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On an elliptic boundary value problem with critical exponent

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Abstract: We study a nonlinear critical second-order PDE with zero Dirichlet boundary condition. We prove existence and compactness results for the equation by using Bahri's method of critical points at infinity.

Key words: Nonlinear elliptic PDE, calculus of variation, critical points at infinity

1. Introduction

In this paper we consider the problem of existence of smooth solutions of the following partial differential equation:

$$\begin{cases} -\Delta u & = & K(x)\, u^{\frac{n+2}{n-2}}, \\ u & > 0 & \text{in } \Omega, \\ u & = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $n \geq 3$, Ω is a smooth bounded domain of $\mathbb{R}^n$, and K is a given function on $\overline{\Omega}$.

The interest in this equation grew from its resemblance to the celebrated Nirenberg–Yamabe problem or more generally the scalar curvature problem on closed Riemannian manifolds. See [1], [3], [6], [7], [9], [10], [11], [12], [13].

Equation (1.1) can be expressed as a variational problem in $H_0^1(\Omega)$. However, the variational structure presents a loss of compactness phenomenon, since $\frac{n+2}{n-2}$ is critical and $H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ is continuous and not compact.

The first contributions to (1.1) concern the case $K = 1$. Indeed, in [14] and [4], the authors proved that the existence of solutions is related to the topology of the domain Ω. For $K \neq 1$, some existence results can be found, for example, in [8], [15], [16], and [17].

Let $G(.,.)$ be the Green function for $(-\Delta)$ on Ω under zero Dirichlet boundary condition and $H(.,.)$ its regular part. Recently, in [16] and [17], we studied the loss of compactness of (1.1) and established existence results under the following three assumptions:

$(f)_\beta$ K is a C^1-positive function such that for any critical point y of K there exists a real number $\beta = \beta(y)$ such that

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$$K(x) = K(y) + \sum_{k=1}^{n} b_k |(x-y)_k|^\beta + o(|x-y|^\beta),$$

for all $x \in B(y, \rho_0)$, where ρ_0 is a fixed positive constant, $b_k = b_k(y) \in \mathbb{R} \setminus \{0\}$, for $k = 1 \dots, n$, and

$$-\frac{n-2}{n}\frac{c_1}{K(y)}\sum_{k=1}^{n} b_k(y) + c_2 \frac{n-2}{2} H(y,y) \neq 0, \text{ for any } y \text{ such that } \beta(y) = n-2,$$

where $c_1 = \int_{\mathbb{R}^n} \frac{|x_1|^{n-2}}{(1+|x|^2)^n} dx$ and $c_2 = \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{\frac{n+2}{2}}}$.

$$(A_0) \qquad \begin{cases} \beta(y) \in (1, n-2], \forall\, y & \text{such that } \nabla K(y) = 0, \\ & \text{or} \\ \beta(y) \in (1, \infty), \forall\, y & \text{such that } \nabla K(y) = 0 \text{ and } K \text{ is assumed close to 1.} \end{cases}$$

Lastly, it is assumed that

$$(A) \qquad \frac{\partial K}{\partial \nu}(x) < 0,\ \forall x \in \partial\Omega, \text{ where } \nu \text{ is the unit outward normal vector on } \partial\Omega.$$

In this paper we continue our study of problem (1.1) and our aim is to describe the loss of compactness of the problem and provide the existence result under the $(f)_\beta$-condition when the flatness order of K at its critical points lies in $[n-2, \infty)$ without assuming any perturbation condition.

Remark 1.1 *We point out that problem (1.1) was studied under the $(f)_\beta$-condition when the β-flatness order $\beta(y) \in (1, \infty)$. See [18] for the three-dimensional case and [17] for the n-dimensional case, $n \geq 4$. However, a strong condition was assumed in these papers:*

$$\|K - 1\|_{L^\infty(\bar{\Omega})} \textit{ is small enough.} \tag{1.2}$$

Such a condition plays a crucial role to prove the "perturbation" results of [18] and [17]. Indeed, it reduces the analysis to $V(1, \varepsilon)$, a neighborhood of the critical points at infinity constructed by only one mass. For this reason, we find in [18] and [17] only a partial description of the critical points at infinity of problem (1.1), namely the critical points at infinity constructed by one mass. The present paper addresses the case where the perturbation condition (1.2) is not satisfied. This leads to an interesting new existence result completely different from those of [18] and [17]. Our proof requires a full description of the loss of compactness of the problem in the whole variational space. Indeed, we prove in Section 3 of this paper the description of the critical points at infinity in all $V(p, \varepsilon), p \geq 1$. This requires new estimates and new analysis at infinity.

Let $\mathcal{K}$ be the set of all critical points of K. For any $\tau_p = (y_{\ell_1}, \dots, y_{\ell_p}) \in \mathcal{K}^p, 1 \leq p \leq \sharp\mathcal{K}$ such that $y_{\ell_i} \neq y_{\ell_j}, \forall i \neq j$, we define a $p \times p$ symmetric matrix $M(\tau_p) = (m_{ij})_{1 \leq i,j \leq p}$ by

$$m_{ii} = m(y_{\ell_i}, y_{\ell_i}) = \begin{cases} -\dfrac{1}{K(y_{\ell_i})^{\frac{n-2}{2}}}\left(\dfrac{n-2}{n}\dfrac{c_1}{K(y_{\ell_i})}\displaystyle\sum_{k=1}^{n} b_k(y_{\ell_i}) - c_2 \dfrac{n-2}{2} H(y_{\ell_i}, y_{\ell_i})\right) \\ \qquad \text{if } \beta(y_{\ell_i}) = n-2, \\ \dfrac{n-2}{2}\dfrac{c_2}{K(y_{\ell_i})^{\frac{n-2}{2}}} H(y_{\ell_i}, y_{\ell_i}) \ \text{ if } \beta(y_{\ell_i}) > n-2, \end{cases}$$

$\forall i = 1, \ldots, p$, and

$$m_{ij} = m(y_{\ell_i}, y_{\ell_j}) = -\frac{n-2}{2} c_2 \frac{G(y_{\ell_i}, y_{\ell_j})}{\Big(K(y_{\ell_i})K(y_{\ell_j})\Big)^{\frac{n-2}{4}}}, \text{ for } i \neq j.$$

(B) Assume that $\rho(\tau_p)$, the least eigenvalue of $M(\tau_p)$, is nonzero for any $1 \leq p \leq \sharp\mathcal{K}$.
Define

$$\mathcal{C}^\infty := \Big\{\tau_p = (y_{l_1}, ..., y_{l_p}) \in \mathcal{K}^p, 1 \leq p \leq \sharp\mathcal{K}, s.t.\ \ y_{\ell_i} \neq y_{\ell_j} \ \ \forall 1 \leq i \neq j \leq p,$$

$$m(y_{l_i}, y_{l_i}) > 0, \forall i = 1 \ldots, p \text{ and } \rho(\tau_p) > 0\Big\}.$$

For any $\tau_p \in \mathcal{C}^\infty$, we define

$$i(\tau_p) = p - 1 - \sum_{j=1}^{p} (n - \widetilde{i}(y_{\ell_j})),$$

where

$$\widetilde{i}(y) = \sharp\{b_k(y), 1 \leq k \leq n,\ b_k(y) < 0\}.$$

Theorem 1.2 *Under the assumptions* $(A), (B)$*, and* $(f)_\beta, \beta \in [n-2, \infty)$*, if*

$$\sum_{\tau_p \in \mathcal{C}^\infty} (-1)^{i(\tau_p)} \neq 1,$$

then (1.1) *has a solution.*

Equation (1.1) has a variational structure. The solutions are the positive critical points (up to a multiplicative constant) of the functional

$$J(u) = \frac{\displaystyle\int_\Omega |\nabla u|^2}{\Big(\displaystyle\int_\Omega K(x) u^{\frac{2n}{n-2}}\Big)^{\frac{n}{n-2}}},$$

and u belongs to the unit sphere Σ of the Sobolev space $(H_0^1(\Omega), |.|_\Omega)$.
We will argue by contradiction to prove Theorem 1.2. Therefore, we will assume throughout this paper that (1.1) has no solution. Our method uses the theory of critical points at infinity of Bahri [2]. We will describe the lack of compactness of the equation and the concentration phenomenon through the construction of a suitable pseudogradient that satisfies the Palais–Smale condition along its flow lines provided that these flow lines are outside a neighborhood of specific critical points of the function K. Finally, a Euler–Poincaré characteristic argument achieves a contradiction.

The next section will be devoted to recalling the general framework of the associated variational structure and expanding the gradient of J near infinity. We will then use these asymptotic expansions in Section 3 to construct the required pseudogradient and characterize the critical points at infinity of J. Lastly, in Section 4, we will prove Theorem 1.2.

2. The gradient expansion of J

Due to the loss of compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$, the Euler–Lagrange functional J fails to satisfy the Palais–Smale (PS) condition. The sequences failing the PS condition were described in [4] and [19].

For $a \in \Omega$ and $\lambda > 0$, we set

$$\delta_{a,\lambda}(x) = c_0\left(\frac{\lambda}{1+\lambda^2|x-a|^2}\right)^{\frac{n-2}{2}}, \tag{2.1}$$

where c_0 is a fixed positive constant. The family $\delta_{a,\lambda}, a \in \Omega$ and $\lambda > 0$, are the only solutions of

$$\begin{cases} -\Delta u &= \quad u^{\frac{n+2}{n-2}}, \\ u &> \quad 0 \quad \text{in } \mathbb{R}^n. \end{cases} \tag{2.2}$$

Define $P\delta_{a,\lambda}$ on Ω to be the unique solution of

$$\begin{cases} -\Delta u &= \quad \delta_{a,\lambda}^{\frac{n+2}{n-2}} \\ u &> \quad 0 \;\; \text{in } \Omega, \\ u &= \quad 0 \;\; \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

By the regularity theorem, $P\delta_{a,\lambda} \in C^\infty(\Omega)$. For $p \in \mathbb{N}$ and $\varepsilon > 0$, we define $V(p,\varepsilon)$, the set of all functions $u \in \Sigma^+ := \{u \in \Sigma,\; u \geq 0\}$ such that there exist $(a_1,\dots,a_p) \in \Omega^p, \lambda_1,\dots,\lambda_p > \varepsilon^{-1}$ and $\alpha_1,\dots,\alpha_p > 0$ satisfying

$$\Big|u - \sum_{i=1}^{p} \alpha_i P\delta_{a_i,\,\lambda_i}\Big| < \varepsilon,$$

with $\left|J(u)^{\frac{n}{n-2}}\alpha_i^{\frac{4}{n-2}}K(a_i) - 1\right| < \varepsilon$ and $\varepsilon_{ij} = \left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2\right)^{\frac{-(n-2)}{2}} < \varepsilon \;\; \forall i \neq j.$

It is known (see [4] and [19]) that for any $(u_k)_k$ in Σ^+ that fails the PS condition, there exist $p \in \mathbb{N}$, a positive sequence $(\varepsilon_k)_k$ tending to zero, and an extracted subsequence $(u_{k_\ell})_\ell$ such that $u_{k_\ell} \in V(p,\varepsilon_{k_\ell}), \forall \ell \in \mathbb{N}$.

The parametrization of the set $V(p,\varepsilon)$ is given in [4]. Indeed, for any $u \in V(p,\varepsilon)$, u can be written as

$$u = \sum_{i=1}^{p} \bar{\alpha}_i P\delta_{\bar{a}_i,\bar{\lambda}_i} + v,$$

where $v \in H_0^1(\Omega), |v| < \varepsilon$, and satisfies

$$(V_0) : \Big\langle v, \psi \Big\rangle = 0 \; for \; \psi \in \{P\delta_{a_i,\lambda_i}, \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i}, \frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i}, i = 1, ..., p\},$$

where $< .,. >$ denotes the inner product on $H_0^1(\Omega)$ associated to the norm $|.|$ and $\bar{\alpha}_i, \bar{a}_i, \bar{\lambda}_i, i = 1,\dots,p$ are the unique solutions (up to a permutation) of

$$\min_{\sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i} \in V(p,\varepsilon)} \Big|u - \sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i}\Big|.$$

Following [2], we know that for $u = \sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i} \in V(p,\varepsilon)$, $v \longrightarrow J(u+v)$ admits a unique minimum denoted $\bar{v} = \bar{v}(\alpha_i, a_i, \lambda_i)$ and there exists a change of variable $v - \bar{v} \longrightarrow V$ such that

$$J(u+v) = J(u+\bar{v}) + |V|^2.$$

Moreover, there exists M independent of $(\alpha_i, a_i, \lambda_i)$ such that

$$|\overline{v}| \leq M \sum_{i=1}^{p} \Big[\frac{1}{\lambda_i^{\frac{n}{2}}} + \frac{1}{\lambda_i^{\beta}} + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{(\log \lambda_i)^{\frac{n+2}{2n}}}{\lambda_i^{\frac{n+2}{2}}} \Big] \tag{2.4}$$

$$+ c \begin{cases} \sum_{k \neq r} \varepsilon_{k\,r}^{\frac{n+2}{n-2}} \Big(\log \varepsilon_{kr}^{-1} \Big)^{\frac{n+2}{2n}}, & \text{if } n \geq 6 \\ \sum_{k \neq r} \varepsilon_{k\,r} \Big(\log \varepsilon_{kr}^{-1} \Big)^{\frac{n-2}{n}}, & \text{if } n < 6. \end{cases}$$

Next, we estimate the variation of the functional J in $V(p,\varepsilon)$ with respect to λ_i and $a_i, i = 1, \ldots, p$.

Proposition 2.1 *Let* $u = \sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i} \in V(p,\varepsilon)$.

For any $i = 1, \ldots, p$ *such that* $a_i \in \cup_{y \in \mathcal{K}} B(y, \rho_0)$, *we have:*

$$(a) \quad \Big\langle \partial J(u), \alpha_i \lambda_i \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i} \Big\rangle = -2c_2 \frac{J(u)}{K(a_i)} \sum_{j \neq i} \alpha_i \alpha_j \Big(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-2}{2} \frac{H(a_i,a_j)}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} \Big)$$

$$+ \quad O\Big(\sum_{s=2}^{[\min(n,\beta)]} \frac{|a_i - y_{\ell_i}|^{\beta - s}}{\lambda_i^s} \Big) + O\Big(\frac{1}{\lambda_i^{\min(n,\beta)}} \Big) + o\Big(\sum_{j \neq i} \Big(\varepsilon_{ij} + \frac{H(a_i,a_j)}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} \Big) \Big),$$

$$(b) \quad \Big\langle \partial J(u), \alpha_i \lambda_i \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i} \Big\rangle = -2c_2 J(u) \sum_{j \neq i} \alpha_i \alpha_j \Big(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-2}{2} \frac{H(a_i,a_j)}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} \Big)$$

$$+ \quad 2\alpha_i^2 J(u) K(a_i)^{\frac{n-2}{2}} m(y_{\ell_i}, y_{\ell_i}) + O\Big(|a_i - y_{\ell_i}|^{\beta} \Big) + o\Big(\sum_{j \neq i} \Big(\varepsilon_{ij} + \frac{H(a_i,a_j)}{(\lambda_i \lambda_j)^{\frac{n-2}{2}}} \Big) + \frac{1}{\lambda_i^{n-2}} \Big).$$

Proof. We combine the proof of Proposition 3.1 of [16] and the proof of Proposition 3.1 of [17], Proposition 2.1. □

Proposition 2.2 *Let* $u = \sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i} \in V(p,\varepsilon)$.

For any $i = 1, \ldots, p$ *such that* $a_i \in \cup_{y \in \mathcal{K}} B(y, \rho_0)$, *we have*

$$(i)\Big\langle \partial J(u), \alpha_i \frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial (a_i)_k}\Big\rangle = -(n-2)\alpha_i^2 J(u)\frac{b_k}{\lambda_i K(a_i)}\beta \ sign(a_i-y_{\ell_i})_k|(a_i-y_{\ell_i})_k|^{\beta-1} + O\Big(\sum_{j=2}^{[\min(n,\beta)]}\frac{|a_i-y_{\ell_i}|^{\beta-j}}{\lambda_i^j}\Big)+O\Big(\frac{1}{\lambda_i^{\min(n,\beta)}}\Big)+O\Big(\sum_{j\neq i}|\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}|\Big), \quad (2.5)$$

$$(ii)\Big\langle \partial J(u), \alpha_i \frac{1}{\lambda_i}\frac{\partial P\delta_{a_i,\lambda_i}}{\partial (a_i)_k}\Big\rangle = -(n-2)\alpha_i^2 J(u)\frac{b_k}{K(a_i)\lambda_i^\beta}\int_{\mathbb{R}^n}|z_k+\lambda_i(a_i-y_{\ell_i})_k|^\beta \times \frac{z_k}{(1+|z|^2)^{n+1}}dz+o\Big(\frac{1}{\lambda_i^\beta}\Big)+O\Big(\sum_{j\neq i}|\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}|\Big), \quad (2.6)$$

provided that $\lambda_i|a_i-y_{\ell_i}|$ *is upper-bounded and* $\beta<n+1$.

Proof. The proof is a combination of the proofs of Proposition 3.2 of [16] and Proposition 3.2 of [17]. □

3. Construction of the pseudogradient

This section is devoted to the construction of a decreasing pseudogradient of J, which allows us to identify the critical points at infinity of the variational structure associated to (1.1). These points are the ends of the noncompact flow lines of $(-\partial J)$. According to [2], each noncompact flow line can be written as:

$$u(s)=\sum_{i=1}^p \alpha_i(s)P\delta_{a_i(s),\,\lambda_i(s)}+v(s).$$

Therefore, if we denote $\alpha_i=\lim\limits_{s\longrightarrow+\infty}\alpha_i(s)$ and $y_i=\lim\limits_{s\longrightarrow+\infty}a_i(s)$, then $\sum_{i=1}^p\alpha_iP\delta_{y_i,\infty}$ or $(y_1,...,y_p)_\infty$ denotes a critical point at infinity.

The following theorem provides a full description of the critical points at infinity of the functional J.

Theorem 3.1 *Assume that* (1.1) *has no solution. Under the assumptions* $(A),(B)$, *and* $(f)_\beta, \beta\in[n-2,\infty)$, *the only critical points at infinity of* J *are*

$$(y_{\ell_1},\dots,y_{\ell_p})_\infty:=\sum_{i=1}^p\frac{1}{K(y_{\ell_i})^{\frac{n-2}{2}}}P\delta_{(y_{\ell_i},\infty)},$$

where $(y_{\ell_1},\dots,y_{\ell_p})\in\mathcal{C}^\infty$. *Moreover, the index of a such critical point at infinity* $(y_{\ell_1},\dots,y_{\ell_p})_\infty$ *equals* $i(y_{\ell_1},\dots,y_{\ell_p})_\infty=p-1+\sum_{i=1}^p n-\widetilde{i}(y_{\ell_i})$.

As in [16] for the case where the β-flatness order $\beta(y)\in(1,n-2]$, the proof of Theorem 3.1 follows from the construction of a suitable pseudogradient in $V(p,\varepsilon),p\geq 1$, such that the PS condition is satisfied along its flow lines as long as these flow lines are away from each $(y_{\ell_1},\dots,y_{\ell_p})_\infty$ such that $(y_{\ell_1},\dots,y_{\ell_p})\in\mathcal{C}^\infty$. Since in our statement the β-flatness order lies in $[n-2,\infty)$, our construction will be completely different with respect to the one of [16].

Proposition 3.2 *For any $p\geq 1$ and $\varepsilon>0$ small enough, there exists a pseudogradient W in $V(p,\varepsilon)$ satisfying the following:*

There exists a constant $c>0$ such that for any $u=\sum_{i=1}^p\alpha_iP\delta_{a_i,\lambda_i}\in V(p,\varepsilon)$, we have

$$(i)\quad <\partial J(u),W(u)>\leq -c\Big(\sum_{i=1}^p\Big(\frac{1}{\lambda_i^{\min(n,\beta)}}+\frac{|\nabla K(a_i)|}{\lambda_i}\Big)+\sum_{j\neq i}\varepsilon_{ij}\Big),$$

$$(ii)\quad <\partial J(u+\bar v),W(u)+\frac{\partial\bar v}{\partial(\alpha,a,\lambda)}(W(u))>\leq -c\Big(\sum_{i=1}^p\Big(\frac{1}{\lambda_i^{\min(n,\beta)}}+\frac{|\nabla K(a_i)|}{\lambda_i}\Big)+\sum_{j\neq i}\varepsilon_{ij}\Big).$$

(iii) $|W|$ is bounded and the only case where $\lambda_i(s), i=1,\dots,p, s\geq 0$, tends to ∞ is when $a_i(s)$ tends to $y_{\ell_i},\forall i=1,\dots,p$ with $(y_{\ell_1},\dots,y_{\ell_p})\in\mathcal{C}^\infty$.

The proof of Proposition 3.2 is based on the following results, which describe the concentration phenomenon in specific regions of $V(p,\varepsilon)$. Let $\eta>0$ small enough. Define

$W_1(p,\varepsilon):=\Big\{u=\sum_{i=1}^p\alpha_iP\delta_{a_i,\lambda_i}\in V(p,\varepsilon),\ \text{s.t, } a_i\in B(y_{\ell_i},\rho_0),\lambda^{n-2}|a_j-y_{l_j}|^\beta<\eta,\ \forall i=1,...,p,\ y_{l_i}\neq y_{l_j},\ \forall i\neq j,\ \text{and } \rho(y_{\ell_1},\dots,y_{\ell_p})>0\Big\}$,

$W_2(p,\varepsilon):=\Big\{u=\sum_{i=1}^p\alpha_iP\delta_{a_i,\lambda_i}\in V(p,\varepsilon),\ \text{s.t, } a_i\in B(y_{\ell_i},\rho_0),\lambda^{n-2}|a_j-y_{l_j}|^\beta<\eta, m(y_{\ell_i},y_{\ell_i})>0,\ \forall i=1,...,p,\ y_{l_i}\neq y_{l_j},\ \forall i\neq j,\ \text{and } \rho(y_{\ell_1},\dots,y_{\ell_p})<0\Big\}$,

$W_3(p,\varepsilon):=\Big\{u=\sum_{i=1}^p\alpha_iP\delta_{a_i,\lambda_i}\in V(p,\varepsilon),\ \text{s.t, } a_i\in B(y_{\ell_i},\rho_0),\lambda^{n-2}|a_j-y_{l_j}|^\beta<\eta,\ \forall i=1,...,p,\ y_{l_i}\neq y_{l_j},\ \forall i\neq j,\ \text{and there exists } i_1,\ 1\leq i_1\leq p,\ \text{s.t,} m(y_{\ell_{i_1}},y_{\ell_{i_1}})<0\Big\}$,

$W_4(p,\varepsilon):=\Big\{u=\sum_{i=1}^p\alpha_iP\delta_{a_i,\lambda_i}\in V(p,\varepsilon),\ \text{s.t, } a_i\in B(y_{\ell_i},\rho_0),\ \forall i=1,...,p,\ y_{l_i}\neq y_{l_j},\ \forall i\neq j,\ \text{and there exists } i_1,\ 1\leq i_1\leq p,\ \text{s.t,}\lambda^{n-2}|a_j-y_{l_j}|^\beta\geq\eta,\Big\}$,

$W_5(p,\varepsilon)=V(p,\varepsilon)\setminus\cup_{j=1}^4W_j(p,\varepsilon)$.

Lemma 3.3 *In $W_1(p,\varepsilon)$, there exists a bounded pseudogradient W_1 satisfying (i) of Proposition 3.2. Moreover, for any $i=1,\dots,p$, $\lambda_i(s)$ tends to ∞ when $s\to+\infty$.*

Proof. Let $u=\sum_{i=1}^p\alpha_iP\delta_{a_i,\lambda_i}\in W_1(p,\varepsilon)$. Set

$$W_1(u)=\sum_{i=1}^{p}\alpha_i\lambda_i\frac{\partial P\delta_{(a_i,\lambda_i)}}{\partial\lambda_i}.$$

Using Proposition 2.1 and the following estimates,

$$|a_i-y_{\ell_i}|^\beta=o(\frac{1}{\lambda_i^{n-2}}),\qquad \text{as } \eta \text{ small},$$

$$-\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i}-\frac{n-2}{2}\frac{H(a_i,a_j)}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}=\frac{n-2}{2}\frac{G(y_{\ell_i},y_{\ell_j})}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}+o\Big(\frac{1}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}\Big),$$

we get

$$< \partial J(u),W_1(u) >= 2J(u)\sum_{i=1}^{p}\sum_{j\neq i}\alpha_i\alpha_j\frac{n-2}{2}c_2\frac{G(y_{\ell_i},y_{\ell_j})}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}$$

$$+2J(u)\sum_{i=1}^{p}\alpha_i^2\begin{cases}\dfrac{n-2}{n}c_1\dfrac{\sum_{k=1}^{n}b_k(y_{\ell_i})}{K(a_i)\lambda_i^{\beta(y_{\ell_i})}}-c_2\dfrac{n-2}{2}\dfrac{H(y_{\ell_i},y_{\ell_i})}{\lambda_i^{n-2}}, & \text{if } \beta(y_{\ell_i})=n-2\\ -c_2\dfrac{n-2}{2}\dfrac{H(y_{\ell_i},y_{\ell_i})}{\lambda_i^{n-2}}, & \text{if } \beta(y_{\ell_i})>n-2\end{cases}$$

$$+o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-2}}\Big).$$

Since $J(u)^{\frac{n}{n-2}}\alpha_i^{\frac{4}{n-2}}K(a_i)=1+o(1),\forall i=1,\dots,p$, we obtain

$$< \partial J(u),W_1(u) >= -2J(u)^{\frac{2-n}{2}}\Big[\sum_{i=1}^{p}\sum_{j\neq i}\frac{m(y_{\ell_i},y_{\ell_j})}{(\lambda_i\lambda_j)^{\frac{n-2}{2}}}+\sum_{i=1}^{p}\frac{m(y_{\ell_i},y_{\ell_i})}{\lambda_i^{n-2}}\Big]+o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-2}}\Big)$$

$$=-2J(u)^{\frac{2-n}{2}}\Big(\frac{1}{\lambda_1^{\frac{n-2}{2}}},\dots,\frac{1}{\lambda_p^{\frac{n-2}{2}}}\Big)M(y_{\ell_1},\dots,y_{\ell_p})\Big(\frac{1}{\lambda_1^{\frac{n-2}{2}}},\dots,\frac{1}{\lambda_p^{\frac{n-2}{2}}}\Big)^t$$

$$+o\Big(\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-2}}\Big).$$

Using the fact that $\rho(y_{\ell_1},\dots,y_{\ell_p})>0$, we get

$$< \partial J(u),W_1(u) >\leq -c\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-2}}\leq -c\Big(\sum_{i=1}^{p}\Big(\frac{1}{\lambda_i^{n-2}}+\frac{|\nabla K(a_i)|}{\lambda_i}\Big)+\sum_{j\neq i}\varepsilon_{ij}\Big).$$

Observe that under the action of W_1 each $\lambda_i(s),i=2,\dots,p$ increases with respect to the differential equation $\dot{\lambda}_i=\lambda_i$ and therefore tends to $+\infty$ when $s\to+\infty$. □

Lemma 3.4 *In $W_2(p,\varepsilon)$, there exists a bounded pseudogradient W_2 satisfying (i) of Proposition 3.2 and the PS condition as long as its flow lines remain in $W_2(p,\varepsilon)$.*

Proof. Let $u = \sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i} \in W_2(p,\varepsilon)$ and let $E = (e_1, \ldots, e_p) \in \mathbb{R}^p$ be an eigenvector of $\rho(y_{\ell_1}, \ldots, y_{\ell_p})$ such that $e_i > 0, \forall i = 1, \ldots, p$ and $\|E\|$. Let $\delta > 0$ small enough such that $XM(y_{\ell_1}, \ldots, y_{\ell_p})X^t < \frac{1}{2}\rho(y_{\ell_1}, \ldots, y_{\ell_p}), \forall X \in B(E,\delta) := \{Z \in \mathbb{R}^p, \|Z\| = 1 \text{ and } \|Z - E\| < \delta\}$. Define

$$V(E) = \{Y \in \mathbb{R}^p, Y \neq 0, \frac{Y}{\|Y\|} \in B(E,\delta)\}.$$

If $\Big(\frac{1}{\lambda_1^{\frac{n-2}{2}}}, \ldots, \frac{1}{\lambda_p^{\frac{n-2}{2}}}\Big) \in V(E)$, we set

$$W_2 = -\sum_{i=1}^{p} \alpha_i \lambda_i \frac{\partial P\delta_{(a_i,\lambda_i)}}{\partial \lambda_i}.$$

We find by Proposition 2.1

$$< \partial J(u), W_2(u) >= 2J(u)^{\frac{2-n}{4}} \Big(\frac{1}{\lambda_1^{\frac{n-2}{2}}}, \ldots, \frac{1}{\lambda_p^{\frac{n-2}{2}}}\Big) M(y_{\ell_1}, \ldots, y_{\ell_p}) \Big(\frac{1}{\lambda_1^{\frac{n-2}{2}}}, \ldots, \frac{1}{\lambda_p^{\frac{n-2}{2}}}\Big)^t$$

$$+ o\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}}\Big)$$

$$\leq -c \sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} \leq -c\Big(\sum_{i=1}^{p}\Big(\frac{1}{\lambda_i^{n-2}} + \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + \sum_{j\neq i} \varepsilon_{ij}\Big).$$

If $\Big(\frac{1}{\lambda_1^{\frac{n-2}{2}}}, \ldots, \frac{1}{\lambda_p^{\frac{n-2}{2}}}\Big) \notin V(E)$, we denote by $c(t)$ the segment in $\mathbb{R}^p$ connecting E with $\Lambda := \Big(\frac{1}{\lambda_1^{\frac{n-2}{2}}}, \ldots, \frac{1}{\lambda_p^{\frac{n-2}{2}}}\Big)^t$. Setting

$$W_2(u) = -\sum_{i=1}^{p} |\Lambda| \alpha_i \lambda_i^2 \frac{\partial P\delta_{(a_i,\lambda_i)}}{\partial \lambda_i} \Big(\frac{\|\Lambda\| e_i - \Lambda_i}{\|c(0)\|} - \frac{c_i(0) < \|\Lambda\|\vec{e} - \Lambda, c(0) >}{\|c(0)\|^3}\Big),$$

then

$$< \partial J(u), W_2(u) >\leq \frac{1}{2}\| \Lambda\|^2 \frac{\partial}{\partial t}\Big(\frac{c(t)M(y_{\ell_1}, \ldots, y_{\ell_p})c(t)^t}{\|c(t)\|^2}\Big)_{/t=0} + o\Big(\sum_{j\neq i} \varepsilon_{ij}\Big).$$

Observe that

$$\frac{\partial}{\partial t}\Big(\frac{c(t)M(y_{\ell_1}, \ldots, y_{\ell_p})c(t)^t}{\|c(t)\|^2}\Big)_{/t=0} \leq -c \text{ in } V(E)^c.$$

Indeed,

$$\frac{c(t)M(y_{\ell_1}, \ldots, y_{\ell_p})c(t)^t}{\|c(t)\|^2} = \rho(y_{\ell_1}, \ldots, y_{\ell_p}) + \frac{(1-t)^2}{\|c(t)\|^2}\Big(\Lambda M(y_{\ell_1}, \ldots, y_{\ell_p})\Lambda^t - \rho(y_{\ell_1}, \ldots, y_{\ell_p})\|\Lambda\|^2\Big).$$

This yields

$$\frac{\partial}{\partial t}\Big(\frac{c(t)M(y_{\ell_1}, \ldots, y_{\ell_p})c(t)^t}{\|c(t)\|^2}\Big)_{/t=0} = -\frac{2(1-t)}{\|c(t)\|^4}\Big(\Lambda M(y_{\ell_1}, \ldots, y_{\ell_p})\Lambda^t$$

$$-\rho(y_{\ell_1},\ldots,y_{\ell_p})\|\Lambda\|^2\Big)\Big((1-t)|\Lambda| < E,\Lambda > +t|\Lambda|^2\Big).$$

We have $< E,\Lambda >\geq \inf_{1\leq i\leq p} e_i\|\Lambda\|^2$. From the fact that $c(t)\in V(E)^c$, there exists $\alpha>0$ such that

$$\alpha < (1-t) \text{ and } \Lambda M(y_{\ell_1},\ldots,y_{\ell_p})\Lambda^t - \rho(y_{\ell_1},\ldots,y_{\ell_p})\|\Lambda\|^2 \geq c\|\Lambda\|^2.$$

We obtain

$$\frac{\partial}{\partial t}\Big(\frac{c(t)M(y_{\ell_1},\ldots,y_{\ell_p})c(t)^t}{\|c(t)\|^2}\Big)_{/t=0} \leq \Big(-\frac{2\alpha}{\|c(t)\|^4}c\|\Lambda\|^2\alpha\inf_{1\leq i\leq p} e_i\|\Lambda\|^2\Big)_{/t=0} \leq -c.$$

It follows that

$$< \partial J(u), W_2(u) >\leq -c\sum_{i=1}^{p}\frac{1}{\lambda_i^{n-2}} \leq -c\Big(\sum_{i=1}^{p}\Big(\frac{1}{\lambda_i^{n-2}}+\frac{|\nabla K(a_i)|}{\lambda_i}\Big)+\sum_{j\neq i}\varepsilon_{ij}\Big).$$

Observe that under the action of W_2 either λ_i satisfies $\dot{\lambda_i}=-\lambda_i, \forall i=1,\ldots,p$ or Λ moves along a compact path. Hence, the flow remains in a compact set and therefore the PS condition is satisfied along the flow line. □

Lemma 3.5 *In $W_3(p,\varepsilon)$, there exists a bounded pseudogradient W_3 satisfying (i) of Proposition 3.2 and the PS condition as long as its flow lines remain in $W_3(p,\varepsilon)$.*

Proof. Let $u=\sum_{i=1}^{p}\alpha_i P\delta_{a_i,\lambda_i}\in W_3(p,\varepsilon)$ and let $i_1,\ldots,i_q$ be the indices such that $m(y_{i_j},y_{i_j})<0$. Define

$$L=\{j, 1\leq j\leq p,\ \lambda_j\leq\frac{1}{2}\min_{1\leq j\leq q}\lambda i_j\}.$$

Let M_L be the matrix defined by $(y_{\ell_i})_{\ell_i\in L}$ and let ρ_L be the least eigenvalue of M_L. Setting

$$W_3^1=-\sum_{j=1}^{q}\alpha_{i_j}\lambda i_j\frac{\partial P\delta_{(a_{i_j},\lambda_{i_j})}}{\partial\lambda_{i_j}},$$

we have

$$< \partial J(u), W_3^1(u) >= -2J(u)^{\frac{2-n}{4}}\sum_{j=1}^{q}\Big(\sum_{k\neq i_j}\varepsilon_{i_j k}-\frac{m(y_{\ell_{i_j}},y_{\ell_{i_j}})}{\lambda_{i_j}^{n-2}}\Big)$$

$$\leq -c\sum_{j\notin L}\Big(\sum_{i\neq j}\varepsilon_{ij}+\frac{1}{\lambda_j^{n-2}}\Big).$$

To add the indices j such that $j\in L$, we set

$$W_3^2=\Big((1+\text{sign }\rho_L)W_1(\sum_{i\in L}\alpha_i P\delta_{(a_i,\lambda_i)})+(1-\text{sign }\rho_L)W_2(\sum_{i\in L}\alpha_i P\delta_{(a_i,\lambda_i)})\Big),$$

where $sign(\rho_L) = 1$ if $\rho_L > 0$ and $sign(\rho_L) = -1$ if $\rho_L < 0$. Using the above two lemmas, we have

$$< \partial J(u), W_3^2(u) > \leq -c\Big(\sum_{i\in L}\Big(\frac{1}{\lambda_i^{n-2}} + \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + \sum_{j\neq i, i, j\in L} \varepsilon_{ij}\Big) + O\Big(\sum_{i\in L, j\notin L} \varepsilon_{ij}\Big).$$

Define $W_3 = W_3^1 + mW_3^2$, where $m > 0$ small enough. From the above estimates we have

$$< \partial J(u), W_3(u) > \leq -c\Big(\sum_{i=1}^{p} \frac{1}{\lambda_i^{n-2}} + \sum_{j\neq i} \varepsilon_{ij}\Big) \leq -c\Big(\sum_{i=1}^{p}\Big(\frac{1}{\lambda_i^{n-2}} + \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + \sum_{j\neq i} \varepsilon_{ij}\Big).$$

$\square$

Lemma 3.6 *In $W_4(p, \varepsilon)$, there exists a bounded pseudogradient W_4 satisfying (i) of Proposition 3.2 and the PS condition as long as its flow lines remain in $W_4(p, \varepsilon)$.*

Proof. Let $u = \sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i} \in W_4(p, \varepsilon)$. Suppose that

$$L = \{j,\ 1 \leq j \leq p,\ \eta \leq \lambda_j^{n-2}|a_j - y_{\ell_j}|^{\beta}\}.$$

We claim the following:

(Claim) $\forall j_1 \in L, \exists Y_{j_1}$ a vector field such that

$$< \partial J(u), Y_{j_1}(u) > \leq -c\Big(\frac{1}{\lambda_{j_1}^{\min(\beta,n)}} + \frac{|\nabla K(a_{j_1})|}{\lambda_{j_1}} + \sum_{j\neq j_1} \varepsilon_{jj_1}\Big) + o\Big(\sum_{k\neq r} \varepsilon_{kr}\Big).$$

Indeed, for $i \in L$, we distinguish 3 cases.

Case 1. If $\beta(y_{\ell_i}) = n - 2$ and $\frac{1}{\eta} \geq \lambda_i|a_i - y_{\ell_i}|$. We set

$$X_i(u) = \alpha_i \sum_{k=1}^{n} b_k \int_{\mathbb{R}^n} \frac{|x_k + \lambda_i(a_i - y_{\ell_i})_k|^{\beta} x_k}{(1 + |x|^2)^{n+1}} dx \frac{1}{\lambda_i} \frac{\partial P\delta_{(a_i,\lambda_i)}}{\partial (a_i)_k}.$$

Using the second expansion of Proposition 2.2, we have

$$\Big\langle \partial J(u), X_i(u)\Big\rangle \leq -\frac{c}{\lambda_i^{n-2}}\Big(\int_{\mathbb{R}^n} \frac{|x_{k_a} + \lambda_i(a_i - y_{\ell_i})_{k_a}|^{\beta} x_{k_a}}{(1 + |x|^2)^{n+1}} dx\Big)^2 + o\Big(\frac{1}{\lambda_i^{n-2}}\Big) + o\Big(\sum_{j\neq i} \varepsilon_{ij}\Big).$$

Observe that

$$\Big(\int_{\mathbb{R}^n} \frac{|x_{k_a} + \lambda_i(a_i - y_{\ell_i})_{k_a}|^{\beta} x_{k_a}}{(1 + |x|^2)^{n+1}} dx\Big)^2 \geq c,$$

since $\lambda_i|a_i - y_{\ell_i}| \geq \eta$. Therefore,

$$\Big\langle \partial J(u), X_i(u) \Big\rangle \leq -\frac{c}{\lambda_i^{n-2}} + o\Big(\sum_{j\neq i} \varepsilon_{ij}\Big) \leq -c\Big(\frac{1}{\lambda_i^{n-2}} + \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + o\Big(\sum_{j\neq i} \varepsilon_{ij}\Big).$$

Now we consider $Z_i(u) = -\alpha_i \lambda_i \dfrac{\partial P\delta_{(a_i,\lambda_i)}}{\partial \lambda_i}$. Proposition 2.1 yields

$$\Big\langle \partial J(u), Z_i(u) \Big\rangle \leq -c\sum_{j\neq i} \varepsilon_{ij} + O\Big(\frac{1}{\lambda_i^{n-2}}\Big),$$

since $\lambda_i \dfrac{\partial \varepsilon_{ij}}{\partial \lambda_i} \sim -c\varepsilon_{ij}$ in our statement. Therefore,

$$\Big\langle \partial J(u), X_i(u) + \gamma Z_i(u) \Big\rangle \leq -c\Big(\frac{1}{\lambda_i^{n-2}} + \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{j\neq i} \varepsilon_{ij}\Big) + o\Big(\sum_{k\neq r} \varepsilon_{kr}\Big),$$

for $\gamma > 0$ and small.

Case 2. If $\beta(y_{\ell_i}) = n-2$ and $\frac{1}{\eta} \leq \lambda_i |a_i - y_{\ell_i}|$, define

$$\hat{X}_i(u) = \alpha_i \sum_{k=1}^{n} b_k \text{ sign } (a_i - y_{\ell_i})_k \frac{1}{\lambda_i} \frac{\partial P\delta_{(a_i,\lambda_i)}}{\partial (a_i)_k}.$$

We obtain from the first expansion of Proposition 2.2

$$\Big\langle \partial J(u), \hat{X}_i(u) \Big\rangle \leq -c\sum_{k=1}^{n} b_k^2 \frac{|(a_i - y_{\ell_i})_k|^{\beta-1}}{\lambda_i} + O\Big(\sum_{s=2}^{n-2} \frac{|a_i - y_{\ell_i}|^{\beta-s}}{\lambda_i^s}\Big) + O\Big(\frac{1}{\lambda_i^{\beta}}\Big) + o\Big(\sum_{k\neq r} \varepsilon_{kr}\Big).$$

Using the fact that

$$\frac{|a_i - y_{\ell_i}|^{\beta-s}}{\lambda_i^s} = o\Big(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i}\Big),\ \forall s \geq 2,\ \text{ and } \frac{1}{\lambda_i^{\beta}} = o\Big(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i}\Big),$$

we obtain

$$\Big\langle \partial J(u), \hat{X}_i(u) \Big\rangle \leq -c\frac{|(a_i - y_{\ell_i})_k|^{\beta-1}}{\lambda_i} + o\Big(\sum_{k\neq r} \varepsilon_{kr}\Big).$$

We now apply the pseudogradient $Z_i(u)$. We have

$$\begin{aligned}\Big\langle \partial J(u), Z_i(u) \Big\rangle &\leq -c\sum_{j\neq i} \varepsilon_{ij} + O\Big(\sum_{s=2}^{n-2} \frac{|a_i - y_{\ell_i}|^{\beta-s}}{\lambda_i^s}\Big) + O\Big(\frac{1}{\lambda_i^{\beta}}\Big) + o\Big(\sum_{k\neq r} \varepsilon_{kr}\Big)\\ &\leq -c\sum_{j\neq i} \varepsilon_{ij} + o\Big(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i}\Big) + o\Big(\sum_{k\neq r} \varepsilon_{kr}\Big).\end{aligned}$$

Thus,

$$\Big\langle \partial J(u), \hat{X}_i(u) + Z_i(u) \Big\rangle \leq -c\Big(\sum_{j\neq i} \varepsilon_{ij} + \frac{1}{\lambda_i^{n-2}} + \frac{|\nabla K(a_i)|}{\lambda_i} \Big) + o\Big(\sum_{k\neq r} \varepsilon_{kr} \Big).$$

Case 3. If $\beta(y_{\ell_i}) > n-2$, we use the vector fields $\hat{X}_i$ and Z_i defined in the two above cases. We have

$$\Big\langle \partial J(u), \hat{X}_i(u) \Big\rangle \leq -c\sum_{k=1}^{n} b_k^2 \frac{|(a_i - y_{\ell_i})_k|^{\beta-1}}{\lambda_i} + O\Big(\sum_{s=2}^{[\min(n,\beta)]} \frac{|a_i - y_{\ell_i}|^{\beta-s}}{\lambda_i^s} \Big)$$

$$+O\Big(\frac{1}{\lambda_i^{\min(n,\beta)}} \Big) + o\Big(\sum_{k\neq r} \varepsilon_{kr} \Big).$$

Since

$$\frac{|a_i - y_{\ell_i}|^{\beta-s}}{\lambda_i^s} = o\Big(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i} \Big), \ \forall s \geq 2,$$

$$\frac{1}{\lambda_i^{\beta}} = o\Big(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i} \Big), \text{ and } \frac{1}{\lambda_i^{n}} = o\Big(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i} \Big), \text{as } \lambda \to +\infty,$$

we get

$$Big\langle \partial J(u), \hat{X}_i(u) \Big\rangle \leq -c\frac{|(a_i - y_{\ell_i})_k|^{\beta-1}}{\lambda_i} + o\Big(\sum_{k\neq r} \varepsilon_{kr} \Big).$$

Moreover,

$$\Big\langle \partial J(u), Z_i(u) \Big\rangle \leq -c\sum_{j\neq i} \varepsilon_{ij} + o\Big(\frac{|a_i - y_{\ell_i}|^{\beta-1}}{\lambda_i} \Big) + o\Big(\sum_{k\neq r} \varepsilon_{kr} \Big).$$

Using $Y_i(u) = \hat{X}_i(u) + Z_i(u)$, we find that

$$\Big\langle \partial J(u), Y_i(u) \Big\rangle \leq -c\Big(\sum_{j\neq i} \varepsilon_{ij} + \frac{1}{\lambda_i^{\min(n,\beta)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \Big) + o\Big(\sum_{k\neq r} \varepsilon_{kr} \Big),$$

and hence our claim follows.

Now let

$$\lambda_{i_0}^{\beta} = \min_{i\in L} \lambda_i^{\beta},$$

and define

$$\widetilde{L} = \{j, 1 \leq j \leq p, \ \lambda_j^{\beta} \geq \frac{1}{2}\lambda_{i_0}^{\beta}\}.$$

We have:

$$< \partial J(u), \sum_{i\in L} Y_i(u) > \leq -c\Big(\sum_{i\in \widetilde{L}} \Big(\frac{1}{\lambda_i^{\min(\beta,n)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \Big) + \sum_{i\in L, j\neq i} \varepsilon_{ij} \Big) + o\Big(\sum_{k\neq r} \varepsilon_{kr} \Big),$$

$$< \partial J(u), \sum_{i\in L} Y_i(u) + \gamma \sum_{i\in \widetilde{L}\setminus L} Z_i(u) > \leq -c\Big(\sum_{i\in \widetilde{L}}\Big(\frac{1}{\lambda_i^{\min(\beta,n)}} + \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + \sum_{i\in \widetilde{L}, j\neq i} \varepsilon_{ij}\Big) + o\Big(\sum_{k\neq r}\varepsilon_{kr}\Big),$$

for $\gamma > 0$ small enough. To add the left indices, let $\bar{u} = \sum_{i\notin \widetilde{L}} \alpha_i P\delta_{(a_i,\lambda_i)}$. Observe that $\bar{u} \in V_i(\sharp\widetilde{L}^c, \varepsilon), i = 1,2,3$. Let $\widetilde{W}(u) = W_i(\bar{u})$ where $W_i(\bar{u})$ is the corresponding pseudogradient. We have

$$< \partial J(u), \widetilde{W}(u) > \leq -c\Big(\sum_{i\notin \widetilde{L}}\Big(\frac{1}{\lambda_i^{\min(\beta,n)}} + \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + \sum_{i,j\notin \widetilde{L}, j\neq i} \varepsilon_{ij}\Big) + O\Big(\sum_{i\notin \widetilde{L}, j\in \widetilde{L}}\varepsilon_{ij}\Big).$$

Let $\widetilde{\gamma}$ be a positive constant small enough and let

$$W_4(u) = \widetilde{\gamma}\widetilde{W}(u) + \sum_{i\in L} Y_i(u) + \gamma \sum_{i\in \widetilde{L}\setminus L} Z_i(u).$$

We have

$$< \partial J(u), W_4(u) > \leq -c\Big(\sum_{i=1}^{p}\Big(\frac{1}{\lambda_i^{\min(\beta,n)}} + \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + \sum_{j\neq i}\varepsilon_{ij}\Big).$$

□

Lemma 3.7 *In $W_5(p,\varepsilon)$, there exists a bounded pseudogradient W_5 satisfying (i) of Proposition 3.2 and the PS condition as long as its flow lines remain in $W_5(p,\varepsilon)$.*

Proof. We decompose $W_5(p,\varepsilon)$ in two regions:

$$R_1 = \{u = \sum_{i=1}^{p} \alpha_i P\delta_{(a_i,\lambda_i)} \in V(p,\varepsilon), a_i \in B(y_{\ell_i}, \rho_0), y_{\ell_i} \in \mathcal{K}, \forall i = 1,\dots,p$$

$$\text{and there exists } j \neq i \text{ such that } y_{\ell_i} = y_{\ell_j}\},$$

$$R_2 = \{u = \sum_{i=1}^{p} \alpha_i P\delta_{(a_i,\lambda_i)} \in V(p,\varepsilon), \exists i, 1 \leq i \leq p, a_i \notin \cup_{y,\nabla K(y)=0} B(y,\rho_0)\}.$$

For the construction of the required vector field in R_2, we refer to [8]. We now state the construction in R_1. For any $i, i = 1,\dots,p$, let

$$B_i = \{j, 1 \leq j \leq p,\ a_j \in B(y_{\ell_i}, \rho_0)\}.$$

Suppose that $i_1,\dots,i_\ell$ are all the indices such that $\sharp B_{i_k} > 1, \forall k = 1,\dots,\ell$. Let $\delta > 0$ small enough and ϕ be a smooth positive cut off function: $\phi(t) = 0$ if $|t| < \eta$ and $\phi(t) = 1$ if $|t| \geq \eta$.

For $j \in B_{i_k}$, we set

$$\bar{\phi}(\lambda_j) = \sum_{i\neq j\in B_{i_k}} \phi\Big(\frac{\lambda_j}{\lambda_i}\Big).$$

Set

$$W_5^1(u) = -\sum_{k=1}^{\ell}\sum_{j\in B_{i_k}} \alpha_j \bar{\phi}(\lambda_j)\lambda_j \frac{\partial P\delta_{(a_i,\lambda_i)}}{\partial \lambda_i}.$$

Then

$$\Big\langle \partial J(u), W_5^1(u)\Big\rangle \leq c\sum_{k=1}^{\ell}\Big(\sum_{j\in B_{i_k}} \bar{\phi}(\lambda_j)\lambda_j\frac{\partial \varepsilon_{ij}}{\partial \lambda_j} + \sum_{j\in B_{i_k}} O\Big(\sum_{s=1}^{[\min(\beta,n)]} \frac{|a_j - y_{\ell_j}|^{\beta-s}}{\lambda_j^s}\Big)\Big).$$

For $j\in B_{i_k}$ such that $\bar{\phi}(\lambda_j)\neq 0$, there exists $i_0 \neq j \in B_{i_k}$ such that

$$\frac{1}{\lambda_j^{\beta}} = o(\varepsilon_{ji_0}) \text{ and } \frac{1}{\lambda_j^{n}} = o(\varepsilon_{ji_0}).$$

Observe that if $i\in B_{i_k}^c$, then $|a_i - a_j| \geq \rho_0$. Therefore,

$$\lambda_j\frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \leq -c\varepsilon_{ij} \text{ and } \lambda_i\frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \leq -c\varepsilon_{ij}.$$

Thus,

$$\Big\langle \partial J(u), W_5^1(u)\Big\rangle \leq -c\sum_{k=1}^{\ell}\Big(\sum_{j\in B_{i_k}} \bar{\phi}(\lambda_j)\Big(\sum_{i\neq j}\varepsilon_{ij} + \frac{1}{\lambda_j^{\min(\beta,n)}}\Big) + \sum_{j\in B_{i_k}} O\Big(\sum_{s=1}^{[\min(\beta,n)]} \frac{|a_j - y_{\ell_j}|^{\beta-s}}{\lambda_j^s}\Big)\Big).$$

Denote by j_0 the index such that

$$\lambda_{j_0}^{\min(n,\beta)} = \min\{\lambda_i^{\min(n,\beta)},\ 1\leq i\leq p\}.$$

We have two cases:

(i) If there exists $j\in B_{i_k}, k=1,\dots,\ell$, with $\bar{\phi}(\lambda_j)\neq 0$ such that $\dfrac{\lambda_{j_0}^{\min(n,\beta)}}{\lambda_j^{\min(n,\beta)}} \geq \gamma$. Here γ is a fixed positive constant small enough. In this case, we get:

$$\Big\langle \partial J(u), W_5^1(u)\Big\rangle \leq -c\sum_{i=1}^{p}\Big(\frac{1}{\lambda_i^{\min(\beta,n)}} + \sum_{i\neq j}\varepsilon_{ij}\Big) + \sum_{k=1}^{\ell}\sum_{j\in B_{i_k}} O\Big(\sum_{s=1}^{[\min(\beta,n)]} \frac{|a_j - y_{\ell_j}|^{\beta-s}}{\lambda_j^s}\Big).$$

Therefore,

$$\Big\langle \partial J(u), W_5^1(u) + \gamma_1\sum_{i=1}^{p}\hat{X}_i(u)\Big\rangle \leq -c\Big(\sum_{i=1}^{p}\Big(\frac{1}{\lambda_i^{\min(\beta,n)}} + \frac{|\nabla K(a_i)|}{\lambda_i}\Big) + \sum_{i\neq j}\varepsilon_{ij}\Big),$$

for $\gamma_1 > 0$ small enough.

(ii) If $\forall j \in B_{i_k}, k = 1, \dots, \ell$, we have $\dfrac{\lambda_{j_0}^{\min(n,\beta)}}{\lambda_j^{\min(n,\beta)}} < \gamma$, or if there exists $j \in B_{i_k}, k = 1, \dots, \ell$, with $\dfrac{\lambda_{j_0}^{\min(n,\beta)}}{\lambda_j^{\min(n,\beta)}} \geq \gamma$, we have $\bar{\phi}(\lambda_j) = 0$. Define

$$E = \{k, \ \frac{\lambda_k^{\min(n,\beta)}}{\lambda_{j_0}^{\min(n,\beta)}} < \frac{1}{\gamma}\} \cup \{k, \ \bar{\phi}(\lambda_k) = 0\} \cup \Big(\cup_{k=1}^{\ell} B_{i_k} \Big)^c.$$

For all $k \neq j \in E$, we have $a_k \in B(y_{\ell_k}, \rho_0)$ and $a_j \in B(y_{\ell_j}, \rho_0)$ with $y_{\ell_j} \neq y_{\ell_k}$. Let $\bar{u} = \sum_{i \in E} \alpha_i P\delta_{a_i, \lambda_i}$. $\bar{u}$ lies in $W_i(\sharp E, \varepsilon), i = 1, 2, 3, 4$. We denote by W_i the related pseudogradient. We get

$$\Big\langle \partial J(u), W_i(u) \Big\rangle \leq -c \Big(\sum_{i \in E} \Big(\frac{1}{\lambda_i^{\min(\beta,n)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \Big) + \sum_{i \neq j, i, j \in E} \varepsilon_{ij} \Big) + O \Big(\sum_{i \in E, j \notin E} \varepsilon_{ij} \Big).$$

Therefore,

$$\Big\langle \partial J(u), W_5^1(u) + \gamma_1 (W_i(u) + \sum_{i=1}^{p} \hat{X}_i(u) \Big\rangle \leq -c \Big(\sum_{i=1}^{p} \Big(\frac{1}{\lambda_i^{\min(\beta,n)}} + \frac{|\nabla K(a_i)|}{\lambda_i} \Big) + \sum_{i \neq j} \varepsilon_{ij} \Big).$$

□

Proof of of Proposition 3.2 The required pseudogradient W is a convex combination of $W_i, i = 1, \dots, 5$. Clearly it satisfies (i) and (iii) of Proposition 3.2. Concerning (ii), it follows from the estimates (2.4). □

Proof of of Theorem 3.1 The characterization of the critical points at infinity follows from Proposition 3.2. Concerning the related index, we expand J near each critical point at infinity $(y_{\ell_1}, \dots, y_{\ell_p})_\infty$. As in [8] and [17], for any $u = \sum_{i=1}^{p} \alpha_i \delta_{(a_i, \lambda_i)} \in V(p, \varepsilon)$ such that $a_i \in B(y_{\ell_i}, \rho), \forall i = 1, \dots p$, with $(y_{\ell_1}, \dots, y_{\ell_p}) \in \mathcal{C}^\infty$, there exists a change of variables

$$(a_1, \dots, a_p, \lambda_1, \dots, \lambda_p) \to (a'_1, \dots, a'_p, \lambda'_1, \dots, \lambda'_p)$$

such that

$$J(u) = \frac{S_n \sum_{i=1}^{p} \alpha_i^2}{\Big(S_n \sum_{i=1}^{p} \alpha_i^{\frac{2n}{n-2}} K(a'_i) \Big)^{\frac{n-2}{n}}} \Big(1 + \sum_{i=1}^{p} \frac{1}{\lambda_i'^{\beta}} \Big),$$

where S_n is the best constant of Sobolev.

It follows that the index of J at $(y_{\ell_1}, \dots, y_{\ell_p})_\infty$ corresponds to the index of

$$\psi(\alpha, a') = \frac{S_n \sum_{i=1}^{p} \alpha_i^2}{\Big(S_n \sum_{i=1}^{p} \alpha_i^{\frac{2n}{n-2}} K(a'_i) \Big)^{\frac{n-2}{n}}}$$

at $(\bar{\alpha}_1, \dots, \bar{\alpha}_p, y_{\ell_1}, \dots, y_{\ell_p})$ where $\bar{\alpha}_i = \dfrac{1}{K(y_{\ell_i})^{\frac{n-2}{4}}}, i = 1, \dots, p$. Hence, Theorem 3.1 follows. □

4. Proof of Theorem 1.2

We argue by contradiction: suppose that (1.1) has no solution. By Theorem 3.1, the critical points at infinity of J are $(y_{\ell_1},\dots,y_{\ell_p})_\infty$ such that $(y_{\ell_1},\dots,y_{\ell_p})\in\mathcal{C}^\infty$. Using the deformation lemma of [5], we have

$$\Sigma^+ \cong \bigcup_{(y_{\ell_1},\dots,y_{\ell_p})\in\mathcal{C}^\infty} W_u^\infty((y_{\ell_1},\dots,y_{\ell_p})_\infty).$$

Here $\cong$ denotes retracts by deformation and $W_u^\infty((y_{\ell_1},\dots,y_{\ell_p})_\infty)$ denotes the unstable manifold at infinity of $(y_{\ell_1},\dots,y_{\ell_p})_\infty$. We now compute the Euler–Poincaré characteristic of each manifold and we obtain

$$1 = \sum_{(y_{\ell_1},\dots,y_{\ell_p})\in\mathcal{C}^\infty} (-1)^{i((y_{\ell_1},\dots,y_{\ell_p}))},$$

since Σ^+ is a contractible space. Such an equality contradicts the assumption of Theorem 1.2.

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