Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 1353 – 1371
© TÜBİTAK
doi:10.3906/mat-1902-34

Star-likeness associated with the exponential function

Adiba NAZ[1], **Sumit NAGPAL**[2*], **V. RAVICHANDRAN**[3]
[1]Department of Mathematics, Faculty of Mathematical Sciences, University of Delhi, Delhi, India
[2]Department of Mathematics, Ramanujan College, University of Delhi, Delhi, India
[3]Department of Mathematics, National Institute of Technology, Tiruchirappalli, India

Received: 07.02.2019 • **Accepted/Published Online:** 22.03.2019 • **Final Version:** 29.05.2019

Abstract: Given a domain Ω in the complex plane $\mathbb{C}$ and a univalent function q defined in an open unit disk $\mathbb{D}$ with nice boundary behaviour, Miller and Mocanu studied the class of admissible functions $\Psi(\Omega, q)$ so that the differential subordination $\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z)$ implies $p(z) \prec q(z)$, where p is an analytic function in $\mathbb{D}$ with $p(0) = 1$, $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ and $\Omega = h(\mathbb{D})$. This paper investigates the properties of this class for $q(z) = e^z$. As application, several sufficient conditions for normalized analytic functions f to be in the subclass of star-like functions associated with the exponential function are obtained.

Key words: Univalent functions, star-like functions, differential subordination, exponential function, Janowski star-like function

1. Introduction and preliminaries

Let $\mathcal{H}[a, n]$ denote the class of analytic functions defined in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$ of the form $f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \cdots$, where n is a positive integer and $a \in \mathbb{C}$. Set $\mathcal{H}_1 := \mathcal{H}[1, 1]$. Let $\mathcal{H}$ be the subclass of $\mathcal{H}[0, 1]$ consisting of functions f normalized by the condition $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ be a subclass of $\mathcal{H}$ containing univalent functions. Given any two analytic functions in $\mathbb{D}$, we say that f is subordinate to g, written as $f \prec g$, if there exists a Schwarz function w that is analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In particular, if g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Some special classes of univalent functions are of great significance in geometric function theory due to their geometric properties. By considering the analytic function $\varphi \in \mathcal{H}_1$ with positive real part in $\mathbb{D}$ that maps $\mathbb{D}$ onto regions which are star-like with respect to a point $\varphi(0) = 1$ and symmetric with respect to the real axis, in 1994, Ma and Minda [8] gave a unified treatment of various subclasses of star-like functions in terms of subordination by studying the class

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{H} \colon \frac{zf'(z)}{f(z)} \prec \varphi(z),\, z \in \mathbb{D} \right\}.$$

For special choices of φ, the class $\mathcal{S}^*(\varphi)$ reduces to widely known subclasses of star-like functions. For example, when $-1 \le B < A \le 1$, $\mathcal{S}^*[A, B] := \mathcal{S}^*((1 + Az)/(1 + Bz))$ is the class of Janowski [6] star-like functions,

*Correspondence: sumitnagpal.du@gmail.com
2010 *AMS Mathematics Subject Classification:* 30C45, 30C80

$\mathcal{S}_P^* := \mathcal{S}^*(1 + 2(\log((1+\sqrt{z})/(1-\sqrt{z})))^2/\pi^2)$ is the class consisting of parabolic star-like functions [15], $\mathcal{S}_L^* := \mathcal{S}^*(\sqrt{1+z})$ is the class of lemniscate star-like functions [18] and $\mathcal{S}_q^* := \mathcal{S}^*(z + \sqrt{1+z^2})$ is the class of star-like functions associated with lune [14]. In 2015, Mendiratta et al. [9] also introduced the class $\mathcal{S}_e^* = \mathcal{S}^*(e^z)$ of star-like functions associated with the exponential function satisfying the condition $|\log(zf'(z)/f(z))| < 1$ for $z \in \mathbb{D}$.

The study of differential subordination which is a generalized form of differential inequalites began with a prodigious article "Differential subordination and univalent functions" by Miller and Mocanu [10] in 1981. After that the theory of differential subordination brought a revolutionary change and attracted many researchers to use this technique for the study of univalent functions. Given a complex function $\psi(r,s,t;z)\colon \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ and a univalent function h in $\mathbb{D}$, if p is an analytic function in $\mathbb{D}$ that satisfies the *second-order differential subordination*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \tag{1.1}$$

then p is called a *solution* of the differential subordination. The univalent function q is said to be a *dominant* of the solutions of the differential subordination if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the *best dominant* of (1.1). The best dominant is unique up to a rotation of $\mathbb{D}$. Moreover, let $\mathcal{Q}$ denote the set of analytic and univalent functions q in $\overline{\mathbb{D}} \setminus E(q)$, where

$$E(q) = \{\zeta \in \partial\mathbb{D}\colon \lim_{z\to\zeta} q(z) = \infty\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \setminus E(q)$. The following definition of admissible functions and the fundamental theorem laid the foundation stone in the theory of differential subordination.

Definition 1.1 *[11, p. 27] Let Ω be a domain in $\mathbb{C}$, $q \in \mathcal{Q}$ and n be a positive integer. Define $\Psi_n(\Omega, q)$ to be the* class of admissible functions $\psi\colon \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ *that satisfies the admissibility condition:*

$$\psi(r,s,t;z) \notin \Omega$$

whenever

$$r = q(\zeta) \text{ is finite, } s = m\zeta q'(\zeta) \text{ and } \operatorname{Re}\left(1 + \frac{t}{s}\right) \geq m \operatorname{Re}\left(1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right)$$

where $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D} \setminus E(q)$ and $m \geq n$ is a positive integer. We write $\Psi_1(\Omega, q)$ as $\Psi(\Omega, q)$.

Theorem 1.2 *[11, p. 28] Let $\psi \in \Psi_n(\Omega, q)$ with $q(0) = a$. If $p \in \mathcal{H}[a, n]$ satisfies*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$$

then $p(z) \prec q(z)$.

Miller and Mocanu [11], in their monograph, discussed the class of admissible functions $\Psi(\Omega, q)$ when the function q maps $\mathbb{D}$ onto a disk or a half-plane. These two special classes together with Theorem 1.2 lead to several important and interesting results in the theory of differential subordination. However, the aim of this paper is to consider differential implications with the superordinate function $q(z) = e^z$. In Section 2, the admissibility class $\Psi(\Omega, e^z)$ is obtained, by deriving its admissibility condition. Examples are provided to illustrate the obtained results.

In 2015, Mendiratta et al. [9] estimated bounds on β for which $p(z) \prec e^z$ whenever $1+\beta zp'(z)/p(z)$ is subordinate to e^z, $(1+Az)/(1+Bz)$ and $\sqrt{1+z}$. In 2018, Kumar and Ravichandran [7] extended the result of Mendiratta et al. and obtained bounds on β for $1+\beta zp'(z)/p^j(z)$ $(j=0,2)$. They also estimated the bounds on β such that $p(z) \prec e^z$ whenever $1+\beta zp'(z)/p^j(z)$ $(j=0,1,2)$ is subordinate to $1+\sin z$ and $1+(z(k+z))/(k(k-z))$, where $k=\sqrt{2}+1$. Also, Gandhi et al. [4] obtained bounds on β for which $p(z) \prec e^z$ whenever $1+\beta zp'(z)/p^j(z)$ $(j=0,1,2)$ is subordinate to $z+\sqrt{1+z^2}$. Motivated by their works and that of [1–3, 5, 12, 13, 16, 17, 19], in Section 3, the problem

$$\psi(p(z), zp(z), z^2p''(z); z) \prec h(z) \quad \Rightarrow \quad p(z) \prec e^z$$

is established for special cases of Janowski star-like functions h. In Section 4, the above-mentioned problem is solved for various expressions when h, in particular, is also an exponential function. The results of [9] are not only generalized but new differential implications are also obtained in last two sections. Additionally, the applications of the results obtained yield sufficient conditions for functions $f \in \mathcal{H}$ to belong to the class $\mathcal{S}_e^*$.

2. The admissibility condition

In this section, we describe the admissible class $\Psi(\Omega, q)$ with examples, where Ω is a domain in $\mathbb{C}$ and $q(z)=e^z$. Note that q is a univalent function in $\overline{\mathbb{D}}$ with $q(\mathbb{D})=\Delta$ and $q(0)=1$, where $\Delta := \{w \in \mathbb{C} \colon |\log w| < 1\}$. Thus, $q \in \mathcal{Q}$ with $E(q)=\emptyset$ and hence the class $\Psi(\Omega, q)$ is well defined.

For $|\zeta|=1$, $q(\zeta) \in q(\partial\mathbb{D}) = \partial q(\mathbb{D}) = \{w \in \mathbb{C} : |\log w| = 1\}$. This gives $|\log q(\zeta)|=1$ so that $\log q(\zeta) = e^{i\theta}$, where $\theta \in [0, 2\pi)$ and hence $q(\zeta)=e^{e^{i\theta}}$. However, $q(\zeta)=e^\zeta$ which implies that $\zeta = e^{i\theta}$. Also $\zeta q'(\zeta) = e^{i\theta}e^{e^{i\theta}}$ and

$$\operatorname{Re}\left(1+\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) = \operatorname{Re}(1+e^{i\theta}) = 1+\cos\theta.$$

Thus, the admissibility condition reduces to

$$\left.\begin{aligned} \psi(r,s,t;z) \notin \Omega \quad \text{whenever} \quad r &= q(\zeta) = e^{e^{i\theta}} \\ s &= m\zeta q'(\zeta) = me^{i\theta}r \\ \text{and} \quad \operatorname{Re}(1+t/s) &\geq m(1+\cos\theta) \end{aligned}\right\} \tag{2.1}$$

where $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ and $m \geq 1$. Therefore, the class $\Psi(\Omega, e^z)$ consists of those functions $\psi \colon \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition given by (2.1). If $\psi \colon \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$, then the admissibility condition (2.1) reduces to

$$\psi(e^{e^{i\theta}}, me^{i\theta}e^{e^{i\theta}}; z) \notin \Omega$$

where $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ and $m \geq 1$. As a particular case of Theorem 1.2, we have the following:

Theorem 2.1 *Let $p \in \mathcal{H}_1$.*

(i) If $\psi \in \Psi(\Omega, e^z)$, then

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega \quad \Rightarrow \quad p(z) \prec e^z.$$

(ii) *If* $\psi \in \Psi(\Delta, e^z)$, *then*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec e^z \quad \Rightarrow \quad p(z) \prec e^z.$$

We close this section with some examples illustrating Theorem 2.1.

Example 2.2 *Let* $\psi(r,s,t;z) = r + (1+2e)s$ *and* $h : \mathbb{D} \to \mathbb{C}$ *be defined by*

$$h(z) = 2\left(\frac{2z+1}{2+z}\right).$$

Then $\Omega = h(\mathbb{D}) = \{w \in \mathbb{C} : |w| < 2\}$. *To prove* $\psi \in \Psi(\Omega, e^z)$, *we need to show that the admissibility condition* (2.1) *is satisfied. Consider*

$$\begin{aligned}|\psi(r,s,t;z)| &= e^{\cos\theta}|1 + (1+2e)me^{i\theta}| \\ &\geq e^{-1}((1+2e)m - 1) \geq 2\end{aligned}$$

whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ *and* $\operatorname{Re}(1 + t/s) \geq m(1+\cos\theta)$, *where* $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ *and* $m \geq 1$. *Therefore,* $\psi(r,s,t;z) \notin \Omega$ *and hence* $\psi \in \Psi(\Omega, e^z)$. *By Theorem 2.1, it follows that if* $p \in \mathcal{H}_1$, *then*

$$|p(z) + (1+2e)zp'(z)| < 2 \quad \Rightarrow \quad p(z) \prec e^z.$$

Example 2.3 *If* $\psi(r,s,t;z) = 1 + (1+\sqrt{2})es$ *and* $h(z) = \sqrt{1+z}$, *then* $\Omega = h(\mathbb{D}) = \{w \in \mathbb{C} : |w^2 - 1| < 1\}$. *Consider*

$$\begin{aligned}|(\psi(r,s,t;z))^2 - 1| &= (1+\sqrt{2})e|s||(1+\sqrt{2})es + 2| \\ &\geq (1+\sqrt{2})e|s|((1+\sqrt{2})e|s| - 2) \\ &= (1+\sqrt{2})me^{1+\cos\theta}((1+\sqrt{2})me^{1+\cos\theta} - 2) \\ &\geq (1+\sqrt{2})(\sqrt{2}-1) = 1\end{aligned}$$

whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ *and* $\operatorname{Re}(1 + t/s) \geq m(1+\cos\theta)$, *where* $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ *and* $m \geq 1$. *Thus,* $\psi(r,s,t;z) \notin \Omega$ *and therefore* $\psi \in \Psi(\Omega, e^z)$. *By Theorem 2.1, it is easily seen that if* $p \in \mathcal{H}_1$, *then*

$$|(1 + (1+\sqrt{2})ezp'(z))^2 - 1| < 1 \quad \Rightarrow \quad p(z) \prec e^z.$$

Example 2.4 *Let* $\psi(r,s,t;z) = 1 + s$ *and suppose that* $\Omega = \{w \in \mathbb{C}\colon |w - 1| < e^{-1}\}$. *In order to prove* $\psi \in \Psi(\Omega, e^z)$, *note that*

$$|\psi(r,s,t;z) - 1| = |s| = |me^{i\theta}e^{e^{i\theta}}| = me^{\cos\theta} \geq me^{-1} \geq e^{-1}$$

whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ *and* $\operatorname{Re}(1 + t/s) \geq m(1+\cos\theta)$, *where* $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ *and* $m \geq 1$. *Therefore,* $\psi(r,s,t;z) \notin \Omega$ *which implies* $\psi \in \Psi(\Omega, e^z)$. *For any* $p \in \mathcal{H}_1$, *we obtain*

$$|zp'(z)| < e^{-1} \quad \Rightarrow \quad p(z) \prec e^z.$$

Similarly, if we take $\psi(r,s,t;z) = r^2 - r + (1+e)s + 1$ *with the same* Ω *as defined earlier, then*

$$\begin{aligned}|\psi(r,s,t;z) - 1| &= |r^2 - r + (2+e)s| \\ &= e^{\cos\theta}|e^{e^{i\theta}} - 1 + (2+e)me^{i\theta}| \\ &\geq e^{-1}((2+e)m - |e^{e^{i\theta}} - 1|) \\ &\geq e^{-1}((2+e)m - 1 - e^{\cos\theta}) \\ &\geq e^{-1}((2+e)m - 1 - e) \geq e^{-1}\end{aligned}$$

whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ *and* $\operatorname{Re}(1+t/s) \geq m(1+\cos\theta)$, *where* $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ *and* $m \geq 1$. *By Theorem 2.1, it is easy to deduce that for any* $p \in \mathcal{H}_1$, *we have*

$$|p^2(z) - p(z) + (1+e)zp'(z)| < e^{-1} \quad \Rightarrow \quad p(z) \prec e^z.$$

In the similar fashion, by taking $\psi(r,s,t;z) = 1 + s/r^2$ *and* Ω *as above, it is easily seen that*

$$|\psi(r,s,t;z) - 1| = |s/r^2| = |me^{i\theta}e^{-e^{i\theta}}| = me^{-\cos\theta} \geq me^{-1} \geq e^{-1}$$

whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ *and* $\operatorname{Re}(1+t/s) \geq m(1+\cos\theta)$, *where* $\theta \in [0, 2\pi)$ *and* $m \geq 1$. *This implies that* $\psi(r,s,t;z) \notin \Omega$ *and hence* $\psi \in \Psi(\Omega, e^z)$. *Thus, for any* $p \in \mathcal{H}_1$, *we have*

$$\left|\frac{zp'(z)}{p^2(z)}\right| < \frac{1}{e} \quad \Rightarrow \quad |\log p(z)| < 1.$$

Example 2.5 *Let* $\psi(r,s,t;z) = 1 + s/r$ *and* $\Omega = h(\mathbb{D}) = \{w \in \mathbb{C} \colon |w-1| < 1\}$, *where* $h(z) = 1 + z$. *Consider*

$$|\psi(r,s,t;z) - 1| = |s/r| = |me^{i\theta}| = m \geq 1$$

whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ *and* $\operatorname{Re}(1+t/s) \geq m(1+\cos\theta)$, *where* $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ *and* $m \geq 1$. *Therefore,* $\psi(r,s,t;z) \notin \Omega$ *and hence* $\psi \in \Psi(\Omega, e^z)$. *Using Theorem 2.1, in terms of subordination, the result can be written as*

$$1 + \frac{zp'(z)}{p(z)} \prec 1 + z \quad \Rightarrow \quad p(z) \prec e^z$$

where $p \in \mathcal{H}_1$. *Since* $\psi(q(z), zq'(z), z^2q''(z); z) = 1 + z = h(z)$ *and* $\psi \in \Psi(\Omega, q)$, *where* $q(z) = e^z$, *it follows that* e^z *is the best dominant by [11, Theorem 2.3e, p. 31].*

Example 2.6 *Let* $\psi(r,s,t;z) = 2s + t$ *and* $\Omega = \{w \in \mathbb{C} \colon |w| < 1/e\}$. *Then*

$$\begin{aligned}|\psi(r,s,t;z)| &= |2s + t| = |s|\left|2 + \frac{t}{s}\right| \\ &\geq me^{\cos\theta}\operatorname{Re}\left(2 + \frac{t}{s}\right) \\ &\geq me^{\cos\theta}(1 + m(1+\cos\theta)) \\ &\geq me^{\cos\theta} \geq e^{-1}\end{aligned}$$

whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ *and* $\operatorname{Re}(1 + t/s) \geq m(1 + \cos\theta)$, *where* $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ *and* $m \geq 1$. *This shows that* $\psi(r, s, t; z) \notin \Omega$ *and* $\psi \in \Psi(\Omega, e^z)$. *By Theorem 2.1, the required result is*

$$|2zp'(z) + z^2p''(z)| < 1/e \quad \Rightarrow \quad p(z) \prec e^z.$$

3. Subordination associated with the Janowski function

For $-1 \leq B < A \leq 1$, we consider the subordination $\psi(p(z), zp'(z), z^2p''(z);\ z) \prec (1 + Az)/(1 + Bz)$ implying $p(z) \prec e^z$ for $z \in \mathbb{D}$. In particular, we first estimate the bound on β such that the first order differential subordination $1 + \beta zp'(z)/p^n(z) \prec 1 + (1 - \alpha)z$ (where n is any nonnegative integer and $0 \leq \alpha < 1$) implies $p(z) \prec e^z$. Throughout this paper, we will assume that β is a positive real number and r, s, t are same as referred in the admissibility condition (2.1).

Theorem 3.1 *If* n *is a nonnegative integer,* $0 \leq \alpha < 1$ *and* $p \in \mathcal{H}_1$ *satisfies the subordination*

$$1 + \beta\frac{zp'(z)}{p^n(z)} \prec 1 + (1 - \alpha)z, \quad \textit{where } \beta \geq \begin{cases} e(1 - \alpha) & \textit{when } n = 0 \\ e^{n-1}(1 - \alpha) & \textit{when } n \neq 0 \end{cases}$$

then $p(z) \prec e^z$.

Proof Let $h(z) = 1 + (1 - \alpha)z$, where $z \in \mathbb{D}$, $0 \leq \alpha < 1$ and $\Omega = h(\mathbb{D}) = \{w \in \mathbb{C} \colon |w - 1| < 1 - \alpha\}$.
Case (i). If $n = 0$, consider the function $\psi(r, s, t; z) = 1 + \beta s$. Then

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec 1 + (1 - \alpha)z.$$

Theorem 2.1 is applicable if we show that $\psi \in \Psi(\Omega, e^z)$, that is, $\psi(r, s, t; z) \notin \Omega$ whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ and $\operatorname{Re}(1 + t/s) \geq m(1 + \cos\theta)$, where $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ and $m \geq 1$. A simple calculation yields

$$|\psi(r, s, t; z) - 1| = \beta me^{\cos\theta} \geq \beta e^{-1} \geq 1 - \alpha.$$

Hence, $\psi(r, s, t; z) \notin \Omega$ which gives $\psi \in \Psi(\Omega, e^z)$. Using Theorem 2.1, we get $p(z) \prec e^z$.
Case (ii). When $n \neq 0$, the function $\psi(r, s, t; z) = 1 + \beta s/r^n$ satisfies

$$|\psi(r, s, t; z) - 1| = \beta me^{-(n-1)\cos\theta} \geq \beta e^{-(n-1)} \geq 1 - \alpha$$

whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ and $\operatorname{Re}(1 + t/s) \geq m(1 + \cos\theta)$, where $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ and $m \geq 1$. Therefore, as argued in Case (i), $\psi(r, s, t; z) \notin \Omega$ which implies $\psi \in \Psi(\Omega, e^z)$. Hence, by Theorem 2.1, we have the desired result. □

Remark 3.2 *For the case* $n = 1$, *Theorem 3.1 reduces to [9, Theorem 2.8b, p. 376] when* $A = 1 - \alpha$ *and* $B = 0$.

Consequently, *if a function* $f \in \mathcal{H}$ *satisfies the subordination*

$$1 + \beta\left(\frac{zf'(z)}{f(z)}\right)^{1-n}\left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right) \prec 1 + (1 - \alpha)z$$

where $0 \leq \alpha < 1$ *and the bound on* β *is defined as in Theorem 3.1, then* $f \in \mathcal{S}_e^*$.

Next, the bound on β is determined such that the first order differential subordination $1 + \beta zp'(z)/p^{n+1}(z) \prec (2 + z)/(2 - z)$ (where n is any nonnegative integer) implies $p(z) \prec e^z$.

Theorem 3.3 *If n is any nonnegative integer and $p \in \mathcal{H}_1$ satisfies the subordination*

$$1+\beta\frac{zp'(z)}{p^{n+1}(z)} \prec \frac{2+z}{2-z}, \quad \textit{where } \beta \geq 2e^n$$

then $p(z) \prec e^z$.

Proof By considering the function $\psi(r,s,t;z)=1+\beta s/r^{n+1}$ and $\Omega=\{w\in\mathbb{C}\colon |(2w-2)/(w+1)|<1\}$, it suffices to show $\psi\in\Psi(\Omega,e^z)$. For this, note that

$$\left|\frac{2\psi(r,s,t;z)-2}{\psi(r,s,t;z)+1}\right| \geq \frac{2\beta m e^{-n\cos\theta}}{2+\beta m e^{-n\cos\theta}}.$$

Since the real-valued function $g(x)=2x/(2+x)$ is increasing for $x\geq 0$ and $\beta m e^{-n\cos\theta}\geq 2$, it is easy to deduce that

$$\left|\frac{2\psi(r,s,t;z)-2}{\psi(r,s,t;z)+1}\right| \geq 1$$

whenever $r=e^{e^{i\theta}}$, $s=me^{i\theta}r$ and $\operatorname{Re}(1+t/s) \geq m(1+\cos\theta)$, where $z\in\mathbb{D}$, $\theta\in[0,2\pi)$ and $m\geq 1$. Hence, by making use of Theorem 2.1, we get the required result. □

Remark 3.4 *The case $n=0$ in Theorem 3.3 is similar to [9, Theorem 2.8b, p. 376] for $A=1/2$ and $B=-1/2$.*

As a result, we have

If a function $f\in\mathcal{H}$ satisfies the subordination

$$1+\beta\left(\frac{zf'(z)}{f(z)}\right)^{-n}\left(1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right) \prec \frac{2+z}{2-z}$$

where $\beta\geq 2e^n$ and n is any nonnegative integer, then $f\in\mathcal{S}_e^*$.

The next theorem provides a bound on α and β such that the first order differential subordination $(1-\alpha)p(z)+\alpha p^2(z)+\beta zp'(z)\prec 1+z$ implies $p(z)\prec e^z$.

Theorem 3.5 *Let α, β be positive real numbers satisfying $\alpha(e-1)+\beta e\geq e$ and $p\in\mathcal{H}_1$. If the following subordination*

$$(1-\alpha)p(z)+\alpha p^2(z)+\beta zp'(z)\prec 1+z$$

holds, then $p(z)\prec e^z$.

Proof Let $\Omega=h(\mathbb{D})=\{w\in\mathbb{C}\colon |w-1|<1\}$, where $h(z)=1+z$. If $\psi(r,s,t;z)=(1-\alpha)r+\alpha r^2+\beta s$, the required subordination is proved if we show that $\psi\in\Psi(\Omega,e^z)$ in view of Theorem 2.1. Observe that

$$\begin{aligned}|\psi(r,s,t;z)-1|^2 = {}&\big((1-\alpha)e^{\cos\theta}\cos(\sin\theta)+\alpha e^{2\cos\theta}\cos(2\sin\theta)+\beta m e^{\cos\theta}\cos\theta\cos(\sin\theta)\\ &-\beta m e^{\cos\theta}\sin\theta\sin(\sin\theta)-1\big)^2+\big((1-\alpha)e^{\cos\theta}\sin(\sin\theta)+\alpha e^{2\cos\theta}\sin(2\sin\theta)\\ &+\beta m e^{\cos\theta}\cos\theta\sin(\sin\theta)+\beta m e^{\cos\theta}\sin\theta\cos(\sin\theta)\big)^2 =: g(\theta).\end{aligned}$$

The second derivative test shows that the function g attains its minimum value at $\theta = \pi$ for $\alpha > 0$ and $\beta > 0$. Therefore, for all $\theta \in [0, 2\pi)$

$$\begin{aligned} g(\theta) \geq g(\pi) &= \frac{(\alpha(e-1) + e(e-1+\beta m))^2}{e^4} \\ &\geq \frac{(\alpha(e-1) + e(e-1+\beta))^2}{e^4} \geq 1 \end{aligned}$$

by the given condition $\alpha(e-1) + \beta e \geq e$. Thus, $\psi(r,s,t;z) \notin \Omega$ whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ and $\mathrm{Re}\,(1 + t/s) \geq m(1+\cos\theta)$, where $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ and $m \geq 1$. By Definition 1.1, $\psi \in \Psi(\Omega, e^z)$ and the result is evident by Theorem 2.1. □

Thus, we have

If $\alpha(e-1) + \beta e \geq e$ and $f \in \mathcal{H}$ satisfies the following subordination

$$\left((1-\alpha) + (\alpha - \beta)\frac{zf'(z)}{f(z)}\right)\frac{zf'(z)}{f(z)} + \beta\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec 1 + z$$

then $f \in \mathcal{S}_e^*$.

Next, we determine the bounds on β such that the first order differential subordinations $p(z) + \beta zp'(z)/p^n(z) \prec (2+2z)/(2-z)$, where $n = 0, 1$ implies $p(z) \prec e^z$.

Theorem 3.6 *Let $p \in \mathcal{H}_1$. Then each of the following conditions is sufficient for $p(z) \prec e^z$:*

(a) $p(z) + \beta zp'(z) \prec (2+2z)/(2-z)$ for $\beta \geq (e + 2 - \sqrt{2}(e-1))/(e(\sqrt{2}-1)) \approx 2.0323$.

(b) $p(z) + \beta zp'(z)/p(z) \prec (2+2z)/(2-z)$ for $\beta \geq (e + 2 - \sqrt{2}(e-1))/(\sqrt{2}-1) \approx 5.52436$.

Proof Define $h\colon \mathbb{D} \to \mathbb{C}$ by $h(z) = (2+2z)/(2-z)$ and suppose that $\Omega = h(\mathbb{D}) = \{w \in \mathbb{C}\colon |(w-1)/(w+2)| < 1/2\}$.

(a) As done earlier in the previous results, the function $\psi(r,s,t;z) = r + \beta s$ should satisfy $\psi(r,s,t;z) \notin \Omega$ whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ and $\mathrm{Re}\,(1+t/s) \geq m(1+\cos\theta)$, where $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ and $m \geq 1$. Observe that

$$\left|\frac{\psi(r,s,t;z) - 1}{\psi(r,s,t;z) + 2}\right|^2 = \frac{(1 + \beta m\cos\theta - e^{-\cos\theta}\cos(\sin\theta))^2 + (\beta m \sin\theta + e^{-\cos\theta}\sin(\sin\theta))^2}{(1 + \beta m\cos\theta + 2e^{-\cos\theta}\cos(\sin\theta))^2 + (\beta m\sin\theta - 2e^{-\cos\theta}\sin(\sin\theta))^2}.$$

It is easily verified that the minimum value of the function in the right hand side of the above equation occurs at $\theta = 0$; therefore, we obtain

$$\left|\frac{\psi(r,s,t;z) - 1}{\psi(r,s,t;z) + 2}\right|^2 \geq \frac{(e - 1 + \beta em)^2}{(e + 2 + \beta em)^2} \geq \frac{(e(1+\beta) - 1)^2}{(e(1+\beta) + 2)^2} \geq \frac{1}{2}$$

since $\beta \geq (e + 2 - \sqrt{2}(e-1))/(e(\sqrt{2}-1))$.

(b) The required subordination is proved if we show that the function $\psi(r,s,t;z) = r + \beta s/r$ does not lie in Ω. For $\beta \geq (e+2-\sqrt{2}(e-1))/(\sqrt{2}-1)$, using the same technique as in previous case, we have

$$\left|\frac{\psi(r,s,t;z)-1}{\psi(r,s,t;z)+2}\right|^2 = \frac{(e^{\cos\theta}\cos(\sin\theta)+\beta m\cos\theta-1)^2+(e^{\cos\theta}\sin(\sin\theta)+\beta m\sin\theta)^2}{(e^{\cos\theta}\cos(\sin\theta)+\beta m\cos\theta+2)^2+(e^{\cos\theta}\sin(\sin\theta)+\beta m\sin\theta)^2}$$
$$\geq \left(\frac{\beta m+e-1}{\beta m+e+2}\right)^2 \geq \frac{1}{2}$$

whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ and $\operatorname{Re}(1+t/s) \geq m(1+\cos\theta)$, where $z \in \mathbb{D}$, $\theta \in [0,2\pi)$ and $m \geq 1$. Therefore, using Theorem 2.1, we have $p(z) \prec e^z$. □

As a consequence, we obtain

If a function $f \in \mathcal{H}$ satisfies either of the following subordinations

(a) $$\frac{zf'(z)}{f(z)} + \beta\frac{zf'(z)}{f(z)}\left(1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right) \prec \frac{2+2z}{2-z} \quad \textit{for } \beta \geq \frac{e+2-\sqrt{2}(e-1)}{e(\sqrt{2}-1)}$$

(b) $$\frac{zf'(z)}{f(z)} + \beta\left(1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right) \prec \frac{2+2z}{2-z} \quad \textit{for } \beta \geq \frac{e+2-\sqrt{2}(e-1)}{\sqrt{2}-1}$$

then $f \in \mathcal{S}_e^*$.

4. Subordination associated with the exponential function

In this section, we consider the problem of determining the conditions under which the subordination $\psi(p(z), zp'(z), z^2p''(z); z) \prec e^z$ implies that $p(z) \prec e^z$ also holds. Alternatively, our aim is to show that $\psi \in \Psi\{e^z\} := \Psi(\Delta, e^z)$ for various choices of ψ, where $\Delta := \{w \in \mathbb{C} \colon |\log w| < 1\}$. The first theorem of this section estimates the bound on β such that the first order differential subordination $1 + \beta(zp'(z))^n \prec e^z$ (where n is any positive integer) implies $p(z) \prec e^z$. Recall that, for $z \neq 0$

$$\log z = \ln|z| + i\arg z = \ln(x^2+y^2)^{1/2} + i\tan^{-1}(y/x) \quad \text{for } x > 0.$$

Theorem 4.1 *If n is any positive integer and $p \in \mathcal{H}_1$ satisfies the subordination*

$$1 + \beta(zp'(z))^n \prec e^z, \quad \textit{where } \beta \geq \begin{cases} e^{n+1}+e^n & \textit{when } n \textit{ is odd} \\ e^{n+1}-e^n & \textit{when } n \textit{ is even} \end{cases}$$

then $p(z) \prec e^z$.

Proof The required subordination is proved if we show that the function defined as $\psi(r,s,t;z) = 1 + \beta s^n$ satisfies the condition $\psi(r,s,t;z) \notin \Omega$ whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ and $\operatorname{Re}(1+t/s) \geq m(1+\cos\theta)$, where $z \in \mathbb{D}$, $\theta \in [0,2\pi)$ and $m \geq 1$. Consider

$$|\log\psi(r,s,t;z)|^2 = \frac{1}{4}\ln^2(\mu^2+\nu^2) + \left(\tan^{-1}\frac{\mu}{\nu}\right)^2 =: g(\theta)$$

where

$$\mu = \beta m^n e^{n\cos\theta}\sin n\theta\cos(n\sin\theta) + \beta m^n e^{n\cos\theta}\cos n\theta\sin(n\sin\theta)$$

and

$$\nu = 1 + \beta m^n e^{n\cos\theta}\cos n\theta\cos(n\sin\theta) - \beta m^n e^{n\cos\theta}\sin n\theta\sin(n\sin\theta).$$

Case (i). When n is odd and $\beta \geq e^{n+1} + e^n$, we have

$$g''(\pi) = \frac{\beta m^n n \ln\left(e^{-2n}(-e^n + \beta m^n)^2\right)}{-e^n + \beta m^n} > 0$$

for all $m \geq 1$. Therefore, second derivative verifies that minimum value of g is attained at $\theta = \pi$. If $\beta \geq e^{n+1} + e^n$, we obtain

$$g(\theta) \geq g(\pi) = \frac{1}{4}\ln^2\left(1 - \frac{2\beta m^n}{e^n} + \frac{\beta^2 m^{2n}}{e^{2n}}\right) \geq \frac{1}{4}\ln^2\left(1 - \frac{\beta m^n}{e^n}\right)^2 \geq 1$$

for all $\theta \in [0, 2\pi)$. Thus, $|\log\psi(r,s,t;z)| \geq 1$ and Theorem 2.1 gives $\psi \in \Psi\{e^z\}$.

Case (ii). If n is even and

$$g''(\pi) = \frac{\beta m^n n \ln\left(e^{-2n}(e^n + \beta m^n)^2\right)}{e^n + \beta m^n} > 0$$

for $\beta > 0$, the minimum value of the function g is attained at $\theta = \pi$. Therefore, for all $\theta \in [0, 2\pi)$ and $\beta \geq e^{n+1} - e^n$, we get

$$g(\theta) \geq g(\pi) = \frac{1}{4}\ln^2\left(1 + \frac{2\beta m^n}{e^n} + \frac{\beta^2 m^{2n}}{e^{2n}}\right) \geq 1.$$

This implies that $\psi \in \Psi\{e^z\}$. Hence, Theorem 2.1 gives the desired differential subordination. □

Now, we estimate the bound on β such that the first order differential subordination $1 + \beta z p'(z)/p^{n+1}(z) \prec e^z$ (where n is any nonnegative integer) implies $p(z) \prec e^z$.

Theorem 4.2 *If $p \in \mathcal{H}_1$ satisfies the subordination*

$$1 + \beta\frac{zp'(z)}{p^{n+1}(z)} \prec e^z, \quad \textit{where } \beta \geq e^{n+1} - e^n$$

and n is any nonnegative integer, then $p(z) \prec e^z$.

Proof We apply Theorem 2.1 to show that $\psi \in \Psi\{e^z\}$, where $\psi(r,s,t;z) = 1 + \beta s/r^n$. Whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ and $\operatorname{Re}(1 + t/s) \geq m(1 + \cos\theta)$, where $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ and $m \geq 1$, note that

$$|\log\psi(r,s,t;z)|^2 = \frac{1}{4}\ln^2(\mu^2 + \nu^2) + \left(\tan^{-1}\frac{\mu}{\nu}\right)^2 =: g(\theta)$$

where

$$\mu = \beta m e^{-n\cos\theta}\sin\theta\cos(n\sin\theta) - \beta m e^{-n\cos\theta}\cos\theta\sin(n\sin\theta)$$

and

$$\nu = 1 + \beta m e^{-n\cos\theta}\cos\theta\cos(n\sin\theta) + \beta m e^{-n\cos\theta}\sin\theta\sin(n\sin\theta).$$

Let $u(x) = x(-1+n)^2 - \left(-xn + e^n(1-3n+n^2)\right)\ln\left(e^{-n}(e^n+x)\right)$, where $x > 0$ and n is a nonnegative integer. Natural logarithm being an increasing function implies that $\ln(e^n + x) > \ln(e^n)$ for $x > 0$, that is, $\ln(e^n + x) > n$ for $x > 0$. This gives

$$u(x) > x(1-2n) + n\left(xn - e^n(1-3n+n^2)\right) + ne^n(1-3n+n^2) = x(1-n)^2 > 0$$

for $x > 0$ and $n \neq 1$. In particular for $n = 1$, $u(x) = (x+e)(\ln(x+e) - 1) > 0$ for $x > 0$. Therefore,

$$g''(0) = \frac{\beta m\left(2\beta m(-1+n)^2 - \left(-\beta mn + e^n(1-3n+n^2)\right)\ln\left(e^{-2n}(e^n+\beta m)^2\right)\right)}{(e^n+\beta m)^2} > 0$$

for $\beta > 0$, which implies, using second derivative test, g attains its minimum value at $\theta = 0$. Hence, for all $\theta \in [0, 2\pi)$ and $\beta \geq e^{n+1} - e^n$

$$g(\theta) \geq g(0) = \frac{1}{4}\ln^2\left(1 + \frac{2\beta m}{e^n} + \frac{\beta^2 m^2}{e^{2n}}\right) = \frac{1}{4}\ln^2\left(1 + \frac{\beta m}{e^n}\right)^2 \geq \frac{1}{4}\ln^2\left(1 + \frac{e^{n+1} - e^n}{e^n}\right)^2 = 1.$$

Thus, $|\log\psi(r,s,t;z)| \geq 1$ which implies $\psi \in \Psi\{e^z\}$. □

Remark 4.3 *For $n = 0$, Theorem 4.2 reduces to [9, Theorem 2.8a, p. 376].*

In the next two theorems, the bound on β is computed such that the first order differential subordination $p(z) + \beta zp'(z)/p^{n+1}\ (z) \prec e^z$ (where $n = -1, 0, 1, 2, \ldots$) implies $p(z) \prec e^z$.

Theorem 4.4 *Let $p \in \mathcal{H}_1$, then each of the following subordinations is sufficient for $p(z) \prec e^z$:*

(a) $p(z) + \beta zp'(z) \prec e^z$ for $\beta \geq e^2 + 1 \approx 8.38906$.

(b) $p(z) + \beta zp'(z)/p(z) \prec e^z$ for $\beta \geq e + e^{-1} \approx 3.08616$.

Proof (a) In order to prove the admissibility condition (2.1) for the function $\psi(r,s,t;z) = r + \beta s$, we need to show that $|\log\psi(r,s,t;z)|^2 \geq 1$ whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ and $\operatorname{Re}(1 + t/s) \geq m(1+\cos\theta)$, where $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ and $m \geq 1$. A simple computation gives

$$\begin{aligned}|\log\psi(r,s,t;z)|^2 = &\frac{1}{4}\ln^2(e^{2\cos\theta} + \beta^2 m^2 e^{2\cos\theta} + 2\beta m e^{2\cos\theta}\cos\theta)\\ &+ \left(\tan^{-1}\left(\frac{\sin(\sin\theta) + \beta m\cos\theta\sin(\sin\theta) + \beta m\sin\theta\cos(\sin\theta)}{\cos(\sin\theta) + \beta m\cos\theta\cos(\sin\theta) - \beta m\sin\theta\sin(\sin\theta)}\right)\right)^2 =: g(\theta).\end{aligned}$$

Note that

$$g''(\pi) = \frac{2\beta m(1-\beta m) + (1 - \beta m + \beta^2 m^2)\ln((-1+\beta m)^2)}{(-1+\beta m)^2} > 0$$

for $\beta > \beta^* \approx 3.4446$, where β^* is the root of the equation $x(1-x) + (1-x+x^2)\ln(-1+x) = 0$. Therefore, the minimum value of the function g is clearly attained at $\theta = \pi$ for $\beta \geq e^2 + 1 \approx 8.38906$. In that case, we have

$$g(\theta) \geq g(\pi) = \frac{1}{4}\ln^2\left(\frac{1}{e^2} - \frac{2\beta m}{e^2} + \frac{\beta^2 m^2}{e^2}\right) \geq 1 \quad \text{for all } \theta \in [0, 2\pi).$$

Hence, $\psi \in \Psi\{e^z\}$.

(b) Using the same technique as above, for the function $\psi(r,s,t;z) = r + \beta s/r$, consider

$$|\log \psi(r,s,t;z)|^2 = \frac{1}{4}\ln^2(e^{2\cos\theta} + \beta^2 m^2 + 2\beta m e^{\cos\theta}\cos\theta\cos(\sin\theta) + 2\beta m e^{\cos\theta}\sin\theta\sin(\sin\theta)) + \left(\tan^{-1}\frac{e^{\cos\theta}\sin(\sin\theta) + \beta m\sin\theta}{e^{\cos\theta}\cos(\sin\theta) + \beta m\cos\theta}\right)^2 =: g(\theta).$$

We observe that the second derivative of g is positive on both of its critical points; therefore, the absolute minimum of g is attained at $\theta = \pi$ for $\beta \geq e + e^{-1}$. Hence, we get

$$g(\theta) \geq g(\pi) = \frac{1}{4}\ln^2\left(\frac{1}{e^2} - \frac{2\beta m}{e} + \beta^2 m^2\right) \geq 1 \quad \text{for all } \theta \in [0, 2\pi).$$

Thus, $\psi \in \Psi\{e^z\}$ and Theorem 2.1 completes the proof. □

Theorem 4.5 *Let n be any positive integer and β_n be a positive root of the equation*

$$\left(e^{1+n} - x(-1+n)\right)^2 - \left(e^{2+2n}n - x^2 n + x e^{1+n}(1 - n + n^2)\right)\ln(e + e^{-n}x) = 0. \tag{4.1}$$

If $p \in \mathcal{H}_1$ satisfies the subordination

$$p(z) + \beta\frac{zp'(z)}{p^{n+1}(z)} \prec e^z, \quad \text{for } \beta > \beta_n$$

then $p(z) \prec e^z$.

Proof As argued in other cases, to prove the required subordinaton, it suffices to show that the function $\psi(r,s,t;z) = r + \beta s/r^{n+1}$ satisfies $\psi(r,s,t;z) \notin \Delta$ whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ and $\operatorname{Re}(1 + t/s) \geq m(1+\cos\theta)$, where $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ and $m \geq 1$. Note that

$$|\log\, \psi(r,s,t;z)|^2 = \frac{1}{4}\ln^2(e^{2\cos\theta} + \beta^2 m^2 e^{-2n\cos\theta} + 2\beta m e^{(1-n)\cos\theta}\cos\theta\cos(\sin\theta)\cos(n\sin\theta) + 2\beta m e^{(1-n)\cos\theta}\sin\theta\cos(\sin\theta)\sin(n\sin\theta) + 2\beta m e^{(1-n)\cos\theta}\sin\theta\sin(\sin\theta) \cos(n\sin\theta) - 2\beta m e^{(1-n)\cos\theta}\cos\theta\sin(\sin\theta)\sin(n\sin\theta)) + \left(\tan^{-1}(\chi)\right)^2 =: g(\theta)$$

where

$$\chi = \frac{e^{(n+1)\cos\theta}\sin(\sin\theta) + \beta m\sin\theta\cos(n\sin\theta) - \beta m\cos\theta\sin(n\sin\theta)}{e^{(n+1)\cos\theta}\cos(\sin\theta) + \beta m\cos\theta\cos(n\sin\theta) + \beta m\sin\theta\sin(n\sin\theta)}.$$

If $\beta > \beta_n$, where β_n is a positive root of the equation (4.1), then

$$g''(0) = \frac{2\left(e^{1+n} - \beta m(-1+n)\right)^2 - \left(e^{2+2n}n - \beta^2 m^2 n + \beta e^{1+n}m(1-n+n^2)\right)\ln\left((e + \beta e^{-n}m)^2\right)}{(e^{1+n} + \beta m)^2} > 0.$$

Therefore, the minimum value of g is attained at $\theta = 0$ by the second derivative test which implies for all $\theta \in [0, 2\pi)$ and $\beta > \beta_n > 0$

$$g(\theta) \geq g(0) = \frac{1}{4}\ln^2\left(e^2 + \frac{2\beta m}{e^{n-1}} + \frac{\beta^2 m^2}{e^{2n}}\right) \geq \frac{1}{4}\ln^2(e^2) \geq \frac{1}{4}(2)^2 = 1$$

for all positive integers m and n. Hence, $\psi \in \Psi\{e^z\}$ and Theorem 2.1 gives the desired result. □

Next, the bound on β is determined such that each of the first order differential subordination $p(z) + \beta(zp'(z))^2/p^n(z) \prec e^z$ $(n = 0, 1, 2)$ implies $p(z) \prec e^z$.

Theorem 4.6 *Let $p \in \mathcal{H}_1$. Then each of the following subordinations is sufficient for $p(z) \prec e^z$:*

(a) $p(z) + \beta(zp'(z))^2 \prec e^z$ for $\beta \geq e^3 - e \approx 17.3673$.

(b) $p(z) + \beta(zp'(z))^2/p(z) \prec e^z$ for $\beta \geq e^2 - 1 \approx 6.38906$.

(c) $p(z) + \beta(zp'(z))^2/p^2(z) \prec e^z$ for $\beta \geq e - e^{-1} \approx 2.3504$.

Proof For different choices of ψ, we need to show that $\psi \in \Psi\{e^z\}$, that is, we must verify $|\log \psi(r, s, t; z)| \geq 1$ whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ and $\operatorname{Re}(1 + t/s) \geq m(1 + \cos\theta)$, where $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ and $m \geq 1$.

(a) Let $\psi(r, s, t; z) = r + \beta s^2$ and consider

$$|\log \psi(r, s, t; z)|^2 = \frac{1}{4}\ln^2(\mu^2 + \nu^2) + \left(\tan^{-1}\frac{\mu}{\nu}\right)^2 =: g(\theta)$$

where

$$\mu = e^{\cos\theta}\sin(\sin\theta) + \beta m^2 e^{2\cos\theta}\cos 2\theta \sin(2\sin\theta) + \beta m^2 e^{2\cos\theta}\sin 2\theta\cos(2\sin\theta)$$

and

$$\nu = e^{\cos\theta}\cos(\sin\theta) + \beta m^2 e^{2\cos\theta}\cos 2\theta\cos(2\sin\theta) - \beta m^2 e^{2\cos\theta}\sin 2\theta\sin(2\sin\theta).$$

Since

$$g''(\pi) = \frac{-2(e + 2\beta m^2)^2 + (e^2 + 2\beta e m^2 + 2\beta^2 m^4)\ln\left((e + \beta m^2)^2\right)}{(e + \beta m^2)^2} > 0$$

for $\beta > \beta^* \approx 3.7586$, where β^* is a positive root of the equation $(e + 2x)^2 - (e^2 + 2xe + 2x^2)\ln(e + x) = 0$, we can say that the minimum value of g is obviously attained at $\theta = \pi$ for $\beta \geq e^3 - e$. Therefore, for $\beta \geq e^3 - e$, we have

$$g(\theta) \geq g(\pi) = \frac{1}{4}\ln^2\left(\frac{1}{e^2} + \frac{2\beta m^2}{e^3} + \frac{\beta^2 m^4}{e^4}\right) \geq 1 \quad \text{for all } \theta \in [0, 2\pi).$$

Hence, $\psi(r, s, t; z) \notin \Delta$ and using Theorem 2.1 the result follows.

(b) Let the function be defined by $\psi(r, s, t; z) = r + \beta s^2/r$ and observe

$$|\log \psi(r, s, t; z)|^2 = \frac{1}{4}\ln^2(e^{2\cos\theta} + \beta^2 m^4 e^{2\cos\theta} + 2\beta m^2 e^{2\cos\theta}\cos 2\theta) + \left(\tan^{-1}(\chi)\right)^2 =: g(\theta)$$

where

$$\chi = \frac{\sin(\sin\theta) + \beta m^2\cos 2\theta\sin(\sin\theta) + \beta m^2\sin 2\theta\cos(\sin\theta)}{\cos(\sin\theta) + \beta m^2\cos 2\theta\cos(\sin\theta) - \beta m^2\sin 2\theta\sin(\sin\theta)}.$$

It is easily verified that the minimum value of the function g is attained at $\theta = \pi$ for $\beta \geq e^2 - 1$. In that case, for all $\theta \in [0, 2\pi)$ and $\beta \geq e^2 - 1$

$$g(\theta) \geq g(\pi) = \frac{1}{4}\ln^2\left(\frac{1}{e^2} + \frac{2\beta m^2}{e^2} + \frac{\beta^2 m^4}{e^2}\right) \geq 1.$$

Therefore, $|\log\psi(r,s,t;z)|\geq 1$ whenever $r=e^{e^{i\theta}}$, $s=me^{i\theta}r$ and $\operatorname{Re}(1+t/s)\geq m(1+\cos\theta)$, where $z\in\mathbb{D}$, $\theta\in[0,2\pi)$ and $m\geq 1$. Hence, Theorem 2.1 yields the desired result.

(c) For the function $\psi(r,s,t;z)=r+\beta s^2/r^2$, it is easy to deduce that

$$|\log\psi(r,s,t;z)|^2=\frac{1}{4}\ln^2(e^{2\cos\theta}+\beta^2m^4+2\beta m^2e^{\cos\theta}\cos 2\theta\cos(\sin\theta)+2\beta m^2e^{\cos\theta}\sin 2\theta$$

$$\sin(\sin\theta))+\left(\tan^{-1}\left(\frac{e^{\cos\theta}\sin(\sin\theta)+\beta m^2\sin 2\theta}{e^{\cos\theta}\cos(\sin\theta)+\beta m^2\cos 2\theta}\right)\right)^2=:g(\theta).$$

Since the second derivative of g is positive on both of its critical points, g attains absolute minimum at $\theta=\pi$ for $\beta\geq e-e^{-1}>0$. Therefore, for all $\theta\in[0,2\pi)$ and for $\beta\geq e-e^{-1}$, we get

$$g(\theta)\geq g(\pi)=\frac{1}{4}\ln^2\left(\frac{1}{e^2}+\frac{2\beta m^2}{e}+\beta^2m^4\right)\geq 1.$$

Hence, $\psi(r,s,t;z)\notin\Delta$ and thus $\psi\in\Psi\{e^z\}$. □

Next, we estimate the bound on β such that each of the first order differential subordination $p^2(z)+\beta zp'(z)/p^n(z)\prec e^z$ $(n=0,1,2)$ implies $p(z)\prec e^z$.

Theorem 4.7 *Let $p\in\mathcal{H}_1$. Then each of the following subordinations is sufficient for $p(z)\prec e^z$:*

(a) $p^2(z)+\beta zp'(z)\prec e^z$ for $\beta\geq e^2+e^{-1}\approx 7.75694$.

(b) $p^2(z)+\beta zp'(z)/p(z)\prec e^z$ for $\beta\geq e+e^{-2}\approx 2.85362$.

(c) $p^2(z)+\beta zp'(z)/p^2(z)\prec e^z$ for $\beta>\beta^*\approx 104.122$, where β^* is a positive root of the equation

$$6e^6+5xe^3-x^2+(-2e^6-5xe^3+x^2)\ln(e^3+x)=0.$$

Proof (a) For the function $\psi(r,s,t;z)=r^2+\beta s$, Theorem 2.1 is applicable if we show that the function $\psi\in\Psi\{e^z\}$ whenever $r=e^{e^{i\theta}}$, $s=me^{i\theta}r$ and $\operatorname{Re}(1+t/s)\geq m(1+\cos\theta)$, where $z\in\mathbb{D}$, $\theta\in[0,2\pi)$ and $m\geq 1$. Consider

$$|\log\,\psi(r,s,t;z)|^2=\frac{1}{4}\ln^2(\mu^2+\nu^2)+\left(\tan^{-1}\frac{\mu}{\nu}\right)^2=:g(\theta)$$

where

$$\mu=e^{2\cos\theta}\sin(2\sin\theta)+\beta me^{\cos\theta}\cos\theta\sin(\sin\theta)+\beta me^{\cos\theta}\sin\theta\cos(\sin\theta)$$

and

$$\nu=e^{2\cos\theta}\cos(2\sin\theta)+\beta me^{\cos\theta}\cos\theta\cos(\sin\theta)-\beta me^{\cos\theta}\sin\theta\sin(\sin\theta).$$

To show $|\log\psi(r,s,t;z)|\geq 1$, note that $g''(\pi)>0$ for $\beta>\beta^*\approx 2.9432$, where β^* is a root of the equation $2xe(1+xe)-(2+xe+x^2e^2)\ln(-1+xe)=0$. Therefore, minimum value of the function g is obviously attained at $\theta=\pi$ for $\beta\geq e^2+e^{-1}$ and hence

$$g(\theta)\geq g(\pi)=\frac{1}{4}\ln^2\left(\frac{1}{e^4}-\frac{2\beta m}{e^3}+\frac{\beta^2m^2}{e^2}\right)\geq 1\quad\text{for all }\theta\in[0,2\pi).$$

Consequently, Theorem 2.1 yields the result.
(b) The required subordination is proved if we show that $\psi \in \Psi\{e^z\}$, that is, if the admissibility condition (2.1) is satisfied. For the function $\psi(r,s,t;z) = r^2 + \beta s/r$, observe

$$|\log \psi(r,s,t;z)|^2 = \frac{1}{4}\ln^2(e^{4\cos\theta} + \beta^2 m^2 + 2\beta m e^{2\cos\theta}\cos\theta\cos(2\sin\theta) + 2\beta m e^{2\cos\theta}\sin\theta \sin(2\sin\theta)) + \left(\tan^{-1}\left(\frac{e^{2\cos\theta}\sin(2\sin\theta)+\beta m\sin\theta}{e^{2\cos\theta}\cos(2\sin\theta)+\beta m\cos\theta}\right)\right)^2 =: g(\theta).$$

For $\beta \geq e + e^{-2}$, it is easily verified using second derivative test that g attains its minimum value at $\theta = \pi$ which implies

$$g(\theta) \geq g(\pi) = \frac{1}{4}\ln^2\left(\frac{1}{e^4} - \frac{2\beta m}{e^2} + \beta^2 m^2\right) \geq 1 \quad \text{for all } \theta \in [0, 2\pi).$$

Therefore, $\psi(r,s,t;z) \notin \Delta$ whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ and $\operatorname{Re}(1 + t/s) \geq m(1+\cos\theta)$, where $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ and $m \geq 1$. Using Theorem 2.1 we get $p(z) \prec e^z$.
(c) As done in other cases, we need to show that $\psi(r,s,t;z) \notin \Delta$, where ψ is defined as $\psi(r,s,t;z) = r^2 + \beta s/r^2$. Consider

$$|\log \psi(r,s,t;z)|^2 = \frac{1}{4}\ln^2(\mu^2 + \nu^2) + \left(\tan^{-1}\frac{\mu}{\nu}\right)^2 =: g(\theta)$$

where

$$\mu = e^{2\cos\theta}\sin(2\sin\theta) - \beta m e^{-\cos\theta}\cos\theta\sin(\sin\theta) + \beta m e^{-\cos\theta}\sin\theta\cos(\sin\theta)$$

and

$$\nu = e^{2\cos\theta}\cos(2\sin\theta) + \beta m e^{-\cos\theta}\cos\theta\cos(\sin\theta) + \beta m e^{-\cos\theta}\sin\theta\sin(\sin\theta).$$

We note that the minimum value of g is attained at $\theta = 0$ for $\beta > \beta^* \approx 104.122$, where β^* is a positive root of the equation $6e^6 + 5xe^3 - x^2 + (-2e^6 - 5xe^3 + x^2)\ln(e^3 + x) = 0$. For $\theta \in [0, 2\pi)$ and $\beta > \beta^*$, we have

$$g(\theta) \geq g(0) = \frac{1}{4}\ln^2\left(e^4 + 2\beta m e + \frac{\beta^2 m^2}{e^2}\right) \geq 1.$$

Therefore, $\psi \in \Psi\{e^z\}$ and hence the result is obtained. □

Next, the bound on β is ascertained such that each of the first order differential subordination $p^n(z) + \beta z p(z) p'(z) \prec e^z$ $(n = 1,2,3)$ implies $p(z) \prec e^z$.

Theorem 4.8 *Let $p \in \mathcal{H}_1$, then each of the following subordinations is sufficient for $p(z) \prec e^z$:*

(a) $p(z) + \beta z p(z) p'(z) \prec e^z$ for $\beta \geq e^3 + e \approx 22.8038$.

(b) $p^2(z) + \beta z p(z) p'(z) \prec e^z$ for $\beta \geq e^3 + 1 \approx 21.0855$.

(c) $p^3(z) + \beta z p(z) p'(z) \prec e^z$ for $\beta \geq e^3 + e^{-1} \approx 20.4534$.

Proof The subordination $p(z) \prec e^z$ is satisfied if we show that $\psi \in \Psi\{e^z\}$ for different choices of ψ. Equivalently, we need to verify the admissibility condition (2.1):

$$|\log \psi(r,s,t;z)|^2 \geq 1$$

whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ and $\mathrm{Re}\,(1 + t/s) \geq m(1 + \cos\theta)$, where $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ and $m \geq 1$.
(a) Let $\psi(r, s, t; z) = r + \beta rs$. A simple calculation yields

$$|\log\psi(r, s, t; z)|^2 = \frac{1}{4}\ln^2(\mu^2 + \nu^2) + \left(\tan^{-1}\frac{\mu}{\nu}\right)^2 =: g(\theta)$$

where

$$\mu = e^{\cos\theta}\sin(\sin\theta) + \beta m e^{2\cos\theta}\cos\theta\sin(2\sin\theta) + \beta m e^{2\cos\theta}\sin\theta\cos(2\sin\theta)$$

and

$$\nu = e^{\cos\theta}\cos(\sin\theta) + \beta m e^{2\cos\theta}\cos\theta\cos(2\sin\theta) - \beta m e^{2\cos\theta}\sin\theta\sin(2\sin\theta).$$

It is easily verified that $g''(\pi) > 0$ for $\beta > \beta^* \approx 7.7065$, where β^* is the positive root of the equation $6x - 2e + (e - 2x)\ln((e - x)^2) = 0$. Since we are given that $\beta \geq e^3 + e \approx 22.8038$, the minimum value of g is attained at $\theta = \pi$ and for all $\theta \in [0, 2\pi)$, we have

$$\begin{aligned} g(\theta) \geq g(\pi) &= \frac{1}{4}\ln^2\left(\frac{1}{e^2} - \frac{2\beta m}{e^3} + \frac{\beta^2 m^2}{e^4}\right) \\ &= \frac{1}{4}\ln^2\left(\frac{\beta m}{e^2} - \frac{1}{e}\right)^2 \geq \frac{1}{4}\ln^2\left(\frac{e^3 + e}{e^2} - \frac{1}{e}\right)^2 = 1. \end{aligned}$$

Therefore, $\psi \in \Psi\{e^z\}$.
(b) The function $\psi(r, s, t; z) = r^2 + \beta rs$ satisfies

$$|\log\psi(r, s, t; z)|^2 = \frac{1}{4}\ln^2\left(e^{4\cos\theta}(\mu^2 + \nu^2)\right) + \left(\tan^{-1}\frac{\mu}{\nu}\right)^2 =: g(\theta)$$

where

$$\mu = \sin(2\sin\theta) + \beta m\sin\theta\cos(2\sin\theta) + \beta m\cos\theta\sin(2\sin\theta)$$

and

$$\nu = \cos(2\sin\theta) + \beta m\cos\theta\cos(2\sin\theta) - \beta m\sin\theta\sin(2\sin\theta).$$

Since $g''(\pi) > 0$ for $\beta > \beta^* \approx 6.46722$, where β^* is the root of the equation $x(2-3x)+(2-3x+2x^2)\ln(-1+x) = 0$, the minimum value of $g(\theta)$ is obviously attained at $\theta = \pi$ if $\beta \geq e^3+1 \approx 20.0855$. Therefore, for $\theta \in [0, 2\pi)$, we get

$$g(\theta) \geq g(\pi) = \frac{1}{4}\ln^2\left(\frac{1}{e^4} - \frac{2\beta m}{e^4} + \frac{\beta^2 m^2}{e^4}\right) \geq 1$$

and hence $\psi \in \Psi\{e^z\}$.
(c) Let $\psi(r, s, t; z) = r^3 + \beta rs$. With r, s and t stated above, ψ takes the form $\psi(r, s, t; z) = e^{3e^{i\theta}} + \beta m e^{i\theta} e^{2e^{i\theta}}$ which satisfies

$$|\log\psi(r, s, t; z)|^2 = \frac{1}{4}\ln^2\left(e^{4\cos\theta}(\mu^2 + \nu^2)\right) + \left(\tan^{-1}\frac{\mu}{\nu}\right)^2 =: g(\theta)$$

where

$$\mu = e^{\cos\theta}\sin(3\sin\theta) + \beta m\cos\theta\sin(2\sin\theta) + \beta m\sin\theta\cos(2\sin\theta)$$

and

$$\nu = e^{\cos\theta}\cos(3\sin\theta) + \beta m\cos\theta\cos(2\sin\theta) - \beta m\sin\theta\sin(2\sin\theta).$$

Clearly, g attains its minimum value either at $\theta = 0$ or $\theta = \pi$. Since $g''(\pi) > 0$ for $\beta > \beta^* \approx 5.66489$, where β^* is the root of the equation $xe(3+5xe) - (3 - xe + 2x^2e^2)\ln(-1+xe) = 0$, the minimum value of g is attained at $\theta = \pi$ if $\beta \geq e^3 + e^{-1} \approx 20.4534$. In that case, we have

$$g(\theta) \geq g(\pi) = \frac{1}{4}\ln^2\left(\frac{1}{e^6} - \frac{2\beta m}{e^5} + \frac{\beta^2 m^2}{e^4}\right) \geq 1 \quad \text{for all } \theta \in [0, 2\pi).$$

Therefore, $\psi \in \Psi\{e^z\}$ by Theorem 2.1. □

Now, we estimate the bound on β such that the first order differential subordination $p^2(z)+p(z)-1+\beta zp'(z) \prec e^z$ implies $p(z) \prec e^z$.

Theorem 4.9 *Let* $p \in \mathcal{H}_1$ *and satisfy the subordination*

$$p^2(z) + p(z) - 1 + \beta zp'(z) \prec e^z \quad \textit{for } \beta \geq e^2 + e^{-1} - e + 1 \approx 6.03865.$$

Then $p(z) \prec e^z$.

Proof Proceeding as in the previous theorems, we need to show that the function $\psi(r,s,t;z) = r^2 + r - 1 + \beta s$ satisfies the admissibility condition (2.1). Whenever $r = e^{e^{i\theta}}$, $s = me^{i\theta}r$ and $\operatorname{Re}(1 + t/s) \geq m(1+\cos\theta)$, where $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ and $m \geq 1$, $\psi(r,s,t;z) = e^{2e^{i\theta}} + e^{e^{i\theta}} - 1 + \beta me^{i\theta}e^{e^{i\theta}}$ satisfies

$$|\log\psi(r,s,t;z)|^2 = \frac{1}{4}\ln^2(\mu^2+\nu^2) + \left(\tan^{-1}\frac{\nu}{\mu}\right)^2 =: g(\theta)$$

where

$$\mu = e^{2\cos\theta}\cos(2\sin\theta) + e^{\cos\theta}\cos(\sin\theta) + \beta me^{\cos\theta}\cos\theta\cos(\sin\theta) - \beta me^{\cos\theta}\sin\theta\sin(\sin\theta) - 1$$

and

$$\nu = e^{2\cos\theta}\sin(2\sin\theta) + e^{\cos\theta}\sin(\sin\theta) + \beta me^{\cos\theta}\cos\theta\sin(\sin\theta) + \beta me^{\cos\theta}\sin\theta\cos(\sin\theta).$$

Using second derivative test, it can be easily verified that g attains its minimum value at $\theta = \pi$ for $\beta > 0$. Therefore, for $\theta \in [0, 2\pi)$, we have

$$g(\theta) \geq g(\pi) = \frac{1}{4}\ln^2\left(\frac{\beta m}{e} - \frac{1}{e^2} - \frac{1}{e} + 1\right)^2.$$

Since logarithm is an increasing function and the condition $\beta \geq e^2 + e^{-1} - e + 1$ imply that

$$|\log\psi(r,s,t;z)|^2 \geq \frac{1}{4}\ln^2\left(\frac{\beta}{e} - \frac{1}{e^2} - \frac{1}{e} + 1\right)^2 \geq \frac{1}{4}\ln^2(e^2) = 1.$$

Hence, $\psi \in \Psi\{e^z\}$ and Theorem 2.1 gives the desired result. □

Remark 4.10 *As depicted in the previous section, results proved in this section also provide several sufficient conditions for a normalized analytic function to be in the class* $\mathcal{S}_e^*$. *These sufficient conditions can be obtained by simply putting* $p(z) = zf'(z)/f(z)$, *where* $f \in \mathcal{H}$.

Remark 4.11 *Since we were concerned with the star-likeness property in this paper, therefore we presented applications of our results only for the subclass of* $\mathcal{S}^*$. *However, by setting* $p(z) = f'(z)$, $p(z) = f(z)/z$, $p(z) = 2f(z)/z - 1$, $p(z) = 2\sqrt{f'(z)} - 1$, $p(z) = 2zf'(z)/f(z) - 1$ *and so forth in the theorems obtained, one can obtain many more differential implications.*

References

[1] Ahuja OP, Kumar S, Ravichandran V. Application of first order differential subordination for functions with positive real part. Studia Universitatis Babeș-Bolyai Mathematica 2018; 63 (3): 303-311.

[2] Ali RM, Cho NE, Ravichandran V, Kumar SS. Differential subordination for functions associated with the lemniscate of Bernoulli, Taiwanese Journal of Mathematics 2012; 16 (3): 1017-1026. doi: 10.11650/twjm/1500406676

[3] Ali RM, Ravichandran V, Seenivasagan N. Sufficient conditions for Janowski star-likeness. International Journal of Mathematics and Mathematical Sciences 2007; Article ID 62925, 7. doi: 10.1155/2007/62925

[4] Gandhi S, Kumar S, Ravichandran V. First order differential subordinations for carathéodory functions. Kyungpook Mathematical Journal 2018; 58 (2).

[5] Goodman AW. Univalent Functions Volume I. Tampa, FL, USA: Mariner Publishing Company, Incorporated, 1983.

[6] Janowski W. Extremal problems for a family of functions with positive real part and for some related families. Annales Polonici Mathematici 1970/1971; 23: 159-177. doi: 10.4064/ap-23-2-159-177

[7] Kumar S, Ravichandran V. Subordinations for functions with positive real part. Complex Analysis and Operator Theory 2018; 12 (5): 1179-1191. doi: 10.1007/s11785-017-0690-4

[8] Ma WC, Minda D. A unified treatment of some special classes of univalent functions. In: Proceedings of the Conference on Complex Analysis (Tianjin, 1992) - Conference Proceedings Lecture Notes Anal., I. Cambridge, MA: International Press, 1994. pp. 157-169.

[9] Mendiratta R, Nagpal S, Ravichandran V. On a subclass of strongly star-like functions associated with exponential function. Bulletin of the Malaysian Mathematical Sciences Society 2015; 38 (1): 365-386.

[10] Miller SS, Mocanu PT. Differential subordinations and univalent functions. The Michigan Mathematical Journal 1981; 28 (2): 157-172.

[11] Miller SS, Mocanu PT. Differential Subordinations Volume 225 of Monographs and Textbooks. In: Pure and Applied Mathematics, New York, NY, USA: Marcel Dekker, Incorporated, 2000.

[12] Nunokawa M, Obradović M, Owa S. One criterion for univalency. Proceedings of the American Mathematical Society 1989; 106 (4): 1035-1037. doi: 10.2307/2047290

[13] Omar R, Halim SA. Differential subordination properties of Sokół-Stankiewicz star-like functions. Kyungpook Mathematical Journal 2013; 53 (3): 459-465. doi: 10.5666/KMJ.2013.53.3.459

[14] Raina RK, Sokół J. On coefficient estimates for a certain class of star-like functions. Hacettepe Journal of Mathematics and Statistics 2015; 44 (6): 1427-1433.

[15] Rønning F. Uniformly convex functions and a corresponding class of star-like functions. Proceedings of the American Mathematical Society 1993; 118 (1): 189-196.

[16] Sharma K, Ravichandran V. Applications of subordination theory to star-like functions. Bulletin of Iranian Mathematical Society 2016; 42 (3): 761-777.

[17] Sokół J. On sufficient condition to be in a certain subclass of star-like functions defined by subordination. Applied Mathematics and Computation 2007; 190 (1): 237-241. doi: 10.1016/j.amc.2007.01.034

[18] Sokół J, Stankiewicz J. Radius of convexity of some subclasses of strongly star-like functions, Zeszyty Nauk. Politech. Rzeszowskiej Mat 1996; 19: 101-105.

[19] Tuneski N. On the quotient of the representations of convexity and star-likeness. Mathematische Nachrichten 2003; 248/249: 200-203. doi: 10.1002/mana.200310015