

Some remarks on generalized Kirk's process in Banach spaces and application

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Abstract: In this work, we establish a common fixed point result for mappings satisfying a controllable punctual inequality and we study the convergence (resp. weak convergence) of the generalized Kirk's process associated with them. In addition, our results are applied to investigate the convergence (resp. weak convergence) of Kuhfittig's iterative process to the solution of a nonlinear system of functional equations.

Key words: Uniformly convex Banach space, strictly convex Banach space, convex subset, common fixed point, generalized Kirk's process, convergence, weak convergence, demiclosed mapping, Kuhfittig's iterative process

1. Introduction

In applied sciences, many problems can be modeled by equations of the form

$$u - Tu = f \quad (1)$$

in a convenient Banach space X . If T is a bounded linear mapping such that $I - T$ is invertible or satisfies the Fredholm alternative, then equation (1) has a solution. In the case where T is assumed to be only linear, other results which ensure the existence of the solution are given by several authors (see, for example, [4]). Now, in the nonlinear setting, the situation is quite different. More precisely, if $f = 0$ and T is a contraction mapping, then equation (1) has a solution z which is the unique fixed point of T and the Picard sequence $\{x_n\}_n$ given by $x_n = T^n(x_0)$ converges to z for any $x_0 \in X$. However, this fact does not apply if T is nonexpansive. To see this, it suffices to take $T : [0, 1] \rightarrow [0, 1]$ given by $T(x) = 1 - x$, it is easy to observe that T is an isometry; thus, it is a nonexpansive mapping having $\frac{1}{2}$ as a unique fixed point. Let $x_0 = 0$, the Picard sequence $\{x_n\}_n$ is given by $x_{2k} = 0$ and $x_{2k+1} = 1$ for all integer $k \in \{0\} \cup \mathbb{N}$ which is not convergent to the point $\frac{1}{2}$. Furthermore, it is easy to observe that u is a solution to equation (1) if and only if that u is a fixed point for the mapping T_f given by

$$T_f u = Tu + f. \quad (2)$$

Thus, studying the existence of the solution of (1) is equivalent to investigating fixed points of the mapping T_f . Here we are interested in the study of common fixed points for mappings satisfying controllable punctual inequality. Our contribution is motivated by the fact that several phenomena can be given as a combination

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of finite problems. Our results extend, in particular, those of [10] and they are applied to give necessary and sufficient conditions to solve a nonlinear system of functional equations and to study the convergence of Kuhfittig's (see [12]) iterative process to its solutions.

2. Notations and preliminaries

Before going to the results, we present some definitions.

Let X be a Banach space. The modulus of convexity of X is the function $\delta_X : [0, 2] \rightarrow [0, 1]$ defined by

$$\delta_X(\epsilon) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\}.$$

Definition 2.1 A Banach space X is said to be uniformly convex if $\delta_X(\epsilon) > 0$ for each $\epsilon \in (0, 2]$.

Definition 2.2 A Banach space X is said to be strictly convex if for $x, y, z \in X$

$$\|x - z\| = \|y - z\| = \frac{1}{2}\|x - y\| \implies z = \frac{x + y}{2}.$$

Recall that the Lebesgue spaces $L_p([0, 1]) (1 < p < \infty)$ are a natural prototype of uniformly convex Banach spaces. Additionally, every uniformly convex Banach space is strictly convex, but the converse is not true in general. For further information on these notions, we quote for example [5, 8].

Definition 2.3 Let X be a normed space and C a subset of X . The selfmapping $T : C \rightarrow C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

It is well known (see [3, 6, 10, 13]) that if C is an arbitrary bounded closed convex subset of a uniformly convex Banach space X , then T has at least one fixed point. As a particular case, if X is a Hilbert space, this result holds. However, if $X = L^1([0, 1])$, we can find a bounded closed convex subset $C_0 \subset L^1([0, 1])$ and a nonexpansive mapping $\tilde{T} : C_0 \rightarrow C_0$ such that \tilde{T} is a fixed point free mapping (see [1]).

Definition 2.4 Let C be a convex subset of a Banach space X and let $T : C \rightarrow C$ be a selfmapping. For $x_0 \in C$, define a sequence $\{x_n\}_n \subset C$ by

$$x_{n+1} = \lambda x_n + (1 - \lambda)T(x_n) \quad \lambda \in (0, 1),$$

$\{x_n\}_n$ is called Krasnoselskii's process associated with T .

Definition 2.5 Let C be a convex subset of a Banach space X and let T_1, T_2, \dots, T_k be selfmappings on C . For $x_0 \in C$, define $\{x_n\}_n \subset C$ by

$$x_{n+1} = \lambda_0 x_n + \lambda_1 T_1(x_n) + \dots + \lambda_k T_k(x_n),$$

where $\lambda_i \geq 0$ for $i = 0, \dots, k$ with $\lambda_1 > 0$ and $\sum_{i=0}^k \lambda_i = 1$. $\{x_n\}_n$ is called generalized Kirk's process associated with the mappings T_1, \dots, T_k .

Remark 2.6 If in Definition 2.5, we put $\lambda_2 = \dots = \lambda_k = 0$, then the generalized Kirk's process is reduced to Krasnoselskii's process associated with the mapping T_1 .

Remark 2.7 For a given selfmapping T on C , we denote by T^i the composition $T \circ T \dots \circ T$ (i times). If in Definition 2.5, we take $T_i = T^i$ for all integer $i \geq 1$, then the process is reduced to the classical Kirk's process associated with the mapping T .

Through this paper, $F(T)$ will denote the set of fixed points of the mapping T .

3. Common fixed points formulas

We begin with the following Lemma.

Lemma 3.1 Let C be a convex subset of a Banach space X and let T_1, \dots, T_k be selfmappings on C . For

$(\lambda_i)_{i=0}^k \subset [0, 1]$ with $\sum_{i=0}^k \lambda_i = 1$, we denote by

$$S = \sum_{i=0}^k \lambda_i T_i \text{ (with the notation } T_0 = Id_C \text{),}$$

then

$$\bigcap_{i=1}^k F(T_i) = F(S) \cap \left(\bigcap_{i=1}^k F(T_i S) \right).$$

Proof Let $x_0 \in \bigcap_{i=1}^k F(T_i)$, then $x_0 \in F(T_i)$ for all integer $i = 1, \dots, k$, which proves that $T_i(x_0) = x_0$ for all $i =$

$1, \dots, k$ and hence $S(x_0) = \sum_{i=0}^k \lambda_i T_i(x_0) = x_0$; thus, $x_0 \in F(S)$ and consequently $x_0 \in F(S) \cap \left(\bigcap_{i=1}^k F(T_i S) \right)$.

Conversely, let $x_0 \in F(S) \cap \left(\bigcap_{i=1}^k F(T_i S) \right)$; thus, $S(x_0) = x_0$ and $(T_i S)(x_0) = x_0$ for all integer $i = 1, \dots, k$, by composition the equality $S(x_0) = x_0$ at the left by T_i ($i = 1, \dots, k$), we get

$$(T_i S)(x_0) = T_i x_0 = x_0,$$

Hence, $x_0 \in F(T_i), \forall i = 1, \dots, k$ which gives that $x_0 \in \bigcap_{i=1}^k F(T_i)$ and completes the proof. □

Theorem 3.2 Let C be a convex subset of a Banach space X and let T_1, T_2, \dots, T_k ($k \geq 2$) be self-mappings on C satisfying that for all $x \in C$ and for all integers $i, j = 1, \dots, k, (i < j)$ there exists an integer $1 \leq n(x) < j$ such that

$$\|T_i(x) - T_j(x)\| \leq \|x - T_{n(x)}(x)\|. \tag{3}$$

Let $(\lambda_i)_{i=0}^k \subset [0, 1]$ with $\lambda_1 > 0$ and $\sum_{i=0}^k \lambda_i = 1$. Set $S = \sum_{i=0}^k \lambda_i T_i$ (with the notation $T_0 = I_C$). Then

$$\bigcap_{i=1}^k F(T_i) = F(S).$$

Proof It is easy to show that $\bigcap_{i=1}^k F(T_i) \subseteq F(S)$. To prove the converse, let $x_0 \in F(S)$; thus,

$$S(x_0) = \left(\sum_{i=0}^k \lambda_i T_i \right) (x_0) = x_0,$$

it follows that

$$x_0 = \left(\sum_{i=1}^k \left(\frac{\lambda_i}{1 - \lambda_0} \right) T_i \right) (x_0) \quad (\lambda_0 \neq 1 \text{ since } \lambda_1 > 0).$$

Let $\delta = \sup\{\|T_i(x_0) - T_j(x_0)\|, i, j = 0, \dots, k\}$. Suppose that $\delta > 0$, the assumption (3) implies the existence of a smallest integer $p(x_0) \in \{1, \dots, k\}$ such that

$$\delta = \|x_0 - T_{p(x_0)}(x_0)\|.$$

Since $\sum_{i=1}^k \frac{\lambda_i}{1 - \lambda_0} = 1$, it follows that

$$x_0 = \gamma_0 T_1(x_0) + (1 - \gamma_0)z,$$

where $z \in \text{conv}\{T_2(x_0), \dots, T_k(x_0)\}$. Thus

$$\begin{aligned} \delta &= \|x_0 - T_{p(x_0)}(x_0)\| = \|\gamma_0 T_1(x_0) + (1 - \gamma_0)z - T_{p(x_0)}(x_0)\| \\ &\leq \gamma_0 \|T_1(x_0) - T_{p(x_0)}(x_0)\| + (1 - \gamma_0) \|z - T_{p(x_0)}(x_0)\| \\ &\leq \gamma_0 \delta + (1 - \gamma_0) \delta = \delta. \end{aligned}$$

(i) If $p(x_0) = 1$, this is a contradiction, since in this case, we obtain that $\|T_1(x_0) - T_1(x_0)\| = 0 = \delta$.

(ii) If $p(x_0) > 1$, by the assumption (3), there exists an integer $m(x_0) < p(x_0)$ such that

$$\delta \leq \|T_1(x_0) - T_{p(x_0)}(x_0)\| \leq \|x_0 - T_{m(x_0)}(x_0)\|$$

which gives that $\|x_0 - T_{m(x_0)}(x_0)\| = \delta$ and contradicts the fact that $p(x_0)$ is the smallest integer such that $\delta = \|x - T_{p(x_0)}(x_0)\|$. Necessarily, we get $\delta = 0$. Hence, $\|x_0 - T_i(x_0)\| = 0$ for all integer $i = 1, \dots, k$,

consequently $x_0 \in \bigcap_{i=1}^k F(T_i)$ which completes the proof.

□

Example 3.3 Let C be a convex subset of a Banach space X and let T be a self-mapping on C . If T satisfies one of the following assumptions, then for all integer $n \geq 2$, the self-mappings T, T^2, \dots, T^n satisfy the controllable punctual inequality (3).

1. T nonexpansive;

$$2. \|Tx - Ty\| \leq \frac{\|x - Tx\|\|x - Ty\| + \|y - Ty\|\|y - Tx\|}{\|x - Ty\| + \|y - Tx\|}$$

for all $x, y \in C, x \neq y$ with $\|x - Ty\| + \|y - Tx\| \neq 0$.

Indeed, for the setting of nonexpansive mappings, it suffices to take $n(x) = j - i$ ($i < j$) for all $x \in C$ since we have $\|T^i x - T^j x\| \leq \|x - T^{j-i} x\|$ while if T satisfies 2, then $n(x)$ can be taken equal to 1 for all $x \in C$ since in this case, we have $\|T^i x - T^j x\| \leq \|x - Tx\|$ (for more details, see [14]).

Corollary 3.4 Let C be a convex subset of a Banach space X and let T be a nonexpansive self-mapping on C . Set

$$S = \sum_{i=0}^k \lambda_i T^i \text{ (with the notation } T^0 = I_C),$$

where $(\lambda_i)_{i=0}^k \subset [0, 1]$ together with $\lambda_1 > 0$ and $\sum_{i=0}^k \lambda_i = 1$. Then $F(S) = F(T)$.

Proof The result follows from Theorem 3.2 by taking $T_i = T^i$ for all integer i . In this case, we have

$$\bigcap_{i=1}^k F(T^i) = F(T) \text{ since } F(T) \subset F(T^i) \text{ for all integer } i \geq 1 \text{ which completes the proof.}$$

4. Convergence Results of Generalized Kirk’s Processes

Definition 4.1 Let C be a nonempty subset of a Banach space X and let T be a self-mapping on C . T is said to be asymptotically regular if, for all $x \in C$, we have

$$\lim_{n \rightarrow \infty} \|T^{n+1}(x) - T^n(x)\| = 0.$$

It is easy to show that if T is a contraction, then T is asymptotically regular. However, if T is nonexpansive, then the sequence $\delta_n = \|T^{n+1}(x) - T^n(x)\|$ is decreasing but does not converge necessarily to 0. To see this, it suffices to take $X = \mathbb{R}$ equipped with its usual norm and $T : \mathbb{R} \rightarrow \mathbb{R}$ defined by $T(x) = 1 - x$.

Now, we state the following theorem which will be used in the sequel. □

Theorem 4.2 (see Theorem 4 in [11]) Let C be a closed convex subset of a uniformly convex Banach space X and let T_1, T_2, \dots, T_k be nonexpansive selfmappings on C . Denote by

$$S = \sum_{i=0}^k \lambda_i T_i \text{ (with the notation } T_0 = Id_C),$$

where $(\lambda_i)_{i=0}^k \subset [0, 1]$ and $\lambda_1 > 0$ with $\sum_{i=0}^k \lambda_i = 1$. If $\bigcap_{i=1}^k F(T_i) \neq \emptyset$ then S is asymptotically regular.

The rest of this section is devoted to the study of the convergence results concerning generalized Kirk's and nonstationary generalized Kirk's processes to common fixed points of the given mappings.

Definition 4.3 Let X be a Banach space and let T be a (nonlinear) self-mapping on X . T is said to be compact if T maps bounded subsets of X into precompacts subsets of X .

Our first convergence result in this section goes as follows.

Theorem 4.4 Let X be a uniformly convex Banach space and let $T_1, T_2, \dots, T_k (k \geq 2)$ be nonexpansive compact self-mappings on X satisfying the assumption (3). Denote by S the mapping

$$S = \sum_{i=0}^k \lambda_i T_i \text{ (with the notation } T_0 = Id_X),$$

where $(\lambda_i)_{i=0}^k \subset [0, 1], \lambda_0 > 0$ and $\sum_{i=0}^k \lambda_i = 1$. If $\bigcap_{i=1}^k F(T_i) \neq \emptyset$, then for each $x_0 \in X$ the Picard sequence $\{S^n(x_0)\}$ converges to a common fixed point of the mappings T_1, T_2, \dots, T_k .

Proof By Theorem 4.2, it follows that S is asymptotically regular and $F(S) = \bigcap_{i=1}^k F(T_i) \neq \emptyset$. Now, we will prove that the mapping $I - S$ maps bounded closed subsets of X into closed subsets of X . To do it, let C be an arbitrary bounded closed subset of X and assume that $\lim_{n \rightarrow +\infty} (y_n - S y_n) = y, y_n \in C$. We show that $y \in (I - S)(C)$. Since each $T_i, 1 \leq i \leq k$ is compact, we obtain the existence of a subsequence $(y_{n^i(l)})_l$ such that $T_i(y_{n^i(l)})_l$ converges to $z_i \in X, 1 \leq i \leq k$ which proves the existence of a subsequence $(y_{f(l)})_l$ of $(y_l)_l$ such that $T_i(y_{f(l)})$ converges to $z_i \in X$ for each $1 \leq i \leq k$. Thus,

$$\begin{aligned} (I - S)(y_{f(l)}) &= y_{f(l)} - \sum_{i=0}^k \lambda_i T_i(y_{f(l)}) \\ &= (1 - \lambda_0)y_{f(l)} - \sum_{i=1}^k \lambda_i T_i(y_{f(l)}). \end{aligned}$$

Afterwards, since $y_{f(l)} - S(y_{f(l)})$ converges to $y (l \rightarrow +\infty)$, we get

$$\lim_{l \rightarrow +\infty} (1 - \lambda_0)y_{f(l)} = y + \sum_{i=1}^k \lambda_i z_i,$$

which implies that $\lim_{l \rightarrow +\infty} y_{f(l)} = \frac{y}{1 - \lambda_0} + \sum_{i=1}^k (\frac{\lambda_i}{1 - \lambda_0})z_i \in C$ (since C is closed). Hence, $\lim_{l \rightarrow +\infty} y_{f(l)} = \tilde{y} \in C$;

thus,

$$\tilde{y} - S\tilde{y} = y.$$

This proves that $y \in (I - S)(C)$. Now the result follows from Theorem 6 in [4]. □

Definition 4.5 A self-mapping T on X is said to be demiclosed if for each sequence $\{x_n\} \subset X$ with $x_n \rightharpoonup x$ weakly in X and $Tx_n \rightarrow y$ in norm in X , then $Tx = y$.

Theorem 4.6 Let X be a uniformly convex Banach space, C a closed bounded convex subset of X , and let T_1, T_2, \dots, T_k ($k \geq 2$) be a nonexpansive mappings satisfying the assumption (3). Set

$$S = \sum_{i=0}^k \lambda_i T_i \text{ (with the notation } T_0 = Id_C),$$

where $(\lambda_i)_{i=0}^k \subset [0, 1], \lambda_1 > 0$ and $\sum_{i=0}^k \lambda_i = 1$. Assume that $\bigcap_{i=1}^k F(T_i) = \{z_0\}$. Then for each $x_0 \in C$, the Picard sequence $\{S^n(x_0)\}$ converges weakly to z_0 in C .

Proof Since S is nonexpansive, the mapping $I - S$ is demiclosed (see [2]). Next, let $x_0 \in C$ and let $\{x_n\}_n$ the Picard sequence $x_n = S^n x_0 (n \in \mathbb{N})$. Since X is uniformly convex, then X is reflexive (see [5, 8]), this fact implies the existence of a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ such that x_{n_k} converges weakly to y_0 . On the other hand, Theorem 4.2 implies that S is asymptotically regular; thus,

$$\lim_{k \rightarrow +\infty} (I - S)(x_{n_k}) = \lim_{k \rightarrow +\infty} (S^{n_k}(x_0) - S^{n_k+1}(x_0)) = 0.$$

By definition of the demiclosedness, it follows that

$$(I - S)(y_0) = 0,$$

which proves that y_0 is a fixed point of S . However, $F(S) = \bigcap_{i=1}^k F(T_i)$ (see Theorem 3.2). Hence $y_0 = z_0$ and consequently y_0 is the unique fixed point of S . Therefore, every weakly convergent subsequence of $\{x_n\}$ converges weakly to z_0 . By a standard argument using the reflexivity of X and the fact that the sequence $\{x_n\}_n$ is bounded, we infer that $\{x_n\}_n$ converges weakly to z_0 which is the desired result. \square

Remark 4.7 It is worth to note that Theorems 4.2 and 4.4 are extensions of Corollary and Theorem 3 in [10] by taking $T_i = T^i$ for all integer $i \geq 1$.

Finally, we study the convergence of a nonstationary generalized Kirk’s process involving T_1, T_2, \dots, T_k to a common fixed point for these mappings.

First of all, we recall the following preparatory result.

Lemma 4.8 (see Lemma 1 in [7]) If $\{x_n\}_n$ and $\{y_n\}_n$ are sequences in a uniformly convex space with $\|y_n\| \leq \|x_n\|$ and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n \quad (0 \leq \alpha_n \leq 1),$$

where $\sum_{n=1}^{\infty} \min(\alpha_n, 1 - \alpha_n) = \infty$, then $0 \in \overline{\{x_n - y_n, n \in \mathbb{N}\}}$ (where \overline{C} denotes the closure in norm of the set C).

Let $(\alpha_{ij})_{i=0}^{\infty}$ ($j = 0, 1, \dots, k$) be a set of positive reals such that $0 \leq \alpha_{ij}, 0 < \alpha \leq \alpha_{i1}$ with $\sum_{j=0}^k \alpha_{ij} = 1$ for each

i and $\sum_{i=0}^{\infty} \min(\alpha_{i0}, 1 - \alpha_{i0}) = \infty$.

Define the mappings S_i by

$$S_i = \alpha_{i0}I + \alpha_{i1}T_1 + \dots + \alpha_{ik}T_k \quad (i = 0, 1, 2, \dots)$$

A nonstationary generalized Kirk's process is given by the formula

$$x_{n+1} = S_n x_n \quad (n = 0, 1, 2, \dots) \tag{4}$$

It is easy to observe that if T_1, T_2, \dots, T_k are nonexpansive mappings and if $z_0 \in \bigcap_{i=1}^k F(T_i)$, then

$$\|x_{n+1} - z_0\| = \left\| \sum_{j=0}^k \alpha_{nj} (T_j x_n - T_j z_0) \right\| \leq \|x_n - z_0\|. \tag{5}$$

Proposition 4.9 Let C be a convex subset of uniformly convex Banach space and let T_1, T_2, \dots, T_k be nonexpansive selfmappings on C with $\bigcap_{i=1}^k F(T_i) \neq \emptyset$ and let $\{x_n\}_n$ defined by equation (4). Then $0 \in \overline{\{x_{n+1} - x_n, n \in \mathbb{N}\}}$.

Proof Let $x_0 \in \bigcap_{i=1}^k F(T_i)$. Define $y_n = x_n - x_0$ and

$$z_n = \frac{1}{1 - \alpha_{n0}} \sum_{j=1}^k \alpha_{nj} (T_j x_n - T_j x_0).$$

Hence,

$$\begin{aligned} y_{n+1} &= x_{n+1} - x_0 = S_n x_n - x_0 = \alpha_{n0} x_n + \dots + \alpha_{nk} T_k x_n - \left(\sum_{j=0}^k \alpha_{nj} \right) x_0 \\ &= \alpha_{n0} (x_n - x_0) + \sum_{j=1}^k \alpha_{nj} (T_j x_n - T_j x_0) \\ &= \alpha_{n0} y_n + (1 - \alpha_{n0}) z_n. \end{aligned}$$

By (5), we obtain that $\|z_n\| \leq \|x_n - x_0\| = \|y_n\|$. Now, Lemma 4.8 implies that $0 \in \overline{\{y_n - z_n, n \in \mathbb{N}\}}$. Moreover, we have

$$\begin{aligned} \|y_n - z_n\| &= \|x_n - x_0 - \frac{1}{1 - \alpha_{n0}} \sum_{j=1}^k \alpha_{nj} T_j x_n + x_0\| \\ &= \|x_n - \frac{1}{1 - \alpha_{n0}} \sum_{j=0}^k \alpha_{nj} T_j x_n + \frac{\alpha_{n0}}{1 - \alpha_{n0}} x_n\| \\ &= \frac{1}{1 - \alpha_{n0}} \|x_n - x_{n+1}\| \\ &\geq \|x_n - x_{n+1}\| \text{ since } \frac{1}{1 - \alpha_{n0}} \geq 1, \end{aligned}$$

this proves the existence of a subsequence $\{x_{n_k}\}$ such that $\lim_{k \rightarrow +\infty} \|x_{n_k} - x_{n_{k+1}}\| = 0$, which is the desired result. □

Theorem 4.10 In addition to the hypotheses of Proposition 4.9, assume that the mappings T_1, T_2, \dots, T_k ($k \geq 2$) satisfy the assumption (3) and each T_i ($1 \leq i \leq k$) is compact. Then for each $x_1 \in C$, the sequence $\{x_n\}_n$ defined by the formula (4) converges to a common fixed point for the mappings T_1, T_2, \dots, T_k .

Proof By Proposition 4.9, there exists a subsequence $\{x_{n_l}\}$ with $x_{n_{l+1}} - x_{n_l} \rightarrow 0$. Moreover, from the assumption on the set $\{\alpha_{ij}\}_{i=0}^\infty$ ($j = 0, 1, \dots, k$), we can extract subsequences $\alpha_{m_l j}$ of the sequence $\{\alpha_{m_l j}\}$ ($j = 1, \dots, k$) such that $\lim_{l \rightarrow +\infty} \alpha_{m_l j} = \alpha_j \in [0, 1]$ with $\alpha_1 > 0$. Let

$$S = \alpha_0 I + \alpha_1 T_1 + \dots + \alpha_k T_k.$$

Then

$$x_{m_l} - Sx_{m_l} = x_{m_l} - S_{m_l}x_{m_l} + S_{m_l}x_{m_l} - Sx_{m_l},$$

where

$$x_{m_l} - S_{m_l}x_{m_l} = x_{m_l} - x_{m_{l+1}} \rightarrow 0.$$

If $x_0 \in \bigcap_{i=1}^k F(T_i)$, since the sequence $\{\|x_n - x_0\|\}_n$ is decreasing and the mappings T_1, T_2, \dots, T_k are nonexpansive, we obtain that

$$\|T_j x_{m_l} - x_0\| = \|T_j x_{m_l} - T_j x_0\| \leq \|x_{m_l} - x_0\| \leq \|x_1 - x_0\|.$$

Similarly,

$$\|T_j x_{m_l}\| \leq \|x_1 - x_0\| + \|x_0\| = \gamma \text{ for all } j = 0, 1, \dots, k$$

Thus,

$$\begin{aligned} \|S_{m_l}x_{m_l} - Sx_{m_l}\| &= \left\| \sum_{j=0}^k (\alpha_{m_l j} - \alpha_j) T_j x_{m_l} \right\| \\ &\leq \gamma \sum_{j=0}^k |\alpha_{m_l j} - \alpha_j| \rightarrow 0 \quad (l \rightarrow +\infty). \end{aligned}$$

This leads to $x_{m_l} - Sx_{m_l} \rightarrow 0$ ($l \rightarrow +\infty$). The fact that each T_i ($i \leq 1 \leq k$) is compact together with the proof of Theorem 4.4 shows that $I - S$ maps closed bounded subsets into closed subsets. Consequently, from the decreasesness of the sequence $\{\|x_n - x_0\|\}_n$, we deduce that $\{\overline{x_n, n \in \mathbb{N}}\}$ is closed and bounded. Next, Proposition 4.9 gives that $0 \in (I - S)(\{\overline{x_n, n \in \mathbb{N}}\})$. This fact proves the existence of $y_0 \in \{\overline{x_n, n \in \mathbb{N}}\}$ such that $S(y_0) = y_0$; hence, y_0 is a fixed point of S . Now, by Theorem 3.2, we get $y_0 \in \bigcap_{i=1}^k F(T_i)$. Apply for a second time the decreasesness of the sequence $\{\|x_n - y_0\|\}_n$, it follows that $x_n \rightarrow y_0$ ($n \rightarrow +\infty$), which completes the proof. \square

5. Application

This section is motivated by the fact that many problems in applied sciences are given by a nonlinear system of functional equations and the study of its resolution is reduced to the investigation of common fixed points of specific mappings.

More precisely, let

$$\begin{cases} x - T_1x &= f_1 \\ \dots\dots &= \dots \\ \dots\dots &= \dots \\ \dots\dots &= \dots \\ x - T_kx &= f_k \end{cases} \quad (\star)$$

be the nonlinear system in a Banach space X where $f_i \in X$ for all $i = 1, \dots, k$ and T_1, \dots, T_k are self-mappings on X .

Denote by $B_i, i = 1, \dots, k$ the mapping given by $B_i x = T_i x + f_i$ with the notation $B_0 = Id_X$. For all

$$(\lambda_i)_{i=0}^k \subset [0, 1] \text{ with } \lambda_1 > 0 \text{ and } \sum_{i=0}^k \lambda_i = 1, \text{ set } \gamma_i = \frac{\lambda_i}{1 - \lambda_0} \quad (i = 1, \dots, k), \text{ then we have}$$

Lemma 5.1 Let $z_0 \in X$. Then z_0 is a solution of the system (\star) if and only if z_0 is at the same time the solution of the nonlinear equation

$$x = \sum_{i=1}^k \gamma_i B_i x \tag{6}$$

and the system

$$x = B_i \left(\sum_{j=0}^k \lambda_j B_j \right) x, \quad i = 1, \dots, k \tag{**}$$

Proof It is easy to prove that z_0 is a solution of the system (\star) if and only if z_0 is a common fixed point for the mappings $(B_i)_{i=1}^k$. Now, the result is an immediate consequence of Lemma 3.1. \square

Lemma 5.2 Assume that the mappings $(B_i)_{i=1}^k (k \geq 2)$ satisfy the assumption (3). Then z_0 is a solution of the system (\star) if and only if z_0 is the solution of the nonlinear equation (6).

Proof Follows by using Theorem 3.2 and the same reasoning as given in the proof of Lemma 5.1.

Let X be a Banach space and C a convex subset of X . Suppose that $\{T_i\}_{i=1}^k$ is a finite family of selfmappings on C . For $\alpha \in]0, 1[$, P. Kuhfittig [9, 12] defined the following iterative process

$$x_{n+1} = U_k(x_n), \quad n = 0, 1, \dots,$$

where

$$\begin{cases} U_0 & = I \\ U_1 & = (1 - \alpha)I + \alpha T_1 U_0 \\ \dots & = \dots \\ \dots & = \dots \\ U_k & = (1 - \alpha)I + \alpha T_k U_{k-1} \end{cases}$$

\square

Theorem 5.3 Let C be a convex compact subset of a strictly convex Banach space X and let $\{T_i\}_{i=1}^k (k \geq 2)$ be a family of nonexpansive selfmappings of C . If the nonlinear equation (6) has at least a solution and the mappings $\{B_i\}_{i=1}^k$ satisfy the assumption (3), then for each $x_0 \in C$, the sequence $\{U_k^n x_0\}$ converges in norm to a solution of the system (\star) .

Proof We have T_i is nonexpansive if and only if B_i is nonexpansive for all $i = 1, \dots, k$. Next, Lemma 5.2 implies that z_0 is a solution of the system (\star) if and only if z_0 is a solution of the equation (6). Now, the result follows from Theorem 1 in [12]. \square

From the previous proof and Theorem 2 in [12], we obtain the following.

Theorem 5.4 If X is a Hilbert space and C is a closed convex subset of X . Assume that $\{T_i\}_{i=1}^k (k \geq 2)$ are nonexpansive selfmappings of C . If the nonlinear equation (6) has at least a solution and the mappings $\{B_i\}_{i=1}^k$ satisfy the assumption (3), then for each $z_0 \in C$, the sequence $\{U_k^n z_0\}$ converges weakly to a solution of the system (\star) .

Conclusion

Our results are obtained for not necessarily commuting mappings and so extend and improve those of [10]. Necessary and sufficient conditions are established to solve a nonlinear system of functional equations; we prove in particular that if a given data of functions f_1, \dots, f_k is chosen to assert that the perturbed mappings satisfy

the controllable punctual inequality (3), then we can obtain a convergence result of Kuhfittig's iterative process to the solution.

Open problem

Let C be nonempty subset of a metric space (X, d) . A self-mapping $T : C \rightarrow C$ is said to be asymptotically nonexpansive if there is a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that $d(T^n x, T^n y) \leq k_n d(x, y)$ for all $x, y \in C$ (if $k_n = 1$ for all $n \geq 1$, T becomes nonexpansive).

Extend the results of this paper for asymptotically nonexpansive mappings on a nonlinear domain by following the technique of Khan [9].

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References

- [1] Alspach D. A fixed point free nonexpansive map. *Proceedings of the American Mathematical Society* 1981; 82: 423-424.
- [2] Browder FE. Semicontractive and semiaccretive nonlinear mapping in Banach spaces. *Bulletin of the American Mathematical Society* 1968; 74: 660-665.
- [3] Browder FE. Nonexpansive nonlinear operators in a Banach space. *Proceedings of the National Academy of Sciences of the United States of America* 1965; 54: 1041-1044.
- [4] Browder FE, Petryshyn WV. The solution by iteration of linear functional equations in Banach spaces. *Bulletin of the American Mathematical Society* 1966; 72: 566-570.
- [5] Goebel K, Kirk WA. *Topics in Metric Fixed Point Theory*. Cambridge Studies in Advanced Mathematics. 1st ed. Cambridge, UK: Cambridge University Press, 1990.
- [6] Göhde D. Zum Prinzip der kontraktiven Abbildungen. *Mathematische Nachrichten* 1966; 30: 251-258 (in German).
- [7] Groetsch CW. A nonstationary iterative process for nonexpansive mappings. *Proceedings of the American Mathematical Society* 1974; 1: 155-158.
- [8] Khamsi MA, Kirk WA. *An Introduction to Metric Spaces and Fixed Point Theory*. Pure and Applied Mathematics: A Wiley-Interscience Series of Texts. New York, NY, USA: John Wiley and Sons, Inc., 2001.
- [9] Khan AR. Common fixed point and solution of nonlinear functional equations. *Fixed Point Theory and Applications* 2013. doi: 10.1186/1687-2013-290.
- [10] Kirk WA. On successive approximations for nonexpansive mappings. *Glasgow Mathematical Journal* 1971; 2: 6-9.
- [11] Kitahara S, Takahashi W. Image recovery by convex combinations of sunny nonexpansive retractions. *Topological Methods in Nonlinear Analysis* 1993; 2: 333-342.
- [12] Kuhfittig PF. Common fixed points of nonexpansive mappings by iteration. *Pacific Journal of Mathematics* 1981; 97: 137-139.
- [13] Lau AT. Normal structure and common fixed point properties for semigroups of nonexpansive mappings in Banach spaces. *Fixed Point Theory and Applications* 2010. doi:10.1155/2010/580956.
- [14] Ray BK, Singh SP. Fixed point theorems in Banach spaces. *Indian Journal of Pure and Applied Mathematics* 1978; 9: 216-221.