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An inequality on diagonal F-thresholds over standard-graded complete intersection rings

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Abstract: In a recent paper, De Stefani and Núñez-Betancourt proved that for a standard-graded F-pure k-algebra R, its diagonal F-threshold $c(R)$ is always at least $-a(R)$, where $a(R)$ is the a-invariant. In this paper, we establish a refinement of this result in the setting of complete intersection rings.

Key words: Frobenius power, socle, F-threshold, F-pure threshold, a-invariant

1. Introduction and notations

Let R be a commutative Noetherian ring in prime characteristic $p > 0$. Let q be a power of p. For an ideal I in R, let $I^{[q]}$ be the qth bracket power of I, that is, $I^{[q]} := (a^q : a \in I)$. For a pair of ideals $\mathfrak{a}$ and J in R such that $\mathfrak{a} \subseteq \sqrt{J}$, define $\nu_{\mathfrak{a}}^J(q) = \max\{r \in \mathbb{N} | \mathfrak{a}^r \not\subseteq J^{[q]}\}$. Recently, De Stefani, Núñez-Betancourt, and Pérez [5] proved that the limit of $\{\nu_{\mathfrak{a}}^J(q)/q\}$ as $q \to \infty$ always exists. Such a limit, denoted by $c^J(\mathfrak{a})$, is called the F*-threshold* of the pair $(\mathfrak{a}, J)$. We mention [1–8, 12] for details of some work done regarding this invariant. In the case of a local ring $(R, \mathfrak{m})$ with an $\mathfrak{m}$-primary ideal J, the F-threshold $c^J(\mathfrak{m})$ is also called the diagonal F-threshold (for simplicity, the diagonal F-threshold $c^{\mathfrak{m}}(\mathfrak{m})$ of R is denoted by $c(R)$). For an ideal $\mathfrak{a}$ in a regular local ring $(R, \mathfrak{m})$, the F-threshold $c^{\mathfrak{m}}(\mathfrak{a})$ coincides with the F-pure threshold of $\mathfrak{a}$ (see [12, Remark 1.5]), that is,

$$c^{\mathfrak{m}}(\mathfrak{a}) = \sup\{s \in R_{\geq 0} | \text{ the pair } (R, \mathfrak{a}^s) \text{ is } F\text{-pure }\}$$

In particular, for an element $f \in \mathfrak{m}$ in a regular local ring $(R, \mathfrak{m})$, the F-threshold $c^{\mathfrak{m}}((f))$ coincides with the F-pure threshold of f. We use $\mathrm{fpt}(f)$ to denote this invariant. The theories on F-threshold and F-pure threshold are motived by the relations between $\mathrm{fpt}(f)$ and $\mathrm{lct}(f)$, where $\mathrm{lct}(f)$ is the log canonical threshold, a notion developed in birational geometry. More precisely, if R is a local ring in characteristic 0 and $R_p = R \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$, then

$$\mathrm{fpt}(f_p) \leq \mathrm{lct}(f), \text{ for all } p, \tag{1}$$

and

$$\lim_{p\to\infty} \mathrm{fpt}(f_p) = \mathrm{lct}(f). \tag{2}$$

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Suppose R is also a standard-graded algebra over a field k in prime characteristic p, then the diagonal F-threshold of an $\mathfrak{m}$-primary ideal J is easily seen to be

$$c^J(\mathfrak{m}) = \lim_{q\to\infty} \frac{\mathrm{t.s.d}(R/J^{[q]})}{q}$$

where $\mathrm{t.s.d}(R/J^{[q]})$ is the top socle degree of the Artinian algebra $R/J^{[q]}$.

Let $a(R)$ be the a-invariant of R. It was proved in [11] that when R is a complete intersection ring or a Gorenstein F-pure ring, the inequality

$$c^J(\mathfrak{m}) \geq \mathrm{t.s.d}(R/J) - a(R)$$

holds. In particular, by taking $J = \mathfrak{m}$, we have

$$c(R) \geq -a(R.)$$

Hirose, Watanabe, and Yoshida conjectured that this latter inequality holds for any F-pure ring [9], and this was recently settled in [4, Theorem 4.9].

2. Main result

The following is the main result which improves the inequality $c(R) \geq -a(R)$ for the case of standard-graded complete intersection rings.

Theorem 2.1 *Let $\mathfrak{a} = (f_1, \cdots, f_t)$ be a homogeneous ideal of the standard-graded polynomial ring $S = k[x_1, \cdots, x_n]$, where $f_1, \cdots, f_t$ form a homogeneous S-sequence. Let $|f_i|$ be the degree of f_i, $i = 1, \cdots, t$. Let $\mathfrak{m} = (x_1, \cdots, x_n)$ be the maximal ideal of S. Let R be the complete intersection ring $k[x_1, \cdots, x_n]/(f_1, \cdots, f_t)$. Let $\mathrm{fpt}(f)$ denote the F-pure threshold for a polynomial $f \in S$. Then the following inequality holds*

$$c(R) \geq -a(R) + \sum_i |f_i| \Big(1 - \mathrm{fpt}(f_i)\Big) \tag{3}$$

In particular, if $c(R) = -a(R)$, then $\mathrm{fpt}(f_i) = 1$ for all $i = 1, \cdots, t$.

Before we prove this theorem, we recall the following observation

Lemma 2.2 *(see [10], Observation 1.4) Let R be an Artinian Gorenstein graded k-algebra with socle degree δ, and J a homogeneous ideal of R. If the socle degree of R/J are d_i, then the degrees of the minimal generators of $(0 : J)$ are $\delta - \mathrm{d}_i$.*

Proof [Proof of Theorem 2.1] Let I be an $\mathfrak{m}$-primary reducible ideal of R. Let J be the pre-image of I in S, so that J contains $\mathfrak{a}$. Applying Lemma 2.2 to the Gorenstein algebra $S/J^{[q]}$, and use socdeg to denote the (unique) socle degree of a Gorenstein Artinian algebra, one has that $\mathrm{socdeg}(S/J^{[q]}) = \mathrm{t.s.d}(S/(J^{[q]} + \mathfrak{a})) + M$ where M is the smallest degree of the minimal generators of $(J^{[q]} : \mathfrak{a})/J^{[q]}$.

Since J has finite projective dimension, $\operatorname{socdeg}(S/J^{[q]}) - a(S) = q(\operatorname{socdeg}(S/J) - a(S))$. Therefore,

$$\begin{aligned}\mathrm{t.s.d}(R/I^{[q]}) &= \mathrm{t.s.d}(S/(J^{[q]} + \mathfrak{a})) \\ &= \operatorname{socdeg}(S/J^{[q]}) - M \\ &= q(\operatorname{socdeg}(S/J)) - (q-1)a(S) - M\end{aligned}$$

We need to estimate an upper bound for M. Let h denote $\nu_{\mathfrak{a}}^{J}(q) := \max\{r \in \mathbb{N} : \mathfrak{a}^r \nsubseteq J^{[q]}\}$. Since $\mathfrak{a}^{h+1} \subseteq J^{[q]}$, $\mathfrak{a}^h$ contains an element of the form $f_1^{\alpha_1} \cdots f_t^{\alpha_t}$ with $\alpha_1 + \cdots + \alpha_t = h$, whose image in $(J^{[q]} : \mathfrak{a})/J^{[q]}$ is nonzero. Hence,

$$M \le \sum_{i=1}^{t} |f_i| \alpha_i$$

On the other hand, let $\beta_i(q)$ denote $\nu_{f_i}^{J}(q) := \max\{t \in \mathbb{N} : {f_i}^t \nsubseteq J^{[q]}\}$. It is obvious that

$$\alpha_i \le \beta_i(q), \forall i.$$

It follows that

$$M \le \sum_{i=1}^{t} |f_i| \beta_i(q)$$

On the other hand, the a-invariant $a(R) = \sum_{i=1}^{t} |f_i| - n$ since R is a complete intersection ring, so

$$\begin{aligned}\mathrm{t.s.d}(R/I^{[q]}) &= q(\operatorname{socdeg}(S/J)) - (q-1)(-n) - M \\ &\ge q(\operatorname{socdeg}(S/J)) - (q-1)(-n) - \sum_{i=1}^{t} |f_i| \beta_i(q) \\ &= q(\operatorname{socdeg}(S/J)) - (q-1)(-n) - q\sum_{i=1}^{t} |f_i| \beta_i(q)/q \\ &= q(\operatorname{socdeg}(S/J)) - n + qn - q(\sum_{i=1}^{t} |f_i|) + q(\sum_{i=1}^{t} |f_i|) - q\sum_{i=1}^{t} |f_i| \beta_i(q)/q \\ &= q(\operatorname{socdeg}(S/J)) - n + qn - q(\sum_{i=1}^{t} |f_i|) + q\sum_{i=1}^{t} |f_i| \Big(1 - \beta_i(q)/q\Big) \\ &= q(\operatorname{socdeg}(S/J)) - n + q(-a(R)) + q\sum_{i=1}^{t} |f_i| \Big(1 - \beta_i(q)/q\Big)\end{aligned}$$

Dividing both sides by q and taking the limit, we have

$$c^{\mathfrak{m}}(I) \ge \operatorname{socdeg}(R/I)) - a(R) + \sum_{i=1}^{t} |f_i| \Big(1 - c^J(f_i)\Big)$$

The desired inequality is then obtained by taking I to be the maximal ideal of R, and $J = \mathfrak{m}$.

□

Remark 2.3 *It is easy to see from the proof above that the condition* $c(R) = -a(R)$ *also forces* $c^{\mathfrak{m}}(\mathfrak{a}) = t = \sum_{i=1}^{t} \mathrm{fpt}(f_i)$. *However, we do not know what else can be derived from this.*

Let k be a field in characteristic 0. Let $R = k[x_1, \cdots, x_n]/(f_1, \cdots, f_t)$ where $f_1, \cdots, f_t$ form a homogeneous regular sequence in $k[x_1, \cdots, x_n]$. For a prime number p, let $R_p = R \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})$.

Corollary 2.4 *If* $\lim_{p\to\infty} c(R_p) = -a(R)$, *then the log canonical thresholds* $\mathrm{lct}(f_i) = 1$ *for all* $i = 1, \cdots, t$.

Proof This follows from Theorem 2.1 and (2) immediately. □

3. An example on diagonal hypersurface rings

We use the following computation of Vraciu [13] to study an example of the diagonal hypersurface case.

Theorem 3.1 *[13, Theorem 4.2] Let* $R = k[x_1, \cdots, x_{n+1}]/(x_1^a + \cdots + x_{n+1}^a)$ *where* k *is a field of characteristic* p *and* a *is a positive integer not divisible by* p. *Then*

$$c(R) = n + 1 - a\mathcal{M}$$

where $\mathcal{M}$ *is equal to*

$$\min\left\{\left\lceil \frac{(n+1)\kappa - n + 1}{2} \right\rceil \cdot \frac{1}{p^{e_0}} + \frac{(n+1)s}{ap^{e_0}}, \left\lceil \frac{(n+1)\kappa - n + 2}{2} \right\rceil \cdot \frac{1}{p^{e_0}} + \frac{ns}{ap^{e_0}}, \left\lceil \frac{(n+1)\kappa + 1}{2} \right\rceil \cdot \frac{1}{p^{e_0}} + \frac{s}{ap^{e_0}}, \left\lceil \frac{(n+1)\kappa + 2}{2} \right\rceil \cdot \frac{1}{p^{e_0}}, \frac{1}{p^{e_0-1}}\right\}$$

where e_0 *is the smallest exponent such that* $p^{e_0} \geq a$, $\kappa = \left\lfloor \frac{p^{e_0}}{a} \right\rfloor$, *and* $s = p^{e_0} - \kappa a$ *is the remainder of* p^{e_0} *modulo* a.

The following example then follows from the above theorem immediately by taking $p \to \infty$.

Example 3.2 *Let* $R = k[x_1, \cdots, x_{n+1}]/(x_1^a + \cdots + x_{n+1}^a)$, *where* k *is a field in characteristic* 0. *Let* $R_p = R \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})$. *Then the limit diagonal* F*-threshold*

$$\lim_{p\to\infty} c(R_p) = \begin{cases} \frac{n+1}{2}, & \text{if } \frac{n+1}{2} \leq a \\ n + 1 - a, & \text{if } 2a \leq n \end{cases}$$

Notice that the a*-invariant* $a(R) = -(n + 1 - a)$, *which is characteristic-free, we obtain by Corollary 2.4 that*

$$\mathrm{lct}(x_1^a + \cdots + x_{n+1}^a) = 1,$$

provided $n \geq 2a$.

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