Sigma}^{m,n,\gamma}(h,p)$ introduced in Definition 4 and give coefficient bounds on $|a_2|$ and $|a_3|$. We obtain the improvements of results obtained by Şeker [5] given in Theorems 1.2 and 1.3 as a result of our main theorem (Theorem 2.1).

2. Main theorem

Theorem 2.1 *Let the function* $f(z)$ *given by* (1.1) *be in the bi-univalent function class* $\mathcal{B}_{\Sigma}^{m,n,\gamma}(h,p)$*. Then*

$$|a_2| \leq \min\left\{\sqrt{|\gamma|^2 \frac{|h'(0)|^2 + |p'(0)|^2}{2(2^m - 2^n)^2}}, \sqrt{|\gamma| \frac{|h''(0)| + |p''(0)|}{4[(3^m - 3^n) - 2^n(2^m - 2^n)]}}\right\} \tag{2.1}$$

and

$$
\begin{aligned}
|a_3| \quad \leq \quad &\min\left\{|\gamma|^2 \frac{|h'(0)|^2 + |p'(0)|^2}{2(2^m - 2^n)^2} + |\gamma| \frac{|h''(0)| + |p''(0)|}{4(3^m - 3^n)},\right. \\
&\left. |\gamma| \frac{[2(3^m - 3^n) - 2^n(2^m - 2^n)]|h''(0)| + 2^n(2^m - 2^n)|p''(0)|}{4(3^m - 3^n)[(3^m - 3^n) - 2^n(2^m - 2^n)]}\right\}.
\end{aligned}
\tag{2.2}
$$

Proof Let us consider the functions h and p satisfying the conditions of Definition 4 and having the form

$$h(z) = 1 + h_1 z + h_2 z^2 + \cdots \quad (z \in \mathbb{U})$$

and

$$p(w) = 1 + p_1 w + p_2 w^2 + \cdots \quad (w \in \mathbb{U}),$$

respectively. Since $f \in \mathcal{B}_{\Sigma}^{m,n,\gamma}(h,p)$, we can write the inequalities in (1.11) and (1.12) as

$$1 + \frac{1}{\gamma}\left(\frac{D^m f(z)}{D^n f(z)} - 1\right) = h(z) \quad (z \in \mathbb{U}), \tag{2.3}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{D^m g(w)}{D^n g(w)} - 1\right) = p(w) \quad (w \in \mathbb{U}), \tag{2.4}$$

respectively. By equating the coefficients in the equaitons (2.3) and (2.4), we get

$$(2^m - 2^n) a_2 = \gamma h_1, \tag{2.5}$$

$$(3^m - 3^n) a_3 - 2^n (2^m - 2^n) a_2^2 = \gamma h_2, \tag{2.6}$$

$$-(2^m - 2^n) a_2 = \gamma p_1, \tag{2.7}$$

and

$$-(3^m - 3^n) a_3 + [2(3^m - 3^n) - 2^n (2^m - 2^n)] a_2^2 = \gamma p_2. \tag{2.8}$$

From (2.5) and (2.7), we obtain

$$h_1 = -p_1 \tag{2.9}$$

and

$$2(2^m - 2^n)^2 a_2^2 = \gamma^2 \left(h_1^2 + p_1^2 \right). \tag{2.10}$$

Also, from (2.6) and (2.8), we find that

$$2\left[(3^m - 3^n) - 2^n (2^m - 2^n) \right] a_2^2 = \gamma (h_2 + p_2). \tag{2.11}$$

Therefore, we find from the equations (2.10) and (2.11) that

$$|a_2|^2 \le |\gamma|^2 \frac{|h'(0)|^2 + |p'(0)|^2}{2(2^m - 2^n)^2}$$

and

$$|a_2|^2 \le |\gamma| \frac{|h''(0)| + |p''(0)|}{4\left[(3^m - 3^n) - 2^n (2^m - 2^n) \right]},$$

respectively. From the above equalities, we get the bounds on $|a_2|$ as asserted in (2.1).

Similarly, in order to find the bound on $|a_3|$, we subtract (2.8) from (2.6). Thus, we obtain the equality

$$2(3^m - 3^n) a_3 - 2(3^m - 3^n) a_2^2 = \gamma (h_2 - p_2). \tag{2.12}$$

Upon substituting the value of a_2^2 from (2.10) into (2.12), it follows that

$$a_3 = \frac{\gamma^2 \left(h_1^2 + p_1^2 \right)}{2(2^m - 2^n)^2} + \frac{\gamma (h_2 - p_2)}{2(3^m - 3^n)}.$$

We thus find that

$$|a_3| \le |\gamma|^2 \frac{|h'(0)|^2 + |p'(0)|^2}{2(2^m - 2^n)^2} + |\gamma| \frac{|h''(0)| + |p''(0)|}{4(3^m - 3^n)}.$$

On the other hand, upon substituting the value of a_2^2 from (2.11) into (2.12), it follows that

$$a_3 = \frac{\gamma (h_2 + p_2)}{2\left[(3^m - 3^n) - 2^n (2^m - 2^n) \right]} + \frac{\gamma (h_2 - p_2)}{2(3^m - 3^n)}$$

or equivalently

$$a_3 = \frac{[2(3^m - 3^n) - 2^n(2^m - 2^n)]\gamma h_2 + 2^n(2^m - 2^n)\gamma p_2}{2(3^m - 3^n)[(3^m - 3^n) - 2^n(2^m - 2^n)]}.$$

We thus obtain

$$|a_3| \leq |\gamma| \frac{[2(3^m - 3^n) - 2^n(2^m - 2^n)]|h''(0)| + 2^n(2^m - 2^n)|p''(0)|}{4(3^m - 3^n)[(3^m - 3^n) - 2^n(2^m - 2^n)]}.$$

This evidently completes the proof of Theorem 2.1. $\qquad\qquad\square$

3. Corollaries of the main theorem

In Theorem 2.1, if we choose the functions h and p as asserted in Remark 1, then we obtain following result.

Corollary 3.1 *Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $\mathcal{B}_\Sigma^{m,n,\gamma}(\alpha)$ $(0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \min\left\{\frac{2|\gamma|\alpha}{2^m - 2^n}, \sqrt{\frac{2|\gamma|\alpha^2}{(3^m - 3^n) - 2^n(2^m - 2^n)}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{4|\gamma|^2\alpha^2}{(2^m - 2^n)^2} + \frac{2|\gamma|\alpha^2}{3^m - 3^n}, \frac{2|\gamma|\alpha^2}{(3^m - 3^n) - 2^n(2^m - 2^n)}\right\}.$$

Letting $\gamma = 1$ in Corollary 3.1, we obtain following consequence.

Corollary 3.2 *Let the function $f(z)$ given by (1.1) be in the bi-univalent function class $\mathcal{H}_\Sigma^{m,n}(\alpha)$ $(0 < \alpha \leq 1)$. Then*

$$|a_2| \leq \begin{cases} \frac{2\alpha}{2^m - 2^n} & , \quad 2(3^m - 3^n) \leq 2^{2m} - 2^{2n} \\ \\ \sqrt{\frac{2\alpha^2}{(3^m - 3^n) - 2^n(2^m - 2^n)}} & , \quad 2(3^m - 3^n) \geq 2^{2m} - 2^{2n} \end{cases}$$

and

$$|a_3| \leq \begin{cases} \frac{4\alpha^2}{(2^m - 2^n)^2} + \frac{2\alpha^2}{3^m - 3^n} & , \quad \delta(m,n) \leq 0 \\ \\ \frac{2\alpha^2}{(3^m - 3^n) - 2^n(2^m - 2^n)} & , \quad \delta(m,n) \geq 0 \end{cases},$$

where

$$\delta(m,n) = 2(3^m - 3^n)^2 - 2^n(2^m - 2^n)^3 - 2^{n+1}(2^m - 2^n)(3^m - 3^n).$$

Remark 4 *If we compare the coefficient bounds on $|a_2|$ in Corollary 3.2 with Theorem 1.2, then after some calculations we find that*

$$\frac{2\alpha}{2^m - 2^n} \leq \frac{2\alpha}{\sqrt{2\alpha(3^m - 3^n) + (2^m - 2^n)^2 - \alpha(2^{2m} - 2^{2n})}}$$

when $2\left(3^m - 3^n\right) \leq 2^{2m} - 2^{2n}$ *and*

$$\sqrt{\frac{2\alpha^2}{\left(3^m - 3^n\right) - 2^n\left(2^m - 2^n\right)}} \leq \frac{2\alpha}{\sqrt{2\alpha\left(3^m - 3^n\right) + \left(2^m - 2^n\right)^2 - \alpha\left(2^{2m} - 2^{2n}\right)}}$$

when $2\left(3^m - 3^n\right) \geq 2^{2m} - 2^{2n}$. *Similarly, if we compare the coefficient bounds on* $|a_3|$ *in Corollary* 3.2 *with Theorem* 1.2, *then we see that*

$$\frac{4\alpha^2}{\left(2^m - 2^n\right)^2} + \frac{2\alpha^2}{3^m - 3^n} \leq \frac{2\alpha}{3^m - 3^n} + \frac{4\alpha^2}{\left(2^m - 2^n\right)^2},$$

and

$$\frac{2\alpha^2}{\left(3^m - 3^n\right) - 2^n\left(2^m - 2^n\right)} \leq \frac{2\alpha}{3^m - 3^n} + \frac{4\alpha^2}{\left(2^m - 2^n\right)^2}$$

when $\delta(m,n) \geq 0$. *Therefore, we see that the coefficient bounds obtained in Corollary* 3.2 *is an improvent of the estimates given in Theorem* 1.2.

Letting $m = 1$ and $n = 0$ in Corollary 3.2, we have the following result.

Corollary 3.3 *Let the function* $f(z)$ *given by* (1.1) *be in the bi-univalent function class* $\mathcal{S}_\Sigma^*[\alpha]$ $(0 < \alpha \leq 1)$. *Then*

$$|a_2| \leq \sqrt{2}\alpha \qquad and \qquad |a_3| \leq 2\alpha^2.$$

Remark 5 *Note that the coefficient estimates obtained in Corollary* 3.3 *is an improvement of the estimates obtained by Brannan and Taha* [1, Theorem 2.1].

In Theorem 2.1, if we choose the functions h and p as asserted in Remark 2, then we obtain following result.

Corollary 3.4 *Let the function* $f(z)$ *given by* (1.1) *be in the bi-univalent function class* $\mathcal{B}_\Sigma^{m,n,\gamma}(\beta)$ $(0 \leq \beta < 1)$. *Then*

$$|a_2| \leq \min\left\{\frac{2\left|\gamma\right|(1-\beta)}{2^m - 2^n}, \sqrt{\frac{2\left|\gamma\right|(1-\beta)}{\left(3^m - 3^n\right) - 2^n\left(2^m - 2^n\right)}}\right\}$$

and

$$|a_3| \leq \min\left\{\frac{4\left|\gamma\right|^2(1-\beta)^2}{\left(2^m - 2^n\right)^2} + \frac{2\left|\gamma\right|(1-\beta)}{3^m - 3^n}, \frac{2\left|\gamma\right|(1-\beta)}{\left(3^m - 3^n\right) - 2^n\left(2^m - 2^n\right)}\right\}.$$

Letting $\gamma = 1$ in Corollary 3.4, we obtain following consequence.

Corollary 3.5 *Let the function* $f(z)$ *given by* (1.1) *be in the bi-univalent function class* $\mathcal{H}_\Sigma^{m,n}(\beta)$ $(0 \leq \beta < 1)$. *Then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{\left(3^m - 3^n\right) - 2^n\left(2^m - 2^n\right)}} & , \quad 0 \leq \beta \leq 1 - \frac{\left(2^m - 2^n\right)^2}{2\left[\left(3^m - 3^n\right) - 2^n\left(2^m - 2^n\right)\right]} \\[4mm] \frac{2(1-\beta)}{2^m - 2^n} & , \quad 1 - \frac{\left(2^m - 2^n\right)^2}{2\left[\left(3^m - 3^n\right) - 2^n\left(2^m - 2^n\right)\right]} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \begin{cases} \frac{2(1-\beta)}{(3^m-3^n)-2^n(2^m-2^n)} & , \quad 0 \leq \beta \leq 1 - \frac{2^n(2^m-2^n)^3}{2(3^m-3^n)[(3^m-3^n)-2^n(2^m-2^n)]} \\[3mm] \frac{4(1-\beta)^2}{(2^m-2^n)^2} + \frac{2(1-\beta)}{3^m-3^n} & , \quad 1 - \frac{2^n(2^m-2^n)^3}{2(3^m-3^n)[(3^m-3^n)-2^n(2^m-2^n)]} \leq \beta < 1 \end{cases}.$$

Remark 6 *It is evident that the coefficient bounds obtained in Corollary* 3.5 *is an improvent of the estimates given in Theorem* 1.3*.*

Letting $m = 1$ and $n = 0$ in Corollary 3.5, we have the following result.

Corollary 3.6 *Let the function* $f(z)$ *given by* (1.1) *be in the bi-univalent function class* $\mathcal{S}^*_\Sigma(\beta)$ $(0 \leq \beta < 1)$. *Then*

$$|a_2| \leq \begin{cases} \sqrt{2(1-\beta)} & , \quad 0 \leq \beta \leq \frac{1}{2} \\[2mm] 2(1-\beta) & , \quad \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq \begin{cases} 2(1-\beta) & , \quad 0 \leq \beta \leq \frac{3}{4} \\[2mm] (1-\beta)(5-4\beta) & , \quad \frac{3}{4} \leq \beta < 1 \end{cases}.$$

Remark 7 *Note that the estimates obtained in Corollary* 3.6 *is an improvement of the estimates obtained by Brannan and Taha [*1*, Theorem 3.1].*

Letting $m = 2$ and $n = 1$ in Corollary 3.5, we have the following result.

Corollary 3.7 *Let the function* $f(z)$ *given by* (1.1) *be in the bi-univalent function class* $\mathcal{C}_\Sigma(\beta)$ $(0 \leq \beta < 1)$. *Then*

$$|a_2| \leq 1 - \beta$$

and

$$|a_3| \leq \begin{cases} 1 - \beta & , \quad 0 \leq \beta \leq \frac{1}{3} \\[2mm] (1-\beta)^2 + \frac{1-\beta}{3} & , \quad \frac{1}{3} \leq \beta < 1 \end{cases}.$$

Remark 8 *Note that the estimates obtained in Corollary* 3.7 *is an improvement of the estimates obtained by Brannan and Taha [*1*, Theorem 4.1].*

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