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A class of finite difference schemes for singularly perturbed delay differential equations of second order

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Abstract: In this paper, we proposed a new class of finite difference schemes for solving singularly perturbed delay differential equation of second order. The proposed schemes are oscillation-free and more accurate than conventional schemes on a uniform mesh. These schemes are easily adaptable on special meshes like Shishkin mesh or Bakhvalov mesh and are uniformly convergent with respect to the perturbation parameter. The error analysis has been carried out and numerical examples are presented to show the accuracy and efficiency of the proposed schemes.

Key words: Delay differential, singular perturbation, boundary layer, finite difference method

1. Introduction

Singular perturbation problems (SPPs) arise very frequently in fluid dynamics, elasticity, aerodynamics, plasma dynamics, magneto hydrodynamics, rarefied gas dynamics, oceanography, and other domains of the great world of fluid motion. These problems depend on a small positive parameter in such a way that the solution varies rapidly in some parts of the domain and varies slowly in some other parts of the domain. Thus, typically there are thin transition layers where the solution varies rapidly or jumps abruptly while away from the layers, the solution behaves regularly and varies slowly. If we apply the existing standard numerical methods for solving these problems, large oscillations may arise and pollute the solution in the entire interval because of the boundary layer behavior. A more general type of the differential equations, often called functional differential equations, is one in which the unknown function occur with various different arguments. The simplest functional differential equations are 'delay differential equations'. Delay differential equations are similar to ordinary differential equations, but their evolution involves past values of the state variable. The solution of delay differential equations therefore requires knowledge of not only the current state, but also the state a certain time previously. In the last few decades, there has been a growing interest in the study of delay differential equations due to their occurrence in a wide variety of application fields such as biosciences, control theory, economics, material science, medicine, and robotics. Any system involving a feedback control will almost always involve time delays. These arise because a finite time is required to sense the information and then to react to it. The delays or lags can represent gestation times, incubation periods, transport delays. Delay models also being prominent in describing several aspects of infectious disease dynamics such as primary infection, drug therapy, and immune response etc. Delays have also appeared in the study of chemostat models, circadian rhythms, epidemiology, the respiratory system, tumor growth and neural networks. Statistical analysis of ecological data has shown

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that there is evidence of delay effects in the population dynamics of many species. A singularly perturbed delay differential equation is a differential equation in which the highest order derivative is multiplied by a small parameter and involving at least one delay term. SPPs are generally the first approximation of the considered physical model. Hence, in such cases, a more realistic model should include some of the past and the future states of the system; hence, a real system should be modeled by differential equations with delay or advance. Such type of equation arises frequently in the mathematical modeling of various practical phenomena, for example, in the modeling of the human pupil-light reflex, model of HIV infection, the study of bistable devices in digital electronics, variational problem in control theory, first exist time problems in modeling of activation of neuronal variability, immune response, mathematical ecology, population dynamics, the modeling of biological oscillators and in a variety of models for physiological process. Lange and Miura [11–13] studied the asymptotic analysis of singularly perturbed boundary value problems for differential-difference equations. This study motivated many researchers to work on numerics of singularly perturbed differential-difference equations. Kadalbajoo and Sharma [8–10] gave a series of papers on singularly perturbed delay differential equations with small delay. Amiraliyev and Cimen [1] proposed an exponential fitted difference scheme on a uniform mesh for singularly perturbed differential equations with large delay. The method is found to be first order accurate. Subburayan and Ramanujam [16, 17] developed an initial value method for singularly perturbed delay differential equations on Shishkin mesh. In [18], they proposed an asymptotic initial value method for singularly perturbed delay differential equations in which coefficient of convection-diffusion term is discontinuous.

Standard central difference schemes on uniform mesh are unstable and gives oscillatory solution for these problems. To get an oscillatory-free solution, more mesh points are required in the layer region. If prior knowledge about the location of the layer is available, one can use adaptive meshes developed by Bakhvalov [2], Gartland [6], and Shishkin [14]. Shishkin meshes are used widely because of their simplicity. The major drawback of Shishkin meshes is the requirement of prior information of the location of the layer regions. Since standard finite difference schemes fail to capture the layer region perfectly, here we developed new finite difference schemes on uniform mesh by taking infinite terms in Taylor's expansions. We followed the step of He and Wang [22–24] to propose new finite difference schemes.

The paper is organized as follows: We stated the problem under consideration in Section 2. Construction of finite difference schemes for constant coefficient problems and variable coefficient problems are discussed in Section 3. Error estimates are derived in Section 4. To demonstrate the efficiency and applicability of the proposed schemes, numerical experiments are carried out for four test problems and results are given in Section 5. The paper ends with conclusion in the last section.

2. Statement of the problem

We consider a second-order singularly perturbed delay differential equation of the form:

$$-\varepsilon y^{''}(x) + a(x)y^{'}(x) + b(x)y(x-1) = f(x), \;\; x \in \Omega, \tag{1}$$

subject to the interval and boundary conditions

$$y(x) = \phi(x), \;\; x \in \Omega_0,$$

$$y(2) = \beta, \tag{2}$$

where $\Omega = (0,2)$, $\Omega_0 = [-1,0]$ and ε is a small perturbation parameter $(0 < \varepsilon \ll 1)$, $a(x) \geq \alpha > 0$ with α being a constant and $a(x)$, $b(x)$, $f(x)$ are supposed to be smooth functions on Ω, $\phi(x)$ is a smooth function on Ω_0 and β is a given constant. The boundary value problem (1), along with (2), has unique solution [4]. It also exhibits a boundary layer at $x = 2$.

Lemma 2.1 *[15] Assume that $a(x) \geq \alpha > 0$ and $a(x)$, $b(x)$, $f(x)$, $\phi(x)$ are sufficiently smooth. Then the solution y of (1) with homogeneous Direchlet boundary conditions satisfy*

$$|y^{(n)}(x)| \leq C\Big[1 + \epsilon^{-n}\exp\Big(-\alpha\frac{2-x}{\epsilon}\Big)\Big], \quad n \in Z^+, \quad for\ 0 \leq x \leq 2,$$

where C is a generic constant which is independent of ε.

3. Construction of finite difference schemes

We develop the finite difference methods for constant coefficient problems and variable coefficient problems separately. We divide the interval $\bar{\Omega} = [0,2]$ into $2N$ equal parts with constant mesh size h. We choose the mesh size h such that the delay $x = 1$ must be a mesh point. Let $x_0 = 0, x_1, x_2, ..., x_N = 1, x_{N+1}, x_{N+2}, ..., x_{2N} = 2$ be the mesh points. Namely, $x_i = ih$ for $i = 0, 1, 2, ..., 2N$.

3.1. Constant coefficient problems

In this subsection, we will derive the new class of schemes for problem (1), when $a(x)$ and $b(x)$ are constants. From (1), we have

$$y^{(2)}(x) = \frac{a}{\varepsilon}y^{(1)}(x) + \frac{b}{\varepsilon}y(x-1) - \frac{1}{\varepsilon}f(x),$$

$$y^{(3)}(x) = \frac{a^2}{\varepsilon^2}y^{(1)}(x) + \frac{ab}{\varepsilon^2}y(x-1) + \frac{b}{\varepsilon}y^{(1)}(x-1) - \frac{a}{\varepsilon^2}f(x) - \frac{1}{\varepsilon}f^{(1)}(x),$$

$$y^{(4)}(x) = \frac{a^3}{\varepsilon^3}y^{(1)}(x) + \frac{a^2b}{\varepsilon^3}y(x-1) + \frac{ab}{\varepsilon^2}y^{(1)}(x-1)\frac{b}{\varepsilon}y^{(2)}(x-1) - \frac{a^2}{\varepsilon^3}f(x) - \frac{a}{\varepsilon^2}f^{(1)}(x) - \frac{1}{\varepsilon}f^{(2)}(x),$$

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$$y^{(n)}(x) = \Big(\frac{a}{\varepsilon}\Big)^{n-1}y'(x) + \frac{b}{\varepsilon}\sum_{j=0}^{n-2}\Big(\frac{a}{\varepsilon}\Big)^{n-2-j}y^{(j)}(x-1) - \frac{1}{\varepsilon}\sum_{j=0}^{n-2}\Big(\frac{a}{\varepsilon}\Big)^{n-2-j}f^{(j)}(x).$$

This $y^{(n)}(x)$ can be rewritten as

$$y^{(n)}(x) = \Big(\frac{a}{\varepsilon}\Big)^{n-1}y'(x) + \frac{1}{\varepsilon}\sum_{j=0}^{n-2}\Big(\frac{a}{\varepsilon}\Big)^{n-2-j}[by^{(j)}(x-1) - f^{(j)}(x)]. \tag{3}$$

Taylor's expansions of y at mesh points x_{i+1} and x_{i-1} are respectively given by

$$y_{i+1} = y_i + hy_i^{'} + \frac{h^2}{2!}y_i^{''} + \frac{h^3}{3!}y_i^{'''} + ... + \frac{h^n}{n!}y_i^n + ..., \tag{4}$$

$$y_{i-1} = y_i - hy_i^{'} + \frac{h^2}{2!}y_i^{''} - \frac{h^3}{3!}y_i^{'''} + ... + (-1)^n\frac{h^n}{n!}y_i^n + ... \tag{5}$$

Using (3) in the above Taylor's series expansions, we have

$$y_{i+1} = y_i + hy_i^{'} + \sum_{n=2}^{\infty}\frac{h^n}{n!}\Big[\Big(\frac{a}{\varepsilon}\Big)^{n-1}y'_i + \frac{1}{\varepsilon}\sum_{j=0}^{n-2}\Big(\frac{a}{\varepsilon}\Big)^{n-2-j}(by_{i-N}^{(j)} - f_i^{(j)})\Big]$$

$$= y_i + \sum_{n=1}^{\infty}\frac{h^n}{n!}\Big(\frac{a}{\varepsilon}\Big)^{n-1}y'_i + \sum_{n=2}^{\infty}\frac{h^n}{n!}\Big[\frac{1}{\varepsilon}\sum_{j=0}^{n-2}\Big(\frac{a}{\varepsilon}\Big)^{n-2-j}(by_{i-N}^{(j)} - f_i^{(j)})\Big].$$

Now, we rewrite y_{i+1} as

$$y_{i+1} = y_i + \frac{\varepsilon}{a}(e^r - 1)y_i^{'} + A_i. \tag{6}$$

Similarly,

$$y_{i-1} = y_i + \frac{\varepsilon}{a}(e^{-r} - 1)y_i^{'} + B_i. \tag{7}$$

Here

$$r = \frac{ah}{\varepsilon},$$

$$A_i = \sum_{n=2}^{\infty}\frac{h^n}{n!}\Big[\frac{1}{\varepsilon}\sum_{j=0}^{n-2}\Big(\frac{a}{\varepsilon}\Big)^{n-2-j}(by_{i-N}^{(j)} - f_i^{(j)})\Big]$$

$$= \sum_{n=0}^{\infty}\frac{\varepsilon^{n+1}}{a^{n+2}}\Big[e^r - \sum_{s=0}^{n+1}\frac{r^s}{s!}\Big](by_{i-N}^{(n)} - f_i^{(n)}),$$

and

$$B_i = \sum_{n=0}^{\infty}\frac{\varepsilon^{n+1}}{a^{n+2}}\Big[e^{-r} - \sum_{s=0}^{n+1}\frac{(-r)^s}{s!}\Big](by_{i-N}^{(n)} - f_i^{(n)}).$$

Eliminating $y_i^{'}$, from (6) and (7), we have

$$y_{i-1} - (e^{-r} + 1)y_i + e^{-r}y_{i+1} = H_i, \tag{8}$$

where

$$H_i = \sum_{n=0}^{\infty}\frac{\varepsilon^{n+1}}{a^{n+2}}\Big[\sum_{s=0}^{n+1}\frac{r^s}{s!}[(-1)^{s+1} - e^{-r}]\Big](by_{i-N}^{(n)} - f_i^{(n)}).$$

Let Y_i is the approximate solution of (1), then taking the first m terms of H_i, we have

$$Y_{i-1} - (e^{-r} + 1)Y_i + e^{-r}Y_{i+1} = H_i^* \quad for \quad 1 < i < 2N - 1, \tag{9}$$

where

$$H_i^* = \sum_{n=0}^{m} \frac{\varepsilon^{n+1}}{a^{n+2}} \Big[\sum_{s=0}^{n+1} \frac{r^s}{s!} [(-1)^{s+1} - e^{-r}] \Big] (bY_{i-N}^{(n)} - f_i^{(n)}).$$

For different positive integer values of m, we have different finite difference schemes.

Numerical algorithm:

Step 1: We obtain the reduced problem by setting $\varepsilon = 0$ in (1) with appropriate interval condition. Let $y_0(x)$ be the solution of reduced problem of (1)-(2), i.e.

$$a(x)y_0^{'}(x) + b(x)y_0(x-1) = f(x),$$

with interval condition

$$y_0(x) = \phi(x), \qquad -1 \leq x \leq 0.$$

By using the Runge–Kutta method, we solved the above problem in $0 \leq x \leq 1$ to obtain $y_0(1)$.

Step 2: To obtain the solution on $0 < x < 1$, we consider the numerical scheme from (9) which is of the form

$$Y_{i-1} - (e^{-r} + 1)Y_i + e^{-r}Y_{i+1} = H_i^* \quad for \quad 1 < i < N-1,$$

where

$$H_i^* = \sum_{n=0}^{m} \frac{\varepsilon^{n+1}}{a^{n+2}} \Big[\sum_{s=0}^{n+1} \frac{r^s}{s!} [(-1)^{s+1} - e^{-r}] \Big] (b\phi_{i-N}^{(n)} - f_i^{(n)}).$$

We solve the above system with the boundary conditions

$$Y(0) = \phi(0), \quad Y(N) = y_0(N),$$

using Thomas Algorithm [7].

Step 3: Now, to obtain the solution on $1 < x < 2$, we consider the numerical scheme from (9) which is of the form

$$Y_{i-1} - (e^{-r} + 1)Y_i + e^{-r}Y_{i+1} = H_i^* \quad for \quad N+1 < i < 2N-1,$$

where

$$H_i^* = \sum_{n=0}^{m} \frac{\varepsilon^{n+1}}{a^{n+2}} \Big[\sum_{s=0}^{n+1} \frac{r^s}{s!} [(-1)^{s+1} - e^{-r}] \Big] (bY_{i-N}^{(n)} - f_i^{(n)})$$

along with the boundary conditions,

$$Y(N) = y_0(N), \quad Y(2N) = \beta.$$

To apply the above scheme, we need to have the derivatives of y on the interval $(0,1)$. For $n = 1$, we obtain $y_i^{'}$ from (6) and (7) as

$$y_i^{'} = \frac{a}{\varepsilon} \Big(\frac{y_{i+1} - y_{i-1} + B_i - A_i}{e^r - e^{-r}} \Big).$$

3.2. Constant coefficient problems with discontinuous source term

Motivated by the work of [5, 20, 21], we consider the case of discontinuity in source term $f(x)$. We assumed that $f(x)$ has a jump discontinuity at $x = 1$, i.e., $f(1^-) \neq f(1^+)$. Now, the problem under consideration will be of the form:

$$-\varepsilon y^{''}(x) + a(x)y^{'}(x) + b(x)y(x-1) = \begin{cases} f_1(x), & 0 < x < 1, \\ f_2(x), & 1 < x < 2. \end{cases}$$

The numerical scheme (9) will be reduced to

$$\begin{cases} Y_{i-1} - (e^{-r}+1)Y_i + e^{-r}Y_{i+1} = H_i^* \quad for \quad 1 < i < N-1, \\ Y_{i-1} - (e^{-r}+1)Y_i + e^{-r}Y_{i+1} = H_i^{**} \quad for \quad N+1 < i < 2N-1, \end{cases} \tag{10}$$

with

$$H_i^* = \sum_{n=0}^{m} \frac{\varepsilon^{n+1}}{a^{n+2}} \Big[\sum_{s=0}^{n+1} \frac{r^s}{s!} [(-1)^{s+1} - e^{-r}] \Big] (b\phi_{i-N}^{(n)} - f_{1,i}^{(n)}),$$

$$H_i^{**} = \sum_{n=0}^{m} \frac{\varepsilon^{n+1}}{a^{n+2}} \Big[\sum_{s=0}^{n+1} \frac{r^s}{s!} [(-1)^{s+1} - e^{-r}] \Big] (bY_{i-N}^{(n)} - f_{2,i}^{(n)}).$$

To obtain the numerical results, we used the numerical algorithm described in the previous subsection.

3.3. Variable coefficient problems

In this subsection, we will derive schemes for problem (1), when $a(x)$ and $b(x)$ are not constants. From (1), n^{th} order derivative of y can be written as

$$y^{(n)}(x) = \frac{1}{\varepsilon}[a(x)y^{'}(x) + b(x)y(x-1) - f(x)]^{(n-2)}. \tag{11}$$

Using (1) recursively, it is possible to rewrite $y^{(n)}(x)$ in terms of $y'(x)$ and the derivatives of $f(x)$ as follows:

$$y^{(n)}(x) = \sum_{s=1}^{n-1} P^s y^{'}(x) + \sum_{j=0}^{n-3} \Big[\sum_{s=2}^{n-j-1} Q_j^s \Big] f^{(j)}(x) + \sum_{j=0}^{n-3} \Big[\sum_{s=2}^{n-j-1} R_j^s \Big] y^{(j)}(x-1) -$$

$$\frac{1}{\varepsilon}[f^{(n-2)}(x) - b(x)y^{(n-2)}(x-1)],$$

where P^s, Q_j^s and R_j^s are the coefficients with respect to $1/\varepsilon^s$.

To derive finite difference schemes, first of all, we consider all the terms which are with respect to $1/\varepsilon^{n-1}$ for $n \geq 2$, on the RHS of (10) are taken into account. After simplification, we get

$$y^{(n)}(x) \approx \frac{a^{n-1}(x)}{\varepsilon^{n-1}} y^{'}(x) + \frac{a^{n-2}(x)}{\varepsilon^{n-1}} b(x)y(x-1) - \frac{a^{n-2}(x)}{\varepsilon^{n-1}} f(x). \tag{12}$$

For the interior point x_i, substituting (12) into Taylor's expansions (4)-(5), and multiplying them by e^{-r_i}, we get

$$e^{-r_i} y_{i+1} \approx e^{-r_i} y_i + C_i y_i^{'} + D_i(b_i y_{i-N} - f_i), \tag{13}$$

$$e^{-r_i}y_{i-1} \approx e^{-r_i}y_i + C_i^* y_i^{'} + D_i^*(b_i y_{i-N} - f_i), \tag{14}$$

where

$$r_i = \frac{a_i h}{\varepsilon},$$

$$C_i = \frac{\varepsilon}{a_i}(1 - e^{-r_i}), \qquad D_i = \frac{\varepsilon}{a_i^2}(1 - e^{-r_i} - r_i e^{-r_i}),$$

$$C_i^* = \frac{\varepsilon}{a_i}(e^{-2r_i} - e^{-r_i}), \qquad D_i^* = \frac{\varepsilon}{a_i^2}(e^{-2r_i} - e^{-r_i} + r_i e^{-r_i}).$$

By eliminating $y_i^{'}$, from (13) and (14), we get

$$I_i y_{i-1} - J_i y_i + K_i y_{i+1} = M_i(b_i y_{i-N} - f_i) \quad for \quad 1 \le i \le 2N-1, \tag{15}$$

where

$$I_i = -e^{-r_i}C_i, \qquad J_i = e^{-r_i}(C_i^* - C_i),$$

$$K_i = e^{-r_i}C_i^*, \qquad M_i = (C_i D_i^* - C_i^* D_i).$$

To derive another finite difference scheme, we collect the terms which are with respect to $1/\varepsilon^{n-1}$ and $1/\varepsilon^{n-2}$ in $y^{(n)}$, for $n \ge 3$. After simplification, the n^{th} order derivative of y can be expressed as

$$\begin{aligned} y^{(n)}(x) \approx & \Big[\frac{a^{n-1}(x)}{\varepsilon^{n-1}} + \frac{(n-1)(n-2)}{2}\frac{a^{n-3}(x)a^{'}(x)}{\varepsilon^{n-2}}\Big]y^{'}(x) + \\ & \Big[\frac{a^{n-2}(x)}{\varepsilon^{n-1}} + \frac{n(n-3)}{2}\frac{a^{n-4}(x)a^{'}(x)}{\varepsilon^{n-2}}\Big]\big(b(x)\phi(x-1) - f(x)\big) + \\ & \frac{a^{n-3}(x)}{\varepsilon^{n-2}}\big(b^{'}(x)\phi(x-1) + b(x)\phi^{'}(x-1) - f^{'}(x)\big). \end{aligned}$$

For the interior point x_i, substituting $y^{(n)}$ for $n \ge 3$ into Taylor's expansions (4)-(5), and multiplying them by e^{-r_i}, we get

$$e^{-r_i}y_{i+1} \approx e^{-r_i}y_i + (C_i + E_i)y_i^{'} + (D_i + F_i)(b_i y_{i-N} - f_i) + G_i(b_i^{'} y_{i-N} + b_i y_{i-N}^{'} - f_i^{'}), \tag{16}$$

$$e^{-r_i}y_{i-1} \approx e^{-r_i}y_i + (C_i^* + E_i^*)y_i^{'} + (D_i^* + F_i^*)(b_i y_{i-N} - f_i) + G_i^*(b_i^{'} y_{i-N} + b_i y_{i-N}^{'} - f_i^{'}), \tag{17}$$

where

$$E_i = \frac{\varepsilon^2 a_i^{'}}{2a_i^3}[(r_i^2 - 2r_i + 2) - 2e^{-r_i}],$$

$$E_i^* = \frac{\varepsilon^2 a_i^{'}}{2a_i^3}[e^{-2r_i}(r_i^2 + 2r_i + 2) - 2e^{-r_i}],$$

$$F_i = \frac{\varepsilon^2 a_i^{'}}{2a_i^4}[r_i^2 - 2r_i + (r_i^2 + 2r_i)e^{-r_i}],$$

$$F_i^* = \frac{\varepsilon^2 a_i^{'}}{2a_i^4}[(r_i^2 - 2r_i)e^{-r_i} + (r_i^2 + 2r_i)e^{-2r_i}],$$

$$G_i = \frac{\varepsilon^2}{a_i^3}(1 - e^{-r_i} - r_i e^{-r_i} - \frac{r_i^2 e^{-r_i}}{2!}),$$

$$G_i^* = \frac{\varepsilon^2}{a_i^3}(e^{-2r_i} - e^{-r_i} + r_i e^{-r_i} - \frac{r_i^2 e^{-r_i}}{2!}).$$

By eliminating $y_i^{'}$, from (16) and (17), we get

$$I_i y_{i-1} - J_i y_i + K_i y_{i+1} = M_i(b_i y_{i-N} - f_i) + N_i(b_i^{'} y_{i-N} + b_i y_{i-N}^{'} - f_i^{'}), \tag{18}$$

where

$$I_i = -e^{-r_i}(C_i + E_i),$$

$$J_i = e^{-r_i}(C_i^* + E_i^*) - e^{-r_i}(C_i + E_i),$$

$$K_i = e^{-r_i}(C_i^* + E_i^*),$$

$$M_i = (C_i + E_i)(D_i^* + F_i^*) - (C_i^* + E_i^*)(D_i + F_i),$$

$$N_i = (C_i + E_i)G_i^* - (C_i^* + E_i^*)G_i.$$

To apply the above scheme, we need to have the derivatives of y on the interval $(0,1)$. We obtain $y_i^{'}$ from (16) and (17) as

$$y_i^{'} = \frac{G_i y_{i-1} + (G_i + G_i^*)y_i - G_i^* y_{i+1} + (G_i^*(D_i + F_i) - G_i(D_i^* + F_i^*))(b_i y_{i-N} - f_i)}{N_i}.$$

To derive another finite difference scheme, we may collect the terms which are with respect to $1/\varepsilon^{n-1}, 1/\varepsilon^{n-2}$ and $1/\varepsilon^{n-3}$ in $y^{(n)}$, for $n \geq 4$ and proceed as above. Applying the same concept, we may easily obtain different finite difference schemes.

The numerical algorithm which has been described in subsection (3.1) has been applied to get the numerical results.

4. Error analysis

In this section, we derived an error estimate for the finite difference scheme (9).

From (6) and (7), we have

$$y_{i+1} = y_i + \frac{\varepsilon}{a}(e^r - 1)y_i^{'} + A_i,$$

and

$$y_{i-1} = y_i + \frac{\varepsilon}{a}(e^{-r} - 1)y_i^{'} + B_i.$$

After neglecting the higher order terms of ε, y_{i+1} and y_{i-1} can be rewritten as

$$y_{i+1} = y_i + \frac{\varepsilon}{a}(e^r - 1)y_i^{'} + \sum_{n=0}^{2} \frac{\varepsilon^{n+1}}{a^{n+2}} \Big[e^r - \sum_{s=0}^{n+1} \frac{r^s}{s!}\Big](by_{i-N}^{(n)} - f_i^{(n)}) + R_i^*, \tag{19}$$

$$y_{i-1} = y_i + \frac{\varepsilon}{a}(e^{-r} - 1)y_i^{'} + \sum_{n=0}^{2} \frac{\varepsilon^{n+1}}{a^{n+2}} \Big[e^{-r} - \sum_{s=0}^{n+1} \frac{(-r)^s}{s!}\Big](by_{i-N}^{(n)} - f_i^{(n)}) + R_i^{**}, \tag{20}$$

where

$$R_i^* = \frac{\varepsilon^4}{a^5}\Big[e^r - \sum_{s=0}^{4} \frac{r^s}{s!}\Big](by_{i-N}^{(3)} - f_i^{(3)}),$$

$$R_i^{**} = \frac{\varepsilon^4}{a^5}\Big[e^{-r} - \sum_{s=0}^{4} \frac{(-r)^s}{s!}\Big](by_{i-N}^{(3)} - f_i^{(3)}).$$

Adding (19) and (20), we get

$$\begin{aligned} y_{i+1} + y_{i-1} =& 2y_i + \frac{\varepsilon}{a}(e^r + e^{-r} - 2)y_i^{'} + \\ & \sum_{n=0}^{2} \frac{\varepsilon^{n+1}}{a^{n+2}} \Big[e^r e^{-r} - \sum_{s=0}^{n+1} \frac{r^s + (-r)^s}{s!}\Big]\big(by_{i-N}^{(n)} - f_i^{(n)}\big) + R_i^* + R_i^{**}. \end{aligned}$$

After rearranging the terms, we have

$$\begin{aligned} y_{i+1} + y_{i-1} - 2y_i =& \frac{\varepsilon}{a}(e^r + e^{-r} - 2)y_i^{'} + \frac{\varepsilon}{a^2}(e^r + e^{-r} - 2)(by_{i-N} - f_i) + \\ & \sum_{n=1}^{2} \frac{\varepsilon^{n+1}}{a^{n+2}} \Big[e^r + e^{-r} - \sum_{s=0}^{n+1} \frac{r^s + (-r)^s}{s!}\Big]\big(by_{i-N}^{(n)} - f_i^{(n)}\big) + R_i^* + R_i^{**}. \end{aligned}$$

Multiplying this equation with $a^2/\varepsilon(e^r + e^{-r} - 2)$, we have

$$\begin{aligned} \frac{a^2}{\varepsilon}\frac{y_{i+1} + y_{i-1} - 2y_i}{e^r + e^{-r} - 2} =& ay_i^{'} + by_{i-N} - f_i + \\ & \sum_{n=1}^{2} \frac{\varepsilon^n}{a^n}\Big[e^r + e^{-r} - \sum_{s=0}^{n+1} \frac{r^s + (-r)^s}{s!}\Big] \frac{(by_{i-N}^{(n)} - f_i^{(n)})}{(e^r + e^{-r} - 2)} + \frac{a^2}{\varepsilon}\left(\frac{R_i^* + R_i^{**}}{e^r + e^{-r} - 2}\right). \end{aligned}$$

From (1) we have, $ay_i^{'} + by_{i-N} - f_i = \varepsilon y_i^{''}$. Putting this value in above equation, we get

$$\begin{aligned} \frac{a^2}{\varepsilon}\frac{y_{i+1} + y_{i-1} - 2y_i}{e^r + e^{-r} - 2} =& \varepsilon y_i^{''} + \sum_{n=1}^{2} \frac{\varepsilon^n}{a^n}\Big[e^r + e^{-r} - \sum_{s=0}^{n+1} \frac{r^s + (-r)^s}{s!}\Big] \frac{(by_{i-N}^{(n)} - f_i^{(n)})}{(e^r + e^{-r} - 2)} + \\ & \frac{a^2}{\varepsilon}\left(\frac{R_i^* + R_i^{**}}{e^r + e^{-r} - 2}\right). \end{aligned}$$

Now define a new operator D^2 as,

$$D^2 y_i = \frac{a^2}{\varepsilon^2}\frac{y_{i+1}+y_{i-1}-2y_i}{e^r+e^{-r}-2} - \sum_{n=1}^{2}\frac{\varepsilon^{n-1}}{a^n}\left[e^r+e^{-r}-\sum_{s=0}^{n+1}\frac{r^s+(-r)^s}{s!}\right]\frac{(by_{i-N}^{(n)}-f_i^{(n)})}{e^r+e^{-r}-2} - \frac{a^2}{\varepsilon^2}\frac{R_i^*+R_i^{**}}{e^r+e^{-r}-2}. \quad (21)$$

Subtracting (20) from (19), we get

$$y_{i+1}-y_{i-1} = \frac{\varepsilon}{a}(e^r-e^{-r})y_i^{'} + \sum_{n=0}^{2}\frac{\varepsilon^{n+1}}{a^{n+2}}\left[e^r-e^{-r}-\sum_{s=0}^{n+1}\frac{r^s-(-r)^s}{s!}\right](by_{i-N}^{(n)}-f_i^{(n)}) + R_i^* - R_i^{**}.$$

Multiplying this equation with $a/\varepsilon(e^r-e^{-r})$, we have

$$\frac{a}{\varepsilon}\frac{y_{i+1}-y_{i-1}}{e^r-e^{-r}} = y_i^{'} + \sum_{n=0}^{2}\frac{\varepsilon^n}{a^{n+1}}\left[e^r-e^{-r}-\sum_{s=0}^{n+1}\frac{r^s-(-r)^s}{s!}\right]\frac{(by_{i-N}^{(n)}-f_i^{(n)})}{(e^r-e^{-r})} + \frac{a}{\varepsilon}\left(\frac{R_i^*-R_i^{**}}{e^r-e^{-r}}\right).$$

Now define a new operator D as,

$$Dy_i = \frac{a}{\varepsilon}\frac{y_{i+1}-y_{i-1}}{e^r-e^{-r}} - \sum_{n=0}^{2}\frac{\varepsilon^n}{a^{n+1}}\left[e^r-e^{-r}-\sum_{s=0}^{n+1}\frac{r^s-(-r)^s}{s!}\right]\frac{by_{i-N}^{(n)}-f_i^{(n)}}{e^r-e^{-r}} - \frac{a}{\varepsilon}\frac{R_i^*-R_i^{**}}{e^r-e^{-r}}. \quad (22)$$

Putting this operator values in (1), we get

$$-\varepsilon D^2 y_i + aDy_i + by_{i-N} = f_i + R_i, \quad (23)$$

where

$$R_i = -\frac{a^2}{\varepsilon}\frac{R_i^*+R_i^{**}}{e^r+e^{-r}-2} + \frac{a^2}{\varepsilon}\frac{R_i^*-R_i^{**}}{e^r+e^{-r}}.$$

Let Y_i is the numerical solution of (1). Neglecting the remainder terms, we have

$$-\varepsilon D^2 Y_i + aDY_i + bY_{i-N} = f_i. \quad (24)$$

Subtracting (24) from (23) and putting $y_i - Y_i = e_i$, we have

$$-\varepsilon \overline{D}^2 e_i + a\overline{D}e_i = R_i, \quad (25)$$

where

$$\overline{D}^2 e_i = \frac{a^2}{\varepsilon^2}\frac{e_{i-1}-2e_i+e_{i+1}}{e^r+e^{-r}-2}, \qquad \overline{D}e_i = \frac{a}{\varepsilon}\frac{e_{i+1}-e_{i-1}}{e^r-e^{-r}}.$$

From standard central difference operators, we have

$$\delta^2 e_i = \frac{e_{i-1}-2e_i+e_{i+1}}{h^2} = \frac{e^r+e^{-r}-2}{r^2}\overline{D}^2 e_i,$$

$$\delta e_i = \frac{e_{i+1}-e_{i-1}}{2h} = \frac{e^r-e^{-r}}{2r}\overline{D}e_i.$$

Using $\delta^2 e_i$ and δe_i values in (25), we get

$$-\varepsilon \frac{r^2}{e^r + e^{-r} - 2}\delta^2 e_i + \frac{2ar}{e^r - e^{-r}}\delta e_i = R_i. \tag{26}$$

Lemma 4.1 *[24] Assume* $U^0 = \{u \mid u = \{u_i \mid 0 \le i \le 2N\}, u_0 = u_{2N} = 0\}$, *then for all* $u \in U^0$, *the following conclusions hold*

$$h\sum_{i=1}^{2N-1}(-\delta^2 u_i)u_i = h\sum_{i=1}^{2N-1}(-\delta u_{i-1/2})^2 = |u|_1^2,$$

$$\|u\|_\infty = \max_{0\le i\le 2N}|u_i| \le \frac{\sqrt{x_e - x_s}}{2}|u|_1,$$

$$\|u\|_2 = \sqrt{h\left(\frac{u_0^2}{2} + \sum_{i=1}^{2N-1}u_i^2 + \frac{u_{2N}^2}{2}\right)} \le \frac{x_e - x_s}{\sqrt{6}}|u|_1,$$

where $|u|_1$, $\|u\|_\infty$, $\|u\|_2$ *denote* H^1 *semi-norm,* L^∞ *norm and* L^2 *norm respectively and* x_s, x_e *are the starting point and end point of the domain.*

Theorem 4.2 *If* y *is the exact solution of* (1) *and* Y *be the numerical solution obtained using the scheme* (9), *the error at the mesh point* x_i *be* $e_i = y_i - Y_i$, *then, the following inequality holds*

$$\|e\|_\infty \le \frac{Mh^2}{12\sqrt{3}a},$$

where $M = \max_{0\le i\le 2N}|by_{i-N}^{(3)} - f_i^{(3)}|$.

Proof Multiplying (26) with he_i and taking summation from $i = 1$ to $2N-1$, we get

$$\frac{r^2\varepsilon h}{e^r + e^{-r} - 2}\sum_{i=1}^{2N-1}(-\delta^2 e_i)e_i + \frac{2arh}{e^r - e^{-r}}\sum_{i=1}^{2N-1}(\delta e_i)e_i = h\sum_{i=1}^{2N-1}R_i e_i. \tag{27}$$

From Lemma 4.1

$$h\sum_{i=1}^{2N-1}(-\delta^2 e_i)e_i = |e|_1^2$$

and

$$\begin{aligned}\sum_{i=1}^{2N-1}(\delta e_i)e_i &= \frac{1}{2h}[(e_2 - e_0)e_1 + (e_3 - e_1)e_2 + (e_4 - e_2)e_3 + ... + (e_{2N} - e_{2N-2})e_{2N-1}],\\ &= \frac{1}{2h}[e_{2N}e_{2N-1} - e_0 e_1],\\ &= 0.\end{aligned}$$

Using these values in (27), we get

$$\frac{r^2\varepsilon}{e^r+e^{-r}-2}|e|_1^2 = h\sum_{i=1}^{2N-1} R_i e_i \\ \leq \|R\|_2\|e\|_2,$$

using Lemma 4.1

$$\frac{r^2\varepsilon}{e^r+e^{-r}-2}|e|_1^2 \leq \|R\|_2\sqrt{\frac{2}{3}}|e|_1,$$

can be rewritten as

$$|e|_1 \leq \sqrt{\frac{2}{3}}\|R\|_2\frac{e^r+e^{-r}-2}{r^2\varepsilon}.$$

From Lemma 4.1, L^∞ norm is defined as

$$\|e\|_\infty \leq \frac{1}{\sqrt{2}}|e|_1, \\ \leq \frac{1}{\sqrt{3}}\|R\|_\infty\frac{e^r+e^{-r}-2}{r^2\varepsilon}.$$

Let $M = \max_{0\leq i\leq 2N} |by^{(3)}_{i-N} - f^{(3)}_i|$, then

$$\|R\|_\infty = \max_{0\leq i\leq 2N}\left| -\frac{a^2}{\varepsilon}\frac{R_i^* + R_i^{**}}{e^r+e^{-r}-2} + \frac{a^2}{\varepsilon}\frac{R_i^* - R_i^{**}}{e^r+e^{-r}}\right|.$$

Using this value, we get

$$\|e\|_\infty \leq \frac{Ma^2}{\sqrt{3}\varepsilon}\frac{e^r+e^{-r}-2}{r^2\varepsilon}\left|\frac{2r^2+r^4/12}{e^r+e^{-r}-2} - \frac{2r+r^3/3}{e^r-e^{-r}}\right|\frac{\varepsilon^4}{a^5}, \\ \leq \frac{M\varepsilon^2}{\sqrt{3}a^3}\left|1+\frac{r^2}{12} - (\frac{2}{r}+\frac{r}{3})\frac{e^r+e^{-r}-2}{e^r-e^{-r}}\right|.$$

It is trivial that $(\frac{2}{r}+\frac{r}{3}) \geq 2\sqrt{2/3}$ for $r > 0$; thus,

$$\|e\|_\infty \leq \frac{M\varepsilon^2}{\sqrt{3}a^3}\frac{r^2}{12} + \frac{M\varepsilon^2}{\sqrt{3}a^3}\left|1 - \frac{2\sqrt{2}}{\sqrt{3}}\frac{e^r+e^{-r}-2}{e^r-e^{-r}}\right| \\ \leq \frac{M\varepsilon^2}{\sqrt{3}a^3}\frac{r^2}{12} \\ \leq \frac{Mh^2}{12\sqrt{3}a}.$$

Hence, the proof. □

5. Numerical examples

To demonstrate the efficiency and applicability of the proposed finite difference methods, we applied them on four test problems. Maximum pointwise errors are tabulated. Maximum point wise errors are calculated using the following double mesh principle [3] for the problems where the exact solutions are not available:

$$E_\varepsilon^N = \max_{0\le i\le N} \mid U^N(x_i) - U^{2N}(x_{2i}) \mid,$$

where $U^N(x_i)$ denote the numerical solution obtained on a mesh containing N subintervals.

The numerical rate of convergence is calculated using the formula

$$R_\varepsilon^N = \frac{\log\left(E_\varepsilon^N - E_\varepsilon^{2N}\right)}{\log 2}.$$

Example 1[16] Consider the following singularly perturbed delay differential equation with constant coefficient:

$$-\varepsilon y^{''}(x) + 3y^{'}(x) - y(x-1) = 0, \qquad 0 < x < 2,$$

$$y(x) = 1, \quad -1 \le x \le 0, \qquad y(2) = 2.$$

The exact solution of this problem is given by

$$y(x) = \begin{cases} 1 + c_1(e^{\frac{3x}{\varepsilon}} - 1) + \frac{x}{\varepsilon}, & 0 \le x \le 1 \\ c_2 + \frac{x}{\varepsilon} + \frac{(x-1)^2}{18} + \frac{\varepsilon x}{27} - \frac{c_1 x}{3} - \frac{c_1 x}{3} e^{\frac{3(x-1)}{\varepsilon}} + e^{\frac{3(x-2)}{\varepsilon}}\left(\frac{23}{18} - \frac{2\varepsilon}{27} - c_2 + \frac{2c_1}{3} + \frac{2c_1}{3} e^{\frac{3}{\varepsilon}}\right), & 1 \le x \le 2, \end{cases}$$

where

$$c_1 = e^{\frac{-6}{\varepsilon}}\Big[\frac{\frac{4\varepsilon}{9} - \frac{\varepsilon^2}{27} - 3}{3 - 4e^{-\frac{6}{\varepsilon}} + \frac{2\varepsilon}{3}(e^{-\frac{3}{\varepsilon}} - e^{-\frac{3}{\varepsilon}})}\Big],$$

$$c_2 = \frac{1 - \frac{23}{18}e^{-\frac{3}{\varepsilon}} + \frac{2\varepsilon}{27}e^{-\frac{3}{\varepsilon}} - \frac{\varepsilon}{27}}{1 - e^{-\frac{3}{\varepsilon}}} + \frac{c_1 e^{\frac{3}{\varepsilon}}[1 - e^{-\frac{3}{\varepsilon}} - \frac{2}{3}e^{-\frac{6}{\varepsilon}}]}{1 - e^{-\frac{3}{\varepsilon}}}.$$

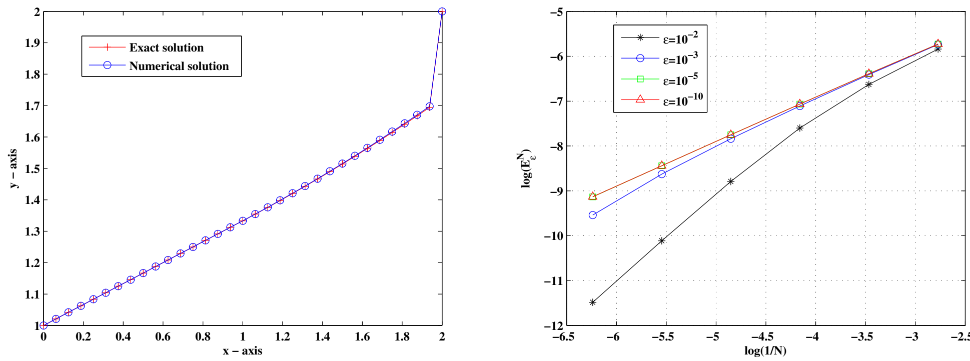
The maximum pointwise errors and rate of convergence are presented in Table 1 for different values of perturbation parameter ε. The numerical solution and exact solution using the scheme (9) with $\varepsilon = 10^{-8}$ and $N = 16$ is plotted in Figure 1a. Convergence order is plotted in Figure 1b.

Example 2 Consider the following singularly perturbed delay differential equation with discontinuous source term:

$$-\varepsilon y^{''}(x) + 5y^{'}(x) - \frac{1}{2}y(x-1) = \begin{cases} e^x & 0 < x \le 1, \\ -e^x & 1 \le x < 2, \end{cases}$$

$$y(x) = 1, \quad -1 \le x \le 0, \qquad y(2) = 2.$$

The maximum pointwise errors and rate of convergence are presented in Table 2 for different values of perturbation parameter ε. The numerical solution using the scheme (9) with $\varepsilon = 10^{-10}$ and $N = 16$ is plotted in Figure 2a. Convergence order is plotted in Figure 2b.



(a) $N = 16, \varepsilon = 10^{-8}$ (b) Log-log plot

Figure 1. Numerical solution of Example 1 and its convergence order.

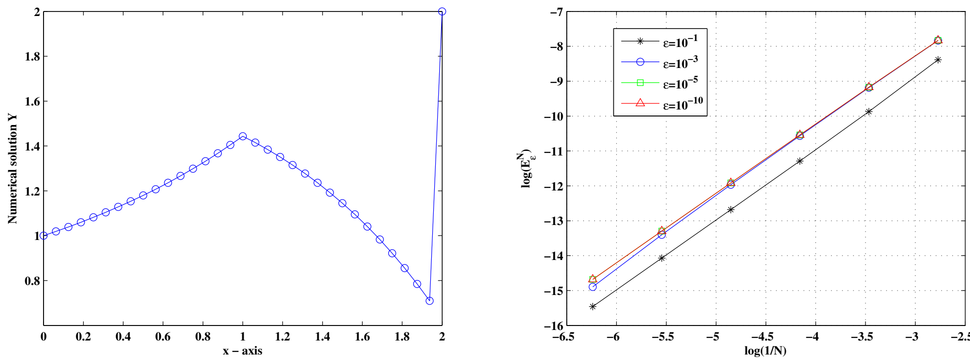
Table 1. Maximum pointwise errors of the solution and corresponding rate of convergence for Example 1 for different values of ε.

ε	N=2^4	N=2^5	N=2^6	N=2^7	N=2^8	N=2^9
10^{-1}	8.7395e-04	2.2767e-04	5.7640e-05	1.4450e-05	3.6159e-06	9.0413e-07
	1.9406	1.9818	1.9960	1.9987	1.9998	
10^{-2}	2.9080e-03	1.3232e-03	5.0106e-04	1.5202e-04	4.0518e-05	1.0300e-05
	1.1360	1.4010	1.7208	1.9076	1.9759	
10^{-3}	3.2205e-03	1.6460e-03	8.1803e-04	3.9389e-04	1.7928e-04	7.1809e-05
	0.9683	1.0087	1.0544	1.1356	1.3200	
10^{-4}	3.2517e-03	1.6783e-03	8.5085e-04	4.2696e-04	2.1248e-04	1.0460e-04
	0.9542	0.9800	0.9948	1.0068	1.0224	
10^{-5}	3.2549e-03	1.6815e-03	8.5413e-04	4.3027e-04	2.1580e-04	1.0793e-04
	0.9528	0.9772	0.9892	0.9956	0.9997	
10^{-6}	3.2552e-03	1.6818e-03	8.5446e-04	4.3060e-04	2.1613e-04	1.0826e-04
	0.9527	0.9769	0.9887	0.9945	0.9974	
10^{-7}	3.2552e-03	1.6819e-03	8.5449e-04	4.3063e-04	2.1616e-04	1.0829e-04
	0.9527	0.9769	0.9886	0.9943	0.9972	
10^{-8}	3.2552e-03	1.6819e-03	8.5449e-04	4.3064e-04	2.1617e-04	1.0829e-04
	0.9527	0.9769	0.9886	0.9943	0.9972	
10^{-9}	3.2552e-03	1.6819e-03	8.5449e-04	4.3064e-04	2.1617e-04	1.0829e-04
	0.9527	0.9769	0.9886	0.9943	0.9972	
10^{-10}	3.2552e-03	1.6819e-03	8.5449e-04	4.3064e-04	2.1617e-04	1.0830e-04
	0.9527	0.9769	0.9886	0.9943	0.9972	

Example 3 Consider the following singularly perturbed delay differential equation with variable coefficient:

$$-\varepsilon y^{''}(x) + (3 + x^2)y^{'}(x) - y(x-1) = e^x, \qquad 0 < x < 2,$$

$$y(x) = e^x, \quad -1 \leq x \leq 0, \qquad y(2) = 2.$$



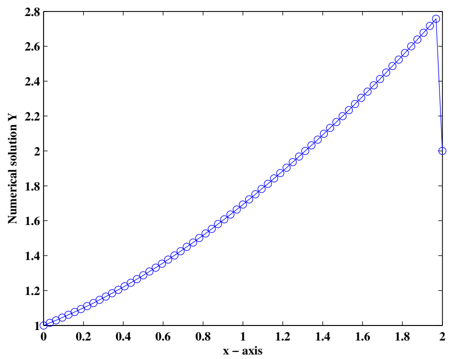
(a) $N = 16, \varepsilon = 10^{-10}$ (b) Log-log plot

Figure 2. Numerical solution of Example 2 and its convergence order.

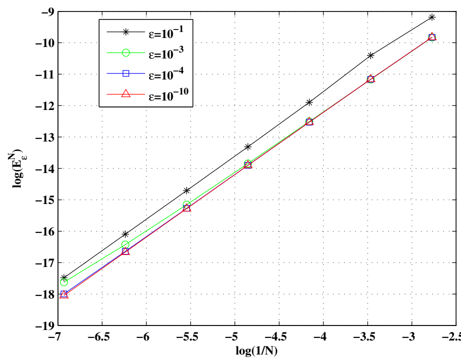
Table 2. Maximum pointwise errors of the solution and corresponding rate of convergence for Example 2 for different values of ε.

ε	N=2^4	N=2^5	N=2^6	N=2^7	N=2^8	N=2^9
10^{-1}	2.2795e-04	5.1521e-05	1.2480e-05	3.0993e-06	7.7311e-07	1.9317e-07
	2.1455	2.0455	2.0096	2.0032	2.0008	
10^{-2}	3.7341e-04	9.0998e-05	2.0214e-05	4.1623e-06	9.1820e-07	2.1894e-07
	2.0368	2.1705	2.2799	2.1805	2.0682	
10^{-3}	3.9536e-04	1.0246e-04	2.5824e-05	6.3578e-06	1.5149e-06	3.3947e-07
	1.9482	1.9882	2.0221	2.0693	2.1579	
10^{-4}	3.9756e-04	1.0360e-04	2.6409e-05	6.6539e-06	1.6637e-06	4.1284e-07
	1.9401	1.9719	1.9888	1.9998	2.0107	
10^{-5}	3.9778e-04	1.0372e-04	2.6468e-05	6.6835e-06	1.6786e-06	4.2030e-07
	1.9393	1.9703	1.9856	1.9933	1.9978	
10^{-6}	3.9780e-04	1.0373e-04	2.6474e-05	6.6865e-06	1.6801e-06	4.2105e-07
	1.9392	1.9702	1.9853	1.9927	1.9965	
10^{-7}	3.9780e-04	1.0373e-04	2.6474e-05	6.6868e-06	1.6802e-06	4.2112e-07
	1.9392	1.9702	1.9852	1.9926	1.9963	
10^{-8}	3.9780e-04	1.0373e-04	2.6475e-05	6.6868e-06	1.6802e-06	4.2113e-07
	1.9392	1.9702	1.9852	1.9926	1.9963	
10^{-9}	3.9780e-04	1.0373e-04	2.6475e-05	6.6868e-06	1.6802e-06	4.2113e-07
	1.9392	1.9702	1.9852	1.9926	1.9963	
10^{-10}	3.9780e-04	1.0373e-04	2.6475e-05	6.6868e-06	1.6802e-06	4.2113e-07
	1.9392	1.9702	1.9852	1.9926	1.9963	

The maximum pointwise errors and rate of convergence are presented in Table 3 for different values of perturbation parameter ε. The numerical solution using the scheme (18) with $\varepsilon = 10^{-5}$ and $N = 32$ is plotted in Figure 3a. Convergence order is plotted in Figure 3b.



(a) $N = 32, \varepsilon = 10^{-5}$



(b) Log-log plot

Figure 3. Numerical solution of Example 3 and its convergence order.

Table 3. Maximum pointwise errors of the solution and corresponding rate of convergence for Test Example 3 for different values of ε.

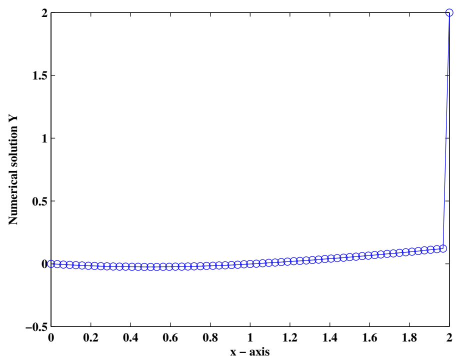
ε	N=2^4	N=2^5	N=2^6	N=2^7	N=2^8	N=2^9
10^{-1}	1.0252e-04	3.0287e-05	6.8160e-06	1.6526e-0	4.0985e-07	1.0226e-07
	1.7592	2.1517	2.0442	2.0115	2.0029	
10^{-2}	5.5691e-05	1.6152e-05	4.7920e-06	1.5496e-06	4.8674e-07	1.0949e-07
	1.7857	1.7530	1.6287	1.6707	2.1524	
10^{-3}	5.3890e-05	1.4153e-05	3.6890e-06	9.7140e-07	2.6162e-07	7.3486e-08
	1.9289	1.9398	1.9251	1.8926	1.8319	
10^{-4}	5.4057e-05	1.4167e-05	3.6233e-06	9.1794e-07	2.3269e-07	5.9270e-08
	1.9319	1.9672	1.9808	1.9800	1.9730	
10^{-5}	5.4070e-05	1.4176e-05	3.6272e-06	9.1715e-07	2.3059e-07	5.7877e-08
	1.9314	1.9665	1.9836	1.9919	1.9942	
10^{-6}	5.4071e-05	1.4176e-05	3.6275e-06	9.1737e-07	2.3065e-07	5.7825e-08
	1.9314	1.9664	1.9834	1.9918	1.9960	
10^{-7}	5.4071e-05	1.4176e-05	3.6276e-06	9.1739e-07	2.3066e-07	5.7831e-08
	1.9314	1.9664	1.9834	1.9917	1.9959	
10^{-8}	5.4071e-05	1.4176e-05	3.6276e-06	9.1739e-07	2.3066e-07	5.7831e-08
	1.9314	1.9664	1.9834	1.9917	1.9959	
10^{-9}	5.4072e-05	1.4176e-05	3.6276e-06	9.1739e-07	2.3066e-07	5.7831e-08
	1.9314	1.9664	1.9834	1.9917	1.9959	
10^{-10}	5.4068e-05	1.4177e-05	3.6275e-06	9.1740e-07	2.3066e-07	5.7831e-08
	1.9312	1.9665	1.9834	1.9917	1.9959	

Example 4[16] Consider the following singularly perturbed delay differential equation with variable coefficient:

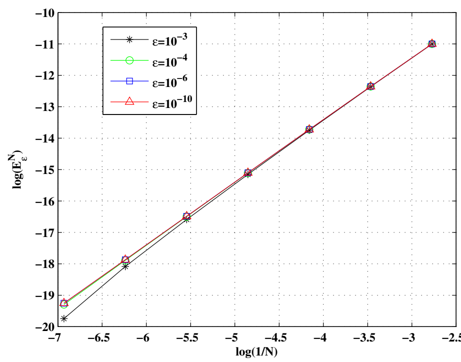
$$-\varepsilon y^{''}(x) + (x+10)y^{'}(x) - y(x-1) = x, \qquad 0 < x < 2,$$

$$y(x) = x, \quad -1 \le x \le 0, \qquad y(2) = 2.$$

The maximum pointwise errors and rate of convergence are presented in Table 5 for different values of perturbation parameter ε. The numerical solution using the scheme (18) with $\varepsilon = 10^{-8}$ and $N = 32$, is plotted in Figure 4a. Convergence order is plotted in Figure 4b.



(a) $N = 32, \varepsilon = 10^{-8}$



(b) Log-log plot

Figure 4. Numerical solution of Example 4 and its convergence order.

Table 4. Maximum pointwise errors of the solution and corresponding rate of convergence for Example 4 for different values of ε.

ε	N=2^4	N=2^5	N=2^6	N=2^7	N=2^8	N=2^9
10^{-1}	7.2946e-06	7.3960e-07	5.6434e-08	6.0406e-09	1.4013e-09	3.5351e-10
	3.3020	3.7121	3.2238	2.1079	1.9870	
10^{-2}	1.5783e-05	3.7988e-06	8.2911e-07	1.4557e-07	1.6997e-08	1.3635e-09
	2.0547	2.1959	2.5098	3.0984	3.6399	
10^{-3}	1.6660e-05	4.2712e-06	1.0702e-06	2.6267e-07	6.2504e-08	1.3956e-08
	1.9637	1.9968	2.0265	2.0712	2.1631	
10^{-4}	1.6729e-05	4.3130e-06	1.0936e-06	2.7478e-07	6.8601e-08	1.7010e-08
	1.9556	1.9796	1.9928	2.0020	2.0118	
10^{-5}	1.6736e-05	4.3164e-06	1.0955e-06	2.7590e-07	6.9202e-08	1.7315e-08
	1.9550	1.9782	1.9894	1.9953	1.9988	
10^{-6}	1.6737e-05	4.3168e-06	1.0957e-06	2.7599e-07	6.9253e-08	1.7344e-08
	1.9550	1.9781	1.9892	1.9947	1.9974	
10^{-7}	1.6737e-05	4.3168e-06	1.0957e-06	2.7599e-07	6.9257e-08	1.7346e-08
	1.9550	1.9781	1.9892	1.9947	1.9973	
10^{-8}	1.6737e-05	4.3168e-06	1.0957e-06	2.7599e-07	6.9258e-08	1.7347e-08
	1.9550	1.9781	1.9892	1.9947	1.9973	
10^{-9}	1.6737e-05	4.3168e-06	1.0957e-06	2.7599e-07	6.9258e-08	1.7347e-08
	1.9550	1.9781	1.9892	1.9947	1.9973	
10^{-10}	1.6737e-05	4.3168e-06	1.0957e-06	2.7599e-07	6.9258e-08	1.7347e-08
	1.9550	1.9781	1.9892	1.9947	1.9973	

Table 5. Maximum errors and corresponding rate of convergence for Example 4.

	$N \rightarrow$	2^6	2^7	2^8	2^9
Proposed method	E^N	1.0957e-06	2.7599e-07	6.9258e-08	1.7347e-08
	R^N	1.9892	1.9947	1.9973	
Method in [16]	E^N	2.6473e-03	8.3944e-04	2.5834e-04	8.0254e-05
	R^N	1.6570	1.7001	1.6866	

6. Conclusion

In this paper, we proposed a class of finite difference schemes to solve singularly perturbed delay differential equation of second order. The proposed schemes have different advantages. They give oscillation free solution on uniform mesh. Results are more accurate than conventional methods. These schemes can keep convergence order stable much better than conventional methods for very small values of perturbation parameter ε. Prior information about the location and width of the layer is not required. These methods are easily extendable for higher dimensional problems. The proposed numerical schemes converges uniformly with respect to the perturbation parameter ε. Numerical results are carried out to show the efficiency and accuracy of the proposed numerical schemes.

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