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A new comprehensive subclass of analytic bi-close-to-convex functions

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Abstract: In a very recent work, Şeker and Sümer Eker [On subclasses of bi-close-to-convex functions related to the odd-starlike functions. Palestine Journal of Mathematics 2017; 6: 215-221] defined two subclasses of analytic bi-close-to-convex functions related to the odd-starlike functions in the open unit disk $\mathbb{U}$. The main purpose of this paper is to generalize and improve the results of Şeker and Sümer Eker (in the aforementioned study) defining a comprehensive subclass of bi-close-to-convex functions. Also, we investigate the Fekete-Szegö type coefficient bounds for functions belonging to this new class.

Key words: Analytic and univalent functions, bi-univalent functions, close-to-convex functions, starlike functions, subordination principle, Fekete-Szegö problem

1. Introduction

Let $\mathcal{A}$ denote the family of analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let $\mathcal{S} := \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{U}\}$.

The family of *starlike functions of order* β $(0 \leq \beta < 1)$ shall be denoted by $\mathcal{S}^*(\beta)$ and is defined by the conditions that $f \in \mathcal{A}$ and

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \qquad (z \in \mathbb{U}).$$

It is well known that

$$\mathcal{S}^*(\beta) \subset \mathcal{S}^*(0) = \mathcal{S}^* \subset \mathcal{S}.$$

A function $f \in \mathcal{A}$ is said to be *close-to-convex* if there exists a function $g \in \mathcal{S}^*$ such that the inequality

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0 \qquad (z \in \mathbb{U})$$

holds. We will denote the class which consists of all functions $f \in \mathcal{A}$ that are close-to-convex by $\mathcal{K}$.

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We have well-known inclusion relations:

$$\mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}.$$

Gao and Zhou [2] introduced the subclass $\mathcal{K}_s$ of close-to-convex analytic functions as follows:

Definition 1 *[2] Let the function f be analytic in $\mathbb{U}$ and normalized by the condition (1.1). We say that $f \in \mathcal{K}_s$ if there exists a function $g \in \mathcal{S}^*(1/2)$ such that*

$$\Re\left(\frac{z^2 f'(z)}{-g(z)g(-z)}\right) > 0 \qquad (z \in \mathbb{U}).$$

In recent years, the subclasses of close-to-convex functions are studied by several authors (see, for example, [3, 5, 11, 13**?** –15]). Motivated by this works, Goyal and Singh [4] defined the general subclass of close-to-convex functions by using the principle of subordination (see [8]) as follows:

Definition 2 *[4] For a function φ with positive real part, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}_s(\lambda,\mu,\varphi)$ if it satisfies the following subordination condition:*

$$\frac{z^2 f'(z) + (\lambda - \mu + 2\lambda\mu)\, z^3 f''(z) + \lambda\mu z^4 f'''(z)}{-g(z)g(-z)} \prec \varphi(z) \quad (z \in \mathbb{U}), \tag{1.2}$$

where $0 \leq \mu \leq \lambda \leq 1$ and $g \in \mathcal{S}^*(1/2)$.

Remark 1 *(i) For $\mu = 0$ and $\varphi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, we get the class $\mathcal{K}_s(\lambda, A, B)$ studied by Wang and Chen [13].*
(ii) For $\mu = \lambda = 0$ and $\varphi(z) = \frac{1+\beta z}{1-\alpha\beta z}$ $(0 \leq \alpha \leq 1,\ 0 < \beta \leq 1)$, we get the class $\mathcal{K}_s(\alpha,\beta)$ studied by Wang et al. [15].
(iii) For $\mu = \lambda = 0$ and $\varphi(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \leq \beta < 1)$, we get the class $\mathcal{K}_s(\beta)$ studied by Kowalczyk and Leś-Bomba [5].
(iv) For $\mu = \lambda = 0$ and $\varphi(z) = \frac{1+z}{1-z}$, we get the class $\mathcal{K}_s$ defined in the Definition 1.

Theorem 1.1 *[1] (Koebe One-Quarter Theorem) The range of every function of class $\mathcal{S}$ contains the disk of radius $\left\{w : |w| < \frac{1}{4}\right\}$.*

Thus, by Theorem 1.1, every function $f \in \mathcal{A}$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}) \qquad \text{and} \qquad f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right).$$

For the inverse function $F = f^{-1}$, we have:

$$F(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{1.3}$$

Definition 3 *[7] If both the function f and its inverse function f^{-1} are univalent in $\mathbb{U}$, then the function f is called bi-univalent. We will denote the class which consists of functions f that are bi-univalent by Σ.*

In a recent paper, Şeker and Sümer Eker [12] defined new subclasses of the bi-univalent function class Σ given in Definitions 4 and 5 as follows:

Definition 4 *(see [12]) A function $f \in \mathcal{A}$ given by (1.1) is said to be in the class $\mathcal{K}_{\Sigma}^{s}(\alpha)$ if there exists a function*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(1/2), \qquad G(w) = w + \sum_{n=2}^{\infty} c_n w^n \in \mathcal{S}^*(1/2)$$

and the following conditions are satisfied:

$$f \in \Sigma \qquad \text{and} \qquad \left| \arg\left(\frac{-z^2 f'(z)}{g(z)g(-z)} \right) \right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ z \in \mathbb{U})$$

and

$$\left| \arg\left(\frac{-w^2 F'(w)}{G(w)G(-w)} \right) \right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ w \in \mathbb{U}),$$

where the function $F = f^{-1}$ is defined by (1.3).

Theorem 1.2 *(see [12]) Let the function $f(z)$ given by (1.1) be in the class $\mathcal{K}_{\Sigma}^{s}(\alpha)$ $(0 < \alpha \leq 1)$, then*

$$|a_2| \leq \sqrt{\frac{\alpha\,(1 + 2\alpha)}{2 + \alpha}} \qquad \text{and} \qquad |a_3| \leq \frac{\alpha\,(3\alpha + 2) + 1}{3}.$$

Definition 5 *(see [12]) A function $f \in \mathcal{A}$ given by (1.1) is said to be in the class $\mathcal{K}_{\Sigma}^{s}(\beta)$ if there exists a function*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(1/2), \qquad G(w) = w + \sum_{n=2}^{\infty} c_n w^n \in \mathcal{S}^*(1/2)$$

and the following conditions are satisfied:

$$f \in \Sigma \qquad \text{and} \qquad \Re\left(\frac{-z^2 f'(z)}{g(z)g(-z)} \right) > \beta \qquad (0 \leq \beta < 1,\ z \in \mathbb{U})$$

and

$$\Re\left(\frac{-w^2 F'(w)}{G(w)G(-w)} \right) > \beta \qquad (0 \leq \beta < 1,\ w \in \mathbb{U}),$$

where the function $F = f^{-1}$ is defined by (1.3).

Theorem 1.3 *(see [12]) Let the function $f(z)$ given by (1.1) be in the class $\mathcal{K}_{\Sigma}^{s}(\beta)$ $(0 \leq \beta < 1)$, then*

$$|a_2| \leq \sqrt{\frac{3 - 2\beta}{3}} \qquad \text{and} \qquad |a_3| \leq \frac{(1-\beta)\,(5 - 3\beta) + 1}{3}.$$

Now we introduce the a new comprehensive subclass of $\mathcal{A}$ which includes Definitions 4 and 5.

Definition 6 *For $0 \leq \mu \leq \lambda \leq 1$ and a function φ with positive real part, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{K}_{\Sigma_s}(\lambda, \mu, \varphi)$ if there exist the functions*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(1/2), \qquad G(w) = w + \sum_{n=2}^{\infty} c_n w^n \in \mathcal{S}^*(1/2)$$

and the following conditions are satisfied:

$$\frac{z^2 f'(z) + (\lambda - \mu + 2\lambda\mu) z^3 f''(z) + \lambda\mu z^4 f'''(z)}{-g(z)g(-z)} \prec \varphi(z) \qquad (z \in \mathbb{U}) \tag{1.4}$$

and

$$\frac{w^2 F'(w) + (\lambda - \mu + 2\lambda\mu) w^3 F''(w) + \lambda\mu w^4 F'''(w)}{-G(w)G(-w)} \prec \varphi(w) \qquad (w \in \mathbb{U}), \tag{1.5}$$

where the function $F = f^{-1}$ is defined by (1.3).

Remark 2 *(i) For $\mu = 0$, we have a new class $\mathcal{K}_{\Sigma_s}(\lambda, \varphi)$ of bi-close-to-convex functions satisfying the conditions*

$$\frac{z^2 f'(z) + \lambda z^3 f''(z)}{-g(z)g(-z)} \prec \varphi(z) \qquad (z \in \mathbb{U})$$

and

$$\frac{w^2 F'(w) + \lambda w^3 F''(w)}{-G(w)G(-w)} \prec \varphi(w) \qquad (w \in \mathbb{U}).$$

(ii) For $\mu = \lambda = 0$, we have a new class $\mathcal{K}_{\Sigma_s}(\varphi)$ of bi-close-to-convex functions satisfying the conditions

$$\frac{z^2 f'(z)}{-g(z)g(-z)} \prec \varphi(z) \qquad (z \in \mathbb{U})$$

and

$$\frac{w^2 F'(w)}{-G(w)G(-w)} \prec \varphi(w) \qquad (w \in \mathbb{U}).$$

(iii) In addition to the conditions given in (ii), if we set

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \qquad (0 < \alpha \leq 1)$$

or

$$\varphi(z) = \frac{1 + (1 - 2\beta) z}{1 - z} \qquad (0 \leq \beta < 1),$$

then we have the classes $\mathcal{K}_{\Sigma}^{s}(\alpha)$ and $\mathcal{K}_{\Sigma}^{s}(\beta)$ defined in Definitions 4 and 5, respectively.

In the light of the work of Şeker and Sümer Eker [12], we obtain initial coefficient estimates for functions $f \in \mathcal{A}$ given by (1.1) belonging to the bi-close-to-convex function class $\mathcal{K}_{\Sigma_s}(\lambda, \mu, \varphi)$ introduced in Definition 6 above. We obtain the improvements of results of Şeker and Sümer Eker [12] given in Theorems 1.2 and 1.3 as a result of our main theorem (Theorem 2.1). Also, we find Fekete-Szegö type coefficient bounds for the bi-close-to-convex functions $f \in \mathcal{K}_{\Sigma_s}(\lambda, \mu, \varphi)$. The following lemmas will be required for proving our main results.

Lemma 1.4 *[2] If $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(1/2)$, then*

$$\psi(z) = \frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in \mathcal{S}^* \subset \mathcal{S}, \tag{1.6}$$

where the coefficients of the odd-starlike function ψ satisfy the condition

$$|B_{2n-1}| = \left| 2b_{2n-1} - 2b_2 b_{2n-2} + \cdots + 2(-1)^n b_{n-1} b_{n+1} + (-1)^{n+1} b_n^2 \right| \leq 1 \qquad (n \geq 2).$$

Lemma 1.5 *[9] Let the function h given by*

$$h(z) = 1 + \sum_{k=1}^{\infty} h_k z^k \qquad (z \in \mathbb{U})$$

be holomorphic in $\mathbb{U}$ and the function q given by

$$q(z) = 1 + \sum_{k=1}^{\infty} q_k z^k \qquad (z \in \mathbb{U})$$

be convex in $\mathbb{U}$. If

$$h(z) \prec q(z) \qquad (z \in \mathbb{U}),$$

then

$$|h_k| \leq |q_1| \qquad (k = 1, 2, \ldots).$$

Lemma 1.6 *[17] Let $k, l \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1| < R$ and $|z_2| < R$, then*

$$|(k+l) z_1 + (k-l) z_2| \leq \begin{cases} 2R|k| & , \quad |k| \geq |l| \\ 2R|l| & \quad |k| \leq |l| \end{cases}.$$

2. Initial coefficient estimates

Throughout this paper, we assume that $0 \leq \mu \leq \lambda \leq 1$ and φ be a function with positive real part.

Theorem 2.1 *If the function $f(z)$ given by (1.1) be in the function class $\mathcal{K}_{\Sigma_s}(\lambda, \mu, \varphi)$, then*

$$|a_2| \leq \min \left\{ \frac{|\varphi'(0)|}{2(1+\lambda-\mu+2\lambda\mu)}, \sqrt{\frac{1+|\varphi'(0)|}{3[1+2(\lambda-\mu)+6\lambda\mu]}} \right\} \tag{2.1}$$

and

$$|a_3| \leq \frac{1+|\varphi'(0)|}{3[1+2(\lambda-\mu)+6\lambda\mu]}. \tag{2.2}$$

Proof Firstly, we will re-arrange the relations in (1.4) and (1.5) as follows:

$$\begin{aligned} h(z) &= \frac{z^2 f'(z) + (\lambda - \mu + 2\lambda\mu)\, z^3 f''(z) + \lambda\mu z^4 f'''(z)}{-g(z)g(-z)} \\ &= \frac{z f'(z) + (\lambda - \mu + 2\lambda\mu)\, z^2 f''(z) + \lambda\mu z^3 f'''(z)}{\frac{-g(z)g(-z)}{z}} \\ &= \frac{z f'(z) + (\lambda - \mu + 2\lambda\mu)\, z^2 f''(z) + \lambda\mu z^3 f'''(z)}{\psi(z)} \qquad (2.3) \\ &\prec \varphi(z) \qquad (z \in \mathbb{U}) \qquad (2.4) \end{aligned}$$

and

$$\begin{aligned} p(w) &= \frac{w^2 F'(w) + (\lambda - \mu + 2\lambda\mu)\, w^3 F''(w) + \lambda\mu w^4 F'''(w)}{-G(w)G(-w)} \\ &= \frac{w F'(w) + (\lambda - \mu + 2\lambda\mu)\, w^2 F''(w) + \lambda\mu w^3 F'''(w)}{\frac{-G(w)G(-w)}{w}} \\ &= \frac{w F'(w) + (\lambda - \mu + 2\lambda\mu)\, w^2 F''(w) + \lambda\mu w^3 F'''(w)}{\Omega(w)} \qquad (2.5) \\ &\prec \varphi(w) \qquad (w \in \mathbb{U}), \qquad (2.6) \end{aligned}$$

respectively, where

$$\psi(z) := \frac{-g(z)g(-z)}{z} \qquad \text{and} \qquad \Omega(w) := \frac{-G(w)G(-w)}{w}.$$

Here h and p are two functions with positive real part defined by

$$h(z) := 1 + h_1 z + h_2 z^2 + \cdots$$

and

$$p(w) := 1 + p_1 w + p_2 w^2 + \cdots,$$

respectively. The relations (2.4) and (2.6) imply by Lemma 1.5 that for all $k = 1, 2, \ldots,$

$$|h_k| \leq |\varphi'(0)| \qquad (2.7)$$

and

$$|p_k| \leq |\varphi'(0)|. \qquad (2.8)$$

Furthermore, by Lemma 1.4, we have following equations:

$$\psi(z) = \frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in \mathcal{S}^* \quad \text{and} \quad |B_{2n-1}| \leq 1, \qquad (2.9)$$

$$\Omega(w) = \frac{-G(w)G(-w)}{w} = w + \sum_{n=2}^{\infty} C_{2n-1} w^{2n-1} \in \mathcal{S}^* \quad \text{and} \quad |C_{2n-1}| \leq 1. \qquad (2.10)$$

Now, upon equating the coefficients in (2.3) and (2.5), we obtain

$$2\left(1+\lambda-\mu+2\lambda\mu\right)a_2 = h_1 \tag{2.11}$$

$$3\left[1+2\left(\lambda-\mu\right)+6\lambda\mu\right]a_3 - B_3 = h_2 \tag{2.12}$$

$$-2\left(1+\lambda-\mu+2\lambda\mu\right)a_2 = p_1 \tag{2.13}$$

$$3\left[1+2\left(\lambda-\mu\right)+6\lambda\mu\right]\left(2a_2^2-a_3\right) - C_3 = p_2. \tag{2.14}$$

From (2.11) and (2.13), we get

$$h_1 = -p_1$$

and

$$8\left(1+\lambda-\mu+2\lambda\mu\right)^2 a_2^2 = h_1^2 + p_1^2. \tag{2.15}$$

We thus find (by (2.7) – (2.10)) that

$$|a_2| \leq \frac{|\varphi'(0)|}{2\left(1+\lambda-\mu+2\lambda\mu\right)}. \tag{2.16}$$

Furthermore, from the equalities (2.12) and (2.14), we find

$$6\left[1+2\left(\lambda-\mu\right)+6\lambda\mu\right]a_2^2 - B_3 - C_3 = h_2 + p_2. \tag{2.17}$$

Consequently (by (2.7) – (2.10)), we have

$$|a_2| \leq \sqrt{\frac{1+|\varphi'(0)|}{3\left[1+2\left(\lambda-\mu\right)+6\lambda\mu\right]}}. \tag{2.18}$$

Hence, we get the desired result on the coefficient a_2 as asserted in (2.1) from the inequalities (2.16) and (2.18).

Now, in order to obtain the bound on the coefficient a_3, we subtract (2.14) from (2.12). We thus get

$$3\left[1+2\left(\lambda-\mu\right)+6\lambda\mu\right]\left(2a_3-2a_2^2\right) - B_3 + C_3 = h_2 - p_2$$

or

$$a_3 = a_2^2 + \frac{h_2 - p_2 + B_3 - C_3}{6\left[1+2\left(\lambda-\mu\right)+6\lambda\mu\right]}. \tag{2.19}$$

Upon substituting the value of a_2^2 from (2.15) into (2.19), it follows that

$$a_3 = \frac{h_1^2 + p_1^2}{8\left(1+\lambda-\mu+2\lambda\mu\right)^2} + \frac{h_2 - p_2 + B_3 - C_3}{6\left[1+2\left(\lambda-\mu\right)+6\lambda\mu\right]}.$$

We thus find (by (2.7) – (2.10)) that

$$|a_3| \leq \frac{|\varphi'(0)|^2}{4\left(1+\lambda-\mu+2\lambda\mu\right)^2} + \frac{1+|\varphi'(0)|}{3\left[1+2\left(\lambda-\mu\right)+6\lambda\mu\right]}. \tag{2.20}$$

On the other hand, upon substituting the value of a_2^2 from (2.17) into (2.19), it follows that

$$a_3 = \frac{h_2 + p_2 + B_3 + C_3}{6\left[1 + 2(\lambda - \mu) + 6\lambda\mu\right]} + \frac{h_2 - p_2 + B_3 - C_3}{6\left[1 + 2(\lambda - \mu) + 6\lambda\mu\right]} = \frac{h_2 + B_3}{3\left[1 + 2(\lambda - \mu) + 6\lambda\mu\right]}.$$

Consequently (by (2.7), (2.8), (2.9) and (2.10)), we have

$$|a_3| \leq \frac{1 + |\varphi'(0)|}{3\left[1 + 2(\lambda - \mu) + 6\lambda\mu\right]}. \tag{2.21}$$

Combining (2.20) and (2.21), we get the desired result on the coefficient a_3 as asserted in (2.2). □

Letting $\mu = 0$ in Theorem 2.1, we have the following consequence.

Corollary 2.2 *If the function $f(z)$ given by (1.1) be in the function class $\mathcal{K}_{\Sigma_s}(\lambda, \varphi)$, then*

$$|a_2| \leq \min\left\{ \frac{|\varphi'(0)|}{2(1+\lambda)}, \sqrt{\frac{1 + |\varphi'(0)|}{3(1+2\lambda)}} \right\}$$

and

$$|a_3| \leq \frac{1 + |\varphi'(0)|}{3(1+2\lambda)}.$$

Letting $\lambda = 0$ in Corollary 2.2, we have the following consequence.

Corollary 2.3 *If the function $f(z)$ given by (1.1) be in the function class $\mathcal{K}_{\Sigma_s}(\varphi)$, then*

$$|a_2| \leq \min\left\{ \frac{|\varphi'(0)|}{2}, \sqrt{\frac{1 + |\varphi'(0)|}{3}} \right\}$$

and

$$|a_3| \leq \frac{1 + |\varphi'(0)|}{3}.$$

Setting

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (0 < \alpha \leq 1)$$

in Corollary 2.3, we have the following result.

Corollary 2.4 *If the function $f(z)$ given by (1.1) be in the function class $\mathcal{K}_{\Sigma}^{s}(\alpha)$, then*

$$|a_2| \leq \alpha \qquad \textit{and} \qquad |a_3| \leq \frac{1+2\alpha}{3}.$$

Remark 3 *Note that Corollary 2.4 is an improvement of the Theorem 1.2.*

Setting

$$\varphi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad (0 \leq \beta < 1)$$

in Corollary 2.3, we have the following result.

Corollary 2.5 *If the function* $f(z)$ *given by* (1.1) *be in the function class* $\mathcal{K}_{\Sigma}^{s}(\beta)$, *then*

$$|a_2| \leq 1-\beta \qquad \textit{and} \qquad |a_3| \leq \frac{3-2\beta}{3}.$$

Remark 4 *Note that Corollary 2.5 is an improvement of the Theorem 1.3.*

3. Fekete-Szegö problem

Theorem 3.1 *If the function* $f(z)$ *given by* (1.1) *be in the function class* $\mathcal{K}_{\Sigma_s}(\lambda, \mu, \varphi)$, *then, for* $\delta \in \mathbb{R}$,

$$\left|a_3 - \delta a_2^2\right| \leq \frac{1+|\varphi'(0)|}{3\left[1+2(\lambda-\mu)+6\lambda\mu\right]}\begin{cases} |1-\delta| & , \quad \delta \in (-\infty, 0] \cup [2, \infty) \\ 1 & , \quad \delta \in [0, 2] \end{cases}.$$

Proof By using the equality (2.19) in the proof of Theorem 2.1, we obtain

$$\begin{aligned} a_3 - \delta a_2^2 &= a_2^2 + \frac{h_2 - p_2 + B_3 - C_3}{6\left[1+2(\lambda-\mu)+6\lambda\mu\right]} - \delta a_2^2 \\ &= (1-\delta)\, a_2^2 + \frac{h_2 - p_2 + B_3 - C_3}{6\left[1+2(\lambda-\mu)+6\lambda\mu\right]}. \end{aligned}$$

Upon substituting the value of a_2^2 from (2.17) into the above equality, it follows that

$$\begin{aligned} a_3 - \delta a_2^2 &= (1-\delta)\frac{h_2 + p_2 + B_3 + C_3}{6\left[1+2(\lambda-\mu)+6\lambda\mu\right]} + \frac{h_2 - p_2 + B_3 - C_3}{6\left[1+2(\lambda-\mu)+6\lambda\mu\right]} \\ &= \frac{1}{6\left[1+2(\lambda-\mu)+6\lambda\mu\right]}\left[(2-\delta)(h_2+B_3) - \delta(p_2+C_3)\right]. \end{aligned}$$

Thus, by Lemma 1.6, we get desired estimate. □

Letting $\mu = 0$ in Theorem 3.1, we have the following consequence.

Corollary 3.2 *If the function* $f(z)$ *given by* (1.1) *be in the function class* $\mathcal{K}_{\Sigma_s}(\lambda, \varphi)$, *then, for* $\delta \in \mathbb{R}$,

$$\left|a_3 - \delta a_2^2\right| \leq \frac{1+|\varphi'(0)|}{3(1+2\lambda)}\begin{cases} |1-\delta| & , \quad \delta \in (-\infty, 0] \cup [2, \infty) \\ 1 & , \quad \delta \in [0, 2] \end{cases}.$$

Letting $\lambda = 0$ in Corollary 3.2, we have the following consequence.

Corollary 3.3 *If the function* $f(z)$ *given by* (1.1) *be in the function class* $\mathcal{K}_{\Sigma_s}(\varphi)$, *then, for* $\delta \in \mathbb{R}$,

$$\left|a_3 - \delta a_2^2\right| \leq \frac{1+|\varphi'(0)|}{3}\begin{cases} |1-\delta| & , \quad \delta \in (-\infty, 0] \cup [2, \infty) \\ 1 & , \quad \delta \in [0, 2] \end{cases}.$$

Setting

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (0 < \alpha \leq 1)$$

in Corollary 3.3, we have the following result.

Corollary 3.4 *If the function* $f(z)$ *given by* (1.1) *be in the function class* $\mathcal{K}^s_\Sigma(\alpha)$*, then, for* $\delta \in \mathbb{R}$*,*

$$\left|a_3 - \delta a_2^2\right| \leq \frac{1+2\alpha}{3} \begin{cases} |1-\delta| & , \quad \delta \in (-\infty, 0] \cup [2, \infty) \\ 1 & , \quad \delta \in [0,2] \end{cases}.$$

Setting

$$\varphi(z) = \frac{1 + (1-2\beta)\, z}{1-z} \quad (0 \leq \beta < 1)$$

in Corollary 3.3, we have the following result.

Corollary 3.5 *If the function* $f(z)$ *given by* (1.1) *be in the function class* $\mathcal{K}^s_\Sigma(\beta)$*, then, for* $\delta \in \mathbb{R}$*,*

$$\left|a_3 - \delta a_2^2\right| \leq \frac{3-2\beta}{3} \begin{cases} |1-\delta| & , \quad \delta \in (-\infty, 0] \cup [2, \infty) \\ 1 & , \quad \delta \in [0,2] \end{cases}.$$

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