Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 1425 – 1446
© TÜBİTAK
doi:10.3906/mat-1805-99

On the Cohen–Macaulayness of tangent cones of monomial curves in $\mathbb{A}^4(K)$

Feza ARSLAN[1,*], **Anargyros KATSABEKIS**[2], **Melissa NALBANDIYAN**[3]
[1]Department of Mathematics, Faculty of Science, Mimar Sinan Fine Arts University, İstanbul, Turkey
[2]Department of Mathematics, Faculty of Science, Bilkent University, Ankara, Turkey
[3]Department of Mathematics, Faculty of Science, Mimar Sinan Fine Arts University, İstanbul, Turkey

Received: 20.05.2018 • **Accepted/Published Online:** 26.03.2019 • **Final Version:** 29.05.2019

Abstract: In this paper we give necessary and sufficient conditions for the Cohen–Macaulayness of the tangent cone of a monomial curve in 4-dimensional affine space. We particularly study the case where C is a Gorenstein noncomplete intersection monomial curve and we generalize some results in the literature. Moreover, by using these results, we construct families supporting Rossi's conjecture, which is still open for monomial curves in 4-dimensional affine space.

Key words: Cohen–Macaulay, tangent cone, monomial curve

1. Introduction

Cohen–Macaulayness of tangent cones of monomial curves has been studied by many authors; see, for instance, [1, 2, 5, 8–10, 13, 15, 19]. It constitutes an important problem, since for example Cohen–Macaulayness of the tangent cone guarantees that the Hilbert function of the local ring associated to the monomial curve is nondecreasing and therefore reduces its computation to the computation of the Hilbert function of an Artin local ring.

Barucci and Fröberg ([2]) used Apery sets of semigroups to give a criterion for checking whether the tangent cone of a monomial curve is Cohen–Macaulay or not. In this article our aim is to provide necessary and sufficient conditions for the Cohen–Macaulayness of the tangent cone of a monomial curve in 4-dimensional affine space by using a minimal generating set for the defining ideal of the curve. This information will allow us to check the Cohen–Macaulay property by just computing a minimal generating set of the ideal.

We first deal with the above problem in the case of a general monomial curve in the 4-dimensional affine space $\mathbb{A}^4(K)$, where K is a field. In Section 2, by using the classification in terms of critical binomials given by Katsabekis and Ojeda [11], we study in detail the problem for Case 1 in this classification and give sufficient conditions for the Cohen–Macaulayness of the tangent cone. Our method can be applied easily to all the remaining cases.

In Section 3, we consider the problem for noncomplete intersection Gorenstein monomial curves. In this case, Bresinsky not only showed that there is a minimal generating set for the defining ideal of the monomial curve consisting of five generators, but also gave the explicit form of these generators [4]. Actually, there are 6 permutations of the above generator set. It is worth noting that Theorem 2.10 in [1] provides a sufficient

*Correspondence: sefa.feza.arslan@msgsu.edu.tr
2010 *AMS Mathematics Subject Classification:* Primary 13H10, 14H20; Secondary 20M14
The second author was supported by the TÜBİTAK 2221 Visiting Scientists and Scientists on Sabbatical Leave Fellowship Program.

condition for the Cohen–Macaulayness of the tangent cone in four of the aforementioned cases. In this paper, we generalize their result and provide a necessary and sufficient condition for the Cohen–Macaulayness of the tangent cone in all six permutations. Finally, we use these results to give some families of Gorenstein monomial curves in $\mathbb{A}^4(K)$ with corresponding local rings having nondecreasing Hilbert function, thus giving a partial answer to Rossi's problem [17]. This problem asks whether the Hilbert function of a Gorenstein local ring of dimension one is nondecreasing. Recently, it has been shown that there are many families of monomial curves giving a negative answer to this problem [14], but one should note that Rossi's conjecture is still open for Gorenstein local rings associated to monomial curves in $\mathbb{A}^4(K)$.

Our paper studies the Cohen–Macaulayness of tangent cones of Gorenstein monomial curves, namely monomial curves associated with symmetric semigroups. It is worth noting that [18] studies the Cohen–Macaulayness of tangent cones of monomial curves associated with pseudosymmetric semigroups.

Let $\{n_1, \ldots, n_d\}$ be a set of all-different positive integers with $\gcd(n_1, \ldots, n_d) = 1$. Let $K[x_1, \ldots, x_d]$ be the polynomial ring in d variables. We shall denote by $\mathbf{x}^{\mathbf{u}}$ the monomial $x_1^{u_1} \cdots x_d^{u_d}$ of $K[x_1, \ldots, x_d]$, with $\mathbf{u} = (u_1, \ldots, u_d) \in \mathbb{N}^d$, where $\mathbb{N}$ stands for the set of nonnegative integers. Consider the affine monomial curve in the d-space $\mathbb{A}^d(K)$ defined parametrically by

$$x_1 = t^{n_1}, \ldots, x_d = t^{n_d}.$$

The toric ideal of C, denoted by $I(C)$, is the kernel of the K-algebra homomorphism $\phi : K[x_1, \ldots, x_d] \to K[t]$ given by

$$\phi(x_i) = t^{n_i} \quad \text{for all} \quad 1 \leq i \leq d.$$

We grade $K[x_1, \ldots, x_d]$ by the semigroup $\mathcal{S} := \{g_1 n_1 + \cdots + g_d n_d | g_i \in \mathbb{N}\}$ setting $\deg_{\mathcal{S}}(x_i) = n_i$ for $i = 1, \ldots, d$. The $\mathcal{S}$-degree of a monomial $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} \cdots x_d^{u_d}$ is defined by

$$\deg_{\mathcal{S}}(\mathbf{x}^{\mathbf{u}}) = u_1 n_1 + \cdots + u_d n_d \in \mathcal{S}.$$

The ideal $I(C)$ is generated by all the binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ such that $\deg_{\mathcal{S}}(\mathbf{x}^{\mathbf{u}}) = \deg_{\mathcal{S}}(\mathbf{x}^{\mathbf{v}})$; see, for example, [20, Lemma 4.1].

For checking the Cohen–Macaulayness of the tangent cone of the monomial curve, the following theorem from [8] is used throughout the article:

Theorem 1.1 *[8] Let $n_1 < n_2 < \cdots < n_d$ and $n_1 + \mathcal{S} = \{n_1 + m | m \in \mathcal{S}\}$. The monomial curve C defined parametrically by $x_1 = t^{n_1}, \ldots, x_d = t^{n_d}$ has Cohen–Macaulay tangent cone at the origin if and only if for all integers $v_2 \geq 0, v_3 \geq 0, \ldots, v_d \geq 0$ such that $\sum_{i=2}^d v_i n_i \in n_1 + \mathcal{S}$, there exist $w_1 > 0$, $w_2 \geq 0, \ldots, w_d \geq 0$ such that $\sum_{i=2}^d v_i n_i = \sum_{i=1}^d w_i n_i$ and $\sum_{i=2}^d v_i \leq \sum_{i=1}^d w_i$.*

Note that $x_1^{\frac{n_i}{\gcd(n_1,n_i)}} - x_i^{\frac{n_1}{\gcd(n_1,n_i)}} \in I(C)$ and also $\frac{n_i}{\gcd(n_1,n_i)} > \frac{n_1}{\gcd(n_1,n_i)}$, for every $2 \leq i \leq d$. Thus, to decide the Cohen–Macaulayness of the tangent cone of C it suffices to consider only such v_i with the extra condition that $v_i < \frac{n_1}{\gcd(n_1,n_i)}$.

The computations of this paper are performed by using CoCoA.*

*CoCoATeam. CoCoA: A system for doing computations in commutative algebra. Available at http://cocoa.dima.unige.it.

2. The general case

Let $A = \{n_1, \ldots, n_4\}$ be a set of relatively prime positive integers.

Definition 2.1 A binomial $x_i^{a_i} - \prod_{j \neq i} x_j^{u_{ij}} \in I(C)$ is called **critical** with respect to x_i if a_i is the least positive integer such that $a_i n_i \in \sum_{j \neq i} \mathbb{N} n_j$. The **critical ideal** of A, denoted by $\mathcal{C}_A$, is the ideal of $K[x_1, \ldots, x_4]$ generated by all the critical binomials of $I(C)$.

The support $\operatorname{supp}(\mathbf{x}^{\mathbf{u}})$ of a monomial $\mathbf{x}^{\mathbf{u}}$ is the set

$$\operatorname{supp}(\mathbf{x}^{\mathbf{u}}) = \{i \in \{1, \ldots, 4\} | x_i \text{ divides } \mathbf{x}^{\mathbf{u}}\}.$$

The support of a binomial $B = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ is the set $\operatorname{supp}(\mathbf{x}^{\mathbf{u}}) \cup \operatorname{supp}(\mathbf{x}^{\mathbf{v}})$. If the support of B equals the set $\{1, \ldots, 4\}$, then we say that B has full support. Let $\mu(\mathcal{C}_A)$ be the minimal number of generators of the ideal $\mathcal{C}_A$.

Theorem 2.2 *[11] After permuting the variables, if necessary, there exists a minimal system of binomial generators S of the critical ideal $\mathcal{C}_A$ of the following form:*

CASE 1: If $a_i n_i \neq a_j n_j$, for every $i \neq j$, then $S = \{x_i^{a_i} - \mathbf{x}^{\mathbf{u}_i},\ i = 1, \ldots, 4\}$.

CASE 2: If $a_1 n_1 = a_2 n_2$ and $a_3 n_3 = a_4 n_4$, then either $a_2 n_2 \neq a_3 n_3$ and

(a) $S = \{x_1^{a_1} - x_2^{a_2}, x_3^{a_3} - x_4^{a_4}, x_4^{a_4} - \mathbf{x}^{\mathbf{u}_4}\}$ when $\mu(\mathcal{C}_A) = 3$,

(b) $S = \{x_1^{a_1} - x_2^{a_2}, x_3^{a_3} - x_4^{a_4}\}$ when $\mu(\mathcal{C}_A) = 2$,

or $a_2 n_2 = a_3 n_3$ and

(c) $S = \{x_1^{a_1} - x_2^{a_2}, x_2^{a_2} - x_3^{a_3}, x_3^{a_3} - x_4^{a_4}\}$.

CASE 3: If $a_1 n_1 = a_2 n_2 = a_3 n_3 \neq a_4 n_4$, then $S = \{x_1^{a_1} - x_2^{a_2}, x_2^{a_2} - x_3^{a_3}, x_4^{a_4} - \mathbf{x}^{\mathbf{u}_4}\}$.

CASE 4: If $a_1 n_1 = a_2 n_2$ and $a_i n_i \neq a_j n_j$ for all $\{i, j\} \neq \{1, 2\}$, then

(a) $S = \{x_1^{a_1} - x_2^{a_2}, x_i^{a_i} - \mathbf{x}^{\mathbf{u}_i} \ \mid \ i = 2, 3, 4\}$ when $\mu(\mathcal{C}_A) = 4$,

(b) $S = \{x_1^{a_1} - x_2^{a_2}, x_i^{a_i} - \mathbf{x}^{\mathbf{u}_i} \ \mid \ i = 3, 4\}$ when $\mu(\mathcal{C}_A) = 3$.

Here, in each case, $\mathbf{x}^{\mathbf{u}_i}$ denotes an appropriate monomial whose support has cardinality greater than or equal to two.

Theorem 2.3 *([11]) The union of S, the set $\mathcal{I}$ of all binomials $x_{i_1}^{u_{i_1}} x_{i_2}^{u_{i_2}} - x_{i_3}^{u_{i_3}} x_{i_4}^{u_{i_4}} \in I(C)$ with $0 < u_{i_j} < a_j$, $j = 1, 2$, $u_{i_3} > 0$, $u_{i_4} > 0$ and $x_{i_3}^{u_{i_3}} x_{i_4}^{u_{i_4}}$ indispensable, and the set $\mathcal{R}$ of all binomials $x_1^{u_1} x_2^{u_2} - x_3^{u_3} x_4^{u_4} \in I(C) \setminus \mathcal{I}$ with full support such that*

- *$u_1 \leq a_1$ and $x_3^{u_3} x_4^{u_4}$ is indispensable, in CASES 2(a) and 4(b),*

- *$u_1 \leq a_1$ and/or $u_3 \leq a_3$ and there is no $x_1^{v_1}x_2^{v_2} - x_3^{v_3}x_4^{v_4} \in I(C)$ with full support such that $x_1^{v_1}x_2^{v_2}$ properly divides $x_1^{u_1+\alpha c_1}x_2^{u_2-\alpha c_2}$ or $x_3^{v_3}x_4^{v_4}$ properly divides $x_3^{u_3+\alpha c_3}x_4^{u_4-\alpha u_4}$ for some $\alpha \in \mathbb{N}$, in CASE 2(b),*

is a minimal system of generators of $I(C)$ (up to permutation of indices).

A binomial $B \in I(C)$ is called indispensable of $I(C)$ if every system of binomial generators of $I(C)$ contains B or $-B$. By Corollary 2.16 in [11] every $f \in \mathcal{I}$ is an indispensable binomial of $I(C)$.

Notation 2.4 Given a monomial $\mathbf{x}^{\mathbf{u}}$ we will write $\deg(\mathbf{x}^{\mathbf{u}}) := \sum_{i=1}^{4} u_i$.

For the rest of this section we will assume that $n_1 < n_2 < n_3 < n_4$. To prove our results we will make repeated use of Theorem 1.1.

Theorem 2.5 *Suppose that $I(C)$ is given as in CASE 1. Let $S = \{x_1^{a_1} - \mathbf{x}^{\mathbf{u}}, x_2^{a_2} - \mathbf{x}^{\mathbf{v}}, x_3^{a_3} - \mathbf{x}^{\mathbf{w}}, x_4^{a_4} - \mathbf{x}^{\mathbf{z}}\}$ be a generating set of $\mathcal{C}_A$ and let $1 \in \operatorname{supp}(\mathbf{x}^{\mathbf{v}})$. Let:*

(C1.1) $a_2 \leq \deg(\mathbf{x}^{\mathbf{v}})$.

(C1.2) $a_3 \leq \deg(\mathbf{x}^{\mathbf{w}})$.

(C1.3) For every binomial $f = M - N \in \mathcal{I}$ with $1 \in \operatorname{supp}(M)$ we have that $\deg(N) \leq \deg(M)$.

(C1.4) For every monomial $M = x_2^{d_2}x_3^{d_3}x_4^{d_4}$, where $d_2 < a_2$ and $d_3 < a_3$, with $d_2n_2 + d_3n_3 + d_4n_4 \in n_1 + \mathcal{S}$, there exists a monomial N with $1 \in \operatorname{supp}(N)$ such that $M - N \in I(C)$ and also $\deg(M) \leq \deg(N)$.

(C1.5) For every monomial $M = x_2^{d_2}x_3^{d_3}x_4^{d_4}$, where $d_2 < a_2$ and $d_4 < a_4$, with $d_2n_2 + d_3n_3 + d_4n_4 \in n_1 + \mathcal{S}$, there exists a monomial N with $1 \in \operatorname{supp}(N)$ such that $M - N \in I(C)$ and also $\deg(M) \leq \deg(N)$.

(C1.6) For every monomial $M = x_2^{d_2}x_3^{d_3}x_4^{d_4}$, where $d_2 < a_2$, with $d_2n_2 + d_3n_3 + d_4n_4 \in n_1 + \mathcal{S}$, there exists a monomial N with $1 \in \operatorname{supp}(N)$ such that $M - N \in I(C)$ and also $\deg(M) \leq \deg(N)$.

In the following cases C has a Cohen–Macaulay tangent cone at the origin.
(i) $1 \in \operatorname{supp}(\mathbf{x}^{\mathbf{w}})$, $1 \in \operatorname{supp}(\mathbf{x}^{\mathbf{z}})$ and the conditions (C1.1), (C1.2), and (C1.3) hold.
(ii) $1 \in \operatorname{supp}(\mathbf{x}^{\mathbf{w}})$, $1 \notin \operatorname{supp}(\mathbf{x}^{\mathbf{z}})$ and the conditions (C1.1), (C1.2), and (C1.4) hold.
(iii) $1 \notin \operatorname{supp}(\mathbf{x}^{\mathbf{w}})$, $1 \in \operatorname{supp}(\mathbf{x}^{\mathbf{z}})$ and the conditions (C1.1) and (C1.5) hold.
(iv) $1 \notin \operatorname{supp}(\mathbf{x}^{\mathbf{w}})$, $1 \notin \operatorname{supp}(\mathbf{x}^{\mathbf{z}})$ and the conditions (C1.1) and (C1.6) hold.

Proof. (i) Let $M = x_2^{d_2}x_3^{d_3}x_4^{d_4}$, where $d_2 \geq 0$, $d_3 \geq 0$ and $d_4 \geq 0$, with $d_2n_2 + d_3n_3 + d_4n_4 \in n_1 + \mathcal{S}$. Thus, there exists a monomial P such that $1 \in \operatorname{supp}(P)$ and $M - P \in I(C)$. Let $d_2 \geq a_2$, and then we consider the monomial $P = x_2^{d_2-a_2}x_3^{d_3}x_4^{d_4}\mathbf{x}^{\mathbf{v}}$. We have that $M - P \in I(C)$ and also

$$\deg(M) = d_2 + d_3 + d_4 \leq d_2 + d_3 + d_4 + (\deg(\mathbf{x}^{\mathbf{v}}) - a_2) = \deg(P).$$

Let $d_3 \geq a_3$, and then we consider the monomial $P = x_2^{d_2}x_3^{d_3-a_3}x_4^{d_4}\mathbf{x}^{\mathbf{w}}$. We have that $M - P \in I(C)$ and also

$$\deg(M) = d_2 + d_3 + d_4 \leq d_2 + d_3 + d_4 + (\deg(\mathbf{x}^{\mathbf{w}}) - a_3) = \deg(P).$$

Let $d_4 \geq a_4$, and then we consider the monomial $P = x_2^{d_2}x_3^{d_3}x_4^{d_4-a_4}\mathbf{x^z}$. We have that $M - P \in I(C)$ and also

$$\deg(M) = d_2 + d_3 + d_4 < d_2 + d_3 + d_4 + (\deg(\mathbf{x^z}) - a_4) = \deg(P).$$

Suppose that $d_2 < a_2$, $d_3 < a_3$ and $d_4 < a_4$. There are 2 cases: (1) there exists a binomial $f = G - H \in \mathcal{I}$ with $1 \in \mathrm{supp}(G)$ such that H divides M, so $M = HH'$. Note that $\deg(H) \leq \deg(G)$. Then $GH' - HH' \in I(C)$ and also

$$\deg(M) = \deg(HH') \leq \deg(G) + \deg(H').$$

(2) There exists no binomial $f = G - H$ in $\mathcal{I}$ such that H divides M. Recall that $M - P \in I(C)$, $S \cup \mathcal{I}$ generates $I(C)$, and also $1 \in \mathrm{supp}(\mathbf{x^v})$, $1 \in \mathrm{supp}(\mathbf{x^w})$, and $1 \in \mathrm{supp}(\mathbf{x^z})$. Then necessarily $\mathbf{x^u}$ divides M, so $M = \mathbf{x^u}M'$. Let $P = M'x_1^{a_1}$; then $M - P \in I(C)$ and also

$$\deg(M) = \deg(\mathbf{x^u}) + \deg(M') < \deg(x_1^{a_1}) + \deg(M') = \deg(P).$$

(ii) Let $M = x_2^{d_2}x_3^{d_3}x_4^{d_4}$, where $d_2 \geq 0$, $d_3 \geq 0$ and $d_4 \geq 0$, with $d_2n_2 + d_3n_3 + d_4n_4 \in n_1 + \mathcal{S}$. Let $d_2 \geq a_2$, and then we consider the monomial $P = x_2^{d_2-a_2}x_3^{d_3}x_4^{d_4}\mathbf{x^v}$. We have that $M - P \in I(C)$ and also

$$\deg(M) = d_2 + d_3 + d_4 \leq d_2 + d_3 + d_4 + (\deg(\mathbf{x^v}) - a_2) = \deg(P).$$

Let $d_3 \geq a_3$, and then we consider the monomial $P = x_2^{d_2}x_3^{d_3-a_3}x_4^{d_4}\mathbf{x^w}$. We have that $M - P \in I(C)$ and also

$$\deg(M) = d_2 + d_3 + d_4 \leq d_2 + d_3 + d_4 + (\deg(\mathbf{x^w}) - a_3) = \deg(P).$$

Suppose that $d_2 < a_2$ and $d_3 < a_3$, and then we are done.

(iii) Let $M = x_2^{d_2}x_3^{d_3}x_4^{d_4}$, where $d_2 \geq 0$, $d_3 \geq 0$ and $d_4 \geq 0$, with $d_2n_2 + d_3n_3 + d_4n_4 \in n_1 + \mathcal{S}$. Let $d_2 \geq a_2$, and then we consider the monomial $P = x_2^{d_2-a_2}x_3^{d_3}x_4^{d_4}\mathbf{x^v}$. We have that $M - P \in I(C)$ and also

$$\deg(M) = d_2 + d_3 + d_4 \leq d_2 + d_3 + d_4 + (\deg(\mathbf{x^v}) - a_2) = \deg(P).$$

Let $d_4 \geq a_4$, and then we consider the monomial $P = x_2^{d_2}x_3^{d_3}x_4^{d_4-a_4}\mathbf{x^z}$. We have that $M - P \in I(C)$ and also

$$\deg(M) = d_2 + d_3 + d_4 < d_2 + d_3 + d_4 + \deg(\mathbf{x^z}) - a_4 = \deg(P).$$

Suppose that $d_2 < a_2$ and $d_4 < a_4$, and then we are done.

(iv) Let $M = x_2^{d_2}x_3^{d_3}x_4^{d_4}$, where $d_2 \geq 0$, $d_3 \geq 0$ and $d_4 \geq 0$, with $d_2n_2 + d_3n_3 + d_4n_4 \in n_1 + \mathcal{S}$. Let $d_2 \geq a_2$, and then we consider the monomial $P = x_2^{d_2-a_2}x_3^{d_3}x_4^{d_4}\mathbf{x^v}$. We have that $M - P \in I(C)$ and also

$$\deg(M) = d_2 + d_3 + d_4 \leq d_2 + d_3 + d_4 + (\deg(\mathbf{x^v}) - a_2) = \deg(P).$$

Suppose that $d_2 < a_2$, and then we are done. □

Theorem 2.6 *Suppose that $I(C)$ is given as in case 1. Let $S = \{x_1^{a_1} - \mathbf{x^u}, x_2^{a_2} - \mathbf{x^v}, x_3^{a_3} - \mathbf{x^w}, x_4^{a_4} - \mathbf{x^z}\}$ be a generating set of $\mathcal{C}_A$ and let $1 \notin \mathrm{supp}(\mathbf{x^v})$. In the following cases, C has a Cohen–Macaulay tangent cone at the origin.*

(i) $1 \in \mathrm{supp}(\mathbf{x^w})$, $1 \in \mathrm{supp}(\mathbf{x^z})$, *and*

1. $a_3 \leq \deg(\mathbf{x}^{\mathbf{w}})$,

2. *for every binomial* $f = M - N \in \mathcal{I}$ *with* $1 \in \operatorname{supp}(M)$ *we have that* $\deg(N) \leq \deg(M)$, *and*

3. *for every monomial* $M = x_2^{d_2} x_3^{d_3} x_4^{d_4}$, *where* $d_2 \geq a_2$, $d_3 < a_3$ *and* $d_4 < a_4$, *with* $d_2 n_2 + d_3 n_3 + d_4 n_4 \in n_1 + \mathcal{S}$, *there exists a monomial* N *with* $1 \in \operatorname{supp}(N)$ *such that* $M - N \in I(C)$ *and also* $\deg(M) \leq \deg(N)$.

(ii) $1 \in \operatorname{supp}(\mathbf{x}^{\mathbf{w}})$, $1 \notin \operatorname{supp}(\mathbf{x}^{\mathbf{z}})$, *and*

1. $a_3 \leq \deg(\mathbf{x}^{\mathbf{w}})$ *and*

2. *for every monomial* $M = x_2^{d_2} x_3^{d_3} x_4^{d_4}$, *where* $d_3 < a_3$, *with* $d_2 n_2 + d_3 n_3 + d_4 n_4 \in n_1 + \mathcal{S}$, *there exists a monomial* N *with* $1 \in \operatorname{supp}(N)$ *such that* $M - N \in I(C)$ *and also* $\deg(M) \leq \deg(N)$.

(iii) $1 \notin \operatorname{supp}(\mathbf{x}^{\mathbf{w}})$, $1 \in \operatorname{supp}(\mathbf{x}^{\mathbf{z}})$, *and for every monomial* $M = x_2^{d_2} x_3^{d_3} x_4^{d_4}$, *where* $d_4 < a_4$, *with* $d_2 n_2 + d_3 n_3 + d_4 n_4 \in n_1 + \mathcal{S}$, *there exists a monomial* N *with* $1 \in \operatorname{supp}(N)$ *such that* $M - N \in I(C)$ *and also* $\deg(M) \leq \deg(N)$.

Proof. (i) Let $M = x_2^{d_2} x_3^{d_3} x_4^{d_4}$, where $d_2 \geq 0$, $d_3 \geq 0$ and $d_4 \geq 0$, with $d_2 n_2 + d_3 n_3 + d_4 n_4 \in n_1 + \mathcal{S}$. Thus, there exists a monomial P such that $1 \in \operatorname{supp}(P)$ and $M - P \in I(C)$. Let $d_3 \geq a_3$, and then we consider the monomial $P = x_2^{d_2} x_3^{d_3 - a_3} x_4^{d_4} \mathbf{x}^{\mathbf{w}}$. We have that $M - P \in I(C)$ and also

$$\deg(M) = d_2 + d_3 + d_4 \leq d_2 + d_3 + d_4 + (\deg(\mathbf{x}^{\mathbf{w}}) - a_3) = \deg(P).$$

Let $d_4 \geq a_4$, and then we consider the monomial $P = x_2^{d_2} x_3^{d_3} x_4^{d_4 - a_4} \mathbf{x}^{\mathbf{z}}$. We have that $M - P \in I(C)$ and also

$$\deg(M) = d_2 + d_3 + d_4 < d_2 + d_3 + d_4 + (\deg(\mathbf{x}^{\mathbf{z}}) - a_4) = \deg(P).$$

Suppose that $d_3 < a_3$ and $d_4 < a_4$. If $d_2 \geq a_2$, then from (3) we are done. Assume that $d_2 < a_2$. There are 2 cases: (1) there exists a binomial $f = G - H \in \mathcal{I}$ with $1 \in \operatorname{supp}(G)$ such that H divides M, so $M = HH'$. Note that $\deg(H) \geq \deg(G)$ from condition (2). Then $GH' - M \in I(C)$ and also $\deg(M) \leq \deg(G) + \deg(H')$. (2) There exists no binomial $f = G - H$ in $\mathcal{I}$ such that H divides M. Recall that $M - P \in I(C)$, $S \cup \mathcal{I}$ generates $I(C)$, and also $1 \in \operatorname{supp}(\mathbf{x}^{\mathbf{w}})$ and $1 \in \operatorname{supp}(\mathbf{x}^{\mathbf{z}})$. Then necessarily $\mathbf{x}^{\mathbf{u}}$ or/and $\mathbf{x}^{\mathbf{v}}$ divides M. Let us suppose that $\mathbf{x}^{\mathbf{u}}$ divides M, so $M = \mathbf{x}^{\mathbf{u}} M'$. Let $P = M' x_1^{a_1}$; then $M - P \in I(C)$ and also

$$\deg(M) = \deg(\mathbf{x}^{\mathbf{u}}) + \deg(M') < \deg(x_1^{a_1}) + \deg(M') = \deg(P).$$

Suppose now that $\mathbf{x}^{\mathbf{v}}$ divides M, so $M = \mathbf{x}^{\mathbf{v}} M'$. Then the binomial $x_2^{a_2} M' - M$ belongs to $I(C)$ and also $\deg_{\mathcal{S}}(x_2^{a_2} M') \in n_1 + \mathcal{S}$. Thus, there exists a monomial N such that $1 \in \operatorname{supp}(N)$, $\deg_{\mathcal{S}}(N) = \deg_{\mathcal{S}}(x_2^{a_2} M')$ and also $\deg(x_2^{a_2} M') \leq \deg(N)$. Consequently,

$$\deg(M) = \deg(\mathbf{x}^{\mathbf{v}} M') < \deg(x_2^{a_2} M') \leq \deg(N).$$

(ii) Let $M = x_2^{d_2} x_3^{d_3} x_4^{d_4}$, where $d_2 \geq 0$, $d_3 \geq 0$ and $d_4 \geq 0$, with $d_2 n_2 + d_3 n_3 + d_4 n_4 \in n_1 + \mathcal{S}$. Let $d_3 \geq a_3$, and then we consider the monomial $P = x_2^{d_2} x_3^{d_3 - a_3} x_4^{d_4} \mathbf{x}^{\mathbf{w}}$. We have that $M - P \in I(C)$ and also

$$\deg(M) = d_2 + d_3 + d_4 \leq d_2 + d_3 + d_4 + (\deg(\mathbf{x}^{\mathbf{w}}) - a_3) = \deg(P).$$

Let $d_3 < a_3$, and then from condition (2) we are done.

(iii) Let $M = x_2^{d_2} x_3^{d_3} x_4^{d_4}$, where $d_2 \geq 0$, $d_3 \geq 0$ and $d_4 \geq 0$, with $d_2 n_2 + d_3 n_3 + d_4 n_4 \in n_1 + \mathcal{S}$. Let $d_4 \geq a_4$, and then we consider the monomial $P = x_2^{d_2} x_3^{d_3} x_4^{d_4 - a_4} \mathbf{x}^{\mathbf{z}}$. We have that $M - P \in I(C)$ and also

$$\deg(M) = d_2 + d_3 + d_4 < d_2 + d_3 + d_4 + (\deg(\mathbf{x}^{\mathbf{z}}) - a_4) = \deg(P).$$

Suppose that $d_4 < a_4$; then, from the assumption, we are done. □

3. The Gorenstein case

In this section we will study the case where C is a noncomplete intersection Gorenstein monomial curve, i.e. the semigroup $\mathcal{S} = \{g_1 n_1 + \cdots + g_4 n_4 | g_i \in \mathbb{N}\}$ is symmetric. Given a polynomial $f \in I(C)$, we let f_* be the homogeneous summand of f of least degree. We shall denote by $I(C)_*$ the ideal generated by the polynomials f_* for f in $I(C)$ and f_* is the homogeneous summand of f of least degree. By [5, Theorem 7] C has Cohen–Macaulay tangent cone if and only if x_1 is not a zero divisor in the ring $K[x_1, \ldots, x_4]/I(C)_*$, where $n_1 = \min\{n_1, \ldots, n_4\}$. Thus, if C has a Cohen–Macaulay tangent cone at the origin, then $I(C)_* : \langle x_1 \rangle = I(C)_*$.

Theorem 3.1 *[4] Let C be a monomial curve having the parametrization*

$$x_1 = t^{n_1}, x_2 = t^{n_2}, x_3 = t^{n_3}, x_4 = t^{n_4}.$$

The semigroup $\mathcal{S}$ is symmetric and C is a noncomplete intersection curve if and only if $I(C)$ is minimally generated by the set

$$G = \{f_1 = x_1^{a_1} - x_3^{a_{13}} x_4^{a_{14}}, f_2 = x_2^{a_2} - x_1^{a_{21}} x_4^{a_{24}}, f_3 = x_3^{a_3} - x_1^{a_{31}} x_2^{a_{32}},$$

$$f_4 = x_4^{a_4} - x_2^{a_{42}} x_3^{a_{43}}, f_5 = x_3^{a_{43}} x_1^{a_{21}} - x_2^{a_{32}} x_4^{a_{14}}\},$$

where the polynomials f_i are unique up to isomorphism and $0 < a_{ij} < a_j$.

Remark 3.2 Bresinsky [4] showed that $\mathcal{S}$ is symmetric and $I(C)$ is as in the previous theorem if and only if $n_1 = a_2 a_3 a_{14} + a_{32} a_{13} a_{24}$, $n_2 = a_3 a_4 a_{21} + a_{31} a_{43} a_{24}$, $n_3 = a_1 a_4 a_{32} + a_{14} a_{42} a_{31}$, $n_4 = a_1 a_2 a_{43} + a_{42} a_{21} a_{13}$ with $\gcd(n_1, n_2, n_3, n_4) = 1$, $a_i > 1, 0 < a_{ij} < a_j$ for $1 \leq i \leq 4$, and $a_1 = a_{21} + a_{31}$, $a_2 = a_{32} + a_{42}$, $a_3 = a_{13} + a_{43}$, $a_4 = a_{14} + a_{24}$.

Remark 3.3 *[1]* The above theorem implies that for any noncomplete intersection Gorenstein monomial curve with embedding dimension four, the variables can be renamed to obtain generators exactly of the given form, and this means that there are six isomorphic possible permutations. which can be considered within three cases:

(1) $f_1 = (1, (3, 4))$

- (a) $f_2 = (2, (1, 4))$, $f_3 = (3, (1, 2))$, $f_4 = (4, (2, 3))$, $f_5 = ((1, 3), (2, 4))$
- (b) $f_2 = (2, (1, 3))$, $f_3 = (3, (2, 4))$, $f_4 = (4, (1, 2))$, $f_5 = ((1, 4), (2, 3))$

(2) $f_1 = (1, (2, 3))$

- (a) $f_2 = (2, (3, 4))$, $f_3 = (3, (1, 4))$, $f_4 = (4, (1, 2))$, $f_5 = ((2, 4), (1, 3))$

(b) $f_2 = (2,(1,4))$, $f_3 = (3,(2,4))$, $f_4 = (4,(1,3))$, $f_5 = ((1,2),(4,3))$

(3) $f_1 = (1,(2,4))$

(a) $f_2 = (2,(1,3))$, $f_3 = (3,(1,4))$, $f_4 = (4,(2,3))$, $f_5 = ((1,2),(3,4))$

(b) $f_2 = (2,(3,4))$, $f_3 = (3,(1,2))$, $f_4 = (4,(1,3))$, $f_5 = ((2,3),(1,4))$

Here, the notations $f_i = (i,(j,k))$ and $f_5 = ((i,j),(k,l))$ denote the generators $f_i = x_i^{a_i} - x_j^{a_{ij}}x_k^{a_{ik}}$ and $f_5 = x_i^{a_{ki}}x_j^{a_{lj}} - x_k^{a_{jk}}x_l^{a_{il}}$. Thus, given a Gorenstein monomial curve C, if we have the extra condition $n_1 < n_2 < n_3 < n_4$, then the generator set of $I(C)$ is exactly given by one of these six permutations.

Remark 3.4 By [11, Corollary 3.13] the toric ideal $I(C)$ of any noncomplete intersection Gorenstein monomial curve C is generated by its indispensable binomials.

First, we use the technique in [16] to compute the Apery set of $\mathcal{S}$ relative to $\{n_1\}$, defined by

$$Q = \{q \in \mathcal{S} | q - n_1 \notin \mathcal{S}\}.$$

Let $lex - inf$ be the total order on the monomials of $K[x_1, \ldots, x_4]$, which is defined as follows:

$$\mathbf{x^u} >_{lex-inf} \mathbf{x^v} \Leftrightarrow \mathbf{x^u} <_{lex} \mathbf{x^v},$$

where lex order is the lexicographic order such that x_1 is the largest variable in $K[x_1, \ldots, x_4]$ with respect to $<_{lex}$.

Proposition 3.5 *The set $G = \{f_1, f_2, f_3, f_4, f_5\}$ is the reduced Gröbner basis of $I(C)$ with respect to an appropriate $lex - inf$ order.*

Proof. Suppose that $I(C)$ is given as in case 1(a). Then

$$f_1 = x_1^{a_1} - x_3^{a_{13}}x_4^{a_{14}}, f_2 = x_2^{a_2} - x_1^{a_{21}}x_4^{a_{24}}, f_3 = x_3^{a_3} - x_1^{a_{31}}x_2^{a_{32}},$$

$$f_4 = x_4^{a_4} - x_2^{a_{42}}x_3^{a_{43}}, f_5 = x_1^{a_{21}}x_3^{a_{43}} - x_2^{a_{32}}x_4^{a_{14}}.$$

With respect to $lex-inf$ such that $x_1 >_{lex} x_2 >_{lex} x_3 >_{lex} x_4$ we have that $lm(f_1) = x_3^{a_{13}}x_4^{a_{14}}$, $lm(f_2) = x_2^{a_2}$, $lm(f_3) = x_3^{a_3}$, $lm(f_4) = x_4^{a_4}$, and $lm(f_5) = x_2^{a_{32}}x_4^{a_{14}}$. We will prove that $S(f_i, f_j) \xrightarrow{G} 0$ for any pair $\{f_i, f_j\}$. Since $lm(f_1)$ and $lm(f_2)$ are relatively prime, we get that $S(f_1, f_2) \xrightarrow{G} 0$. Similarly, $S(f_2, f_3) \xrightarrow{G} 0$, $S(f_2, f_4) \xrightarrow{G} 0$, $S(f_3, f_4) \xrightarrow{G} 0$, and $S(f_3, f_5) \xrightarrow{G} 0$. We have that

$$S(f_1, f_3) = x_1^{a_{31}}x_2^{a_{32}}x_4^{a_{14}} - x_1^{a_1}x_3^{a_{43}} \xrightarrow{f_5} x_1^{a_1}x_3^{a_{43}} - x_1^{a_1}x_3^{a_{43}} = 0,$$

$$S(f_1, f_4) = x_2^{a_{42}}x_3^{a_3} - x_1^{a_1}x_4^{a_{24}} \xrightarrow{f_3} x_1^{a_{31}}x_2^{a_2} - x_1^{a_1}x_4^{a_{24}} \xrightarrow{f_2} x_1^{a_1}x_4^{a_{24}} - x_1^{a_1}x_4^{a_{24}} = 0,$$

$$S(f_1, f_5) = x_1^{a_{21}}x_3^{a_3} - x_1^{a_1}x_2^{a_{32}} \xrightarrow{f_3} x_1^{a_1}x_2^{a_{32}} - x_1^{a_1}x_2^{a_{32}} = 0,$$

$$S(f_2, f_5) = x_1^{a_{21}}x_2^{a_{42}}x_3^{a_{43}} - x_1^{a_{21}}x_4^{a_4} \xrightarrow{f_4} x_1^{a_{21}}x_2^{a_{42}}x_3^{a_{43}} - x_1^{a_{21}}x_2^{a_{42}}x_3^{a_{43}} = 0,$$

$$S(f_4,f_5)=x_1^{a_{21}}x_3^{a_{43}}x_4^{a_{24}}-x_2^{a_2}x_3^{a_{43}}\xrightarrow{f_2}x_1^{a_{21}}x_3^{a_{43}}x_4^{a_{24}}-x_1^{a_{21}}x_3^{a_{43}}x_4^{a_{24}}=0.$$

Thus, G is a Gröbner basis for $I(C)$ with respect to $lex-inf$ such that $x_1>_{lex}x_2>_{lex}x_3>_{lex}x_4$. It is not hard to show that G is a Gröbner basis for $I(C)$ with respect to $lex-inf$ such that

1. $x_1>_{lex}x_2>_{lex}x_3>_{lex}x_4$ in case 1(b).
2. $x_1>_{lex}x_3>_{lex}x_2>_{lex}x_4$ in case 2(a).
3. $x_1>_{lex}x_2>_{lex}x_3>_{lex}x_4$ in case 2(b).
4. $x_1>_{lex}x_2>_{lex}x_3>_{lex}x_4$ in case 3(a).
5. $x_1>_{lex}x_3>_{lex}x_2>_{lex}x_4$ in case 3(b). □

Using Lemma 1.2 in [16], we compute the Apery set Q as follows:

Corollary 3.6 *Let B be the set of monomials $x_2^{u_2}x_3^{u_3}x_4^{u_4}$ in the polynomial ring $K[x_2,x_3,x_4]$, which are not divisible by any of the monomials of the set*

1. $\{x_3^{a_{13}}x_4^{a_{14}},x_2^{a_2},x_3^{a_3},x_4^{a_4},x_2^{a_{32}}x_4^{a_{14}}\}$ *in case 1(a).*
2. $\{x_3^{a_{13}}x_4^{a_{14}},x_2^{a_2},x_3^{a_3},x_4^{a_4},x_2^{a_{42}}x_3^{a_{13}}\}$ *in case 1(b).*
3. $\{x_2^{a_{12}}x_3^{a_{13}},x_2^{a_2},x_3^{a_3},x_4^{a_4},x_2^{a_{12}}x_4^{a_{34}}\}$ *in case 2(a).*
4. $\{x_2^{a_{12}}x_3^{a_{13}},x_2^{a_2},x_3^{a_3},x_4^{a_4},x_3^{a_{13}}x_4^{a_{24}}\}$ *in case 2(b).*
5. $\{x_2^{a_{12}}x_4^{a_{14}},x_2^{a_2},x_3^{a_3},x_4^{a_4},x_3^{a_{23}}x_4^{a_{14}}\}$ *in case 3(a).*
6. $\{x_2^{a_{12}}x_4^{a_{14}},x_2^{a_2},x_3^{a_3},x_4^{a_4},x_2^{a_{12}}x_3^{a_{43}}\}$ *in case 3(b).*

Then

$$Q=\{m\in\mathcal{S}|m=\sum_{i=2}^{4}u_in_i\ \textit{where}\ x_2^{u_2}x_3^{u_3}x_4^{u_4}\in B\}.$$

Theorem 3.7 *Suppose that $I(C)$ is given as in case 1(a). Then C has a Cohen–Macaulay tangent cone at the origin if and only if $a_2\le a_{21}+a_{24}$.*

Proof. In this case $I(C)$ is minimally generated by the set

$$G=\{f_1=x_1^{a_1}-x_3^{a_{13}}x_4^{a_{14}},f_2=x_2^{a_2}-x_1^{a_{21}}x_4^{a_{24}},f_3=x_3^{a_3}-x_1^{a_{31}}x_2^{a_{32}},$$

$$f_4=x_4^{a_4}-x_2^{a_{42}}x_3^{a_{43}},f_5=x_1^{a_{21}}x_3^{a_{43}}-x_2^{a_{32}}x_4^{a_{14}}\}.$$

If $a_2\le a_{21}+a_{24}$, then we have, from Theorem 2.8 in [1], that the curve C has Cohen–Macaulay tangent cone at the origin. Conversely, suppose that C has Cohen–Macaulay tangent cone at the origin. Since $I(C)$ is generated by the indispensable binomials, every binomial f_i, $1\le i\le 5$, is indispensable of $I(C)$. In particular the binomial f_2 is indispensable of $I(C)$. If there exists a monomial $N \neq x_1^{a_{21}}x_4^{a_{24}}$ such that $g = x_2^{a_2} - N$ belongs to $I(C)$, then we can replace f_2 in G by the binomials g and $N - x_1^{a_{21}}x_4^{a_{24}} \in I(C)$, a contradiction to the fact that f_2 is indispensable. Thus, $N = x_1^{a_{21}}x_4^{a_{24}}$, but $a_2n_2 \in n_1 + \mathcal{S}$ and therefore we have, from Theorem 1.1, that $a_2 \leq a_{21} + a_{24}$. □

Remark 3.8 Suppose that $I(C)$ is given as in case 1(b). (1) It holds that $a_1 > a_{13} + a_{14}$ and $a_4 < a_{41} + a_{42}$.
(2) If $a_{42} \leq a_{32}$, then $x_3^{a_3+a_{13}} - x_1^{a_{21}}x_2^{a_{32}-a_{42}}x_4^{2a_{34}} \in I(C)$.
(3) If $a_{14} \leq a_{34}$, then the binomial $x_3^{a_3+a_{13}} - x_1^{a_1}x_2^{a_{32}}x_4^{a_{34}-a_{14}}$ belongs to $I(C)$.

Proposition 3.9 *Suppose that $I(C)$ is given as in case 1(b). Then C has Cohen–Macaulay tangent cone at the origin if and only if*

1. $a_2 \leq a_{21} + a_{23}$,
2. $a_{42} + a_{13} \leq a_{21} + a_{34}$, *and*
3. *for every monomial $M = x_2^{u_2}x_3^{u_3}x_4^{u_4}$, where $u_2 < a_{42}$, $u_3 \geq a_3$, and $u_4 < a_{14}$, with $u_2n_2+u_3n_3+u_4n_4 \in n_1+\mathcal{S}$, there exists a monomial N with $1 \in \operatorname{supp}(N)$ such that $M-N \in I(C)$ and also $\deg(M) \leq \deg(N)$.*

Proof. In this case $I(C)$ is minimally generated by the set

$$G = \{f_1 = x_1^{a_1} - x_3^{a_{13}}x_4^{a_{14}}, f_2 = x_2^{a_2} - x_1^{a_{21}}x_3^{a_{23}}, f_3 = x_3^{a_3} - x_2^{a_{32}}x_4^{a_{34}},$$

$$f_4 = x_4^{a_4} - x_1^{a_{41}}x_2^{a_{42}}, f_5 = x_1^{a_{21}}x_4^{a_{34}} - x_2^{a_{42}}x_3^{a_{13}}\}.$$

Suppose that C has Cohen–Macaulay tangent cone at the origin. Since $I(C)$ is generated by the indispensable binomials, every binomial f_i, $1 \leq i \leq 5$, is indispensable of $I(C)$. In particular the binomials f_2 and f_5 are indispensable of $I(C)$. Therefore, both inequalities $a_2 \leq a_{21}+a_{23}$ and $a_{42}+a_{13} \leq a_{21}+a_{34}$ hold. By Theorem 1.1, condition (3) is also true.

Conversely, from Theorem 2.5 (iii), it is enough to consider a monomial $M = x_2^{u_2}x_3^{u_3}x_4^{u_4}$, where $u_2 < a_2$, $u_3 \geq 0$ and $u_4 < a_4$, with the following property: there exists at least one monomial P such that $1 \in \operatorname{supp}(P)$ and also $M - P$ is in $I(C)$. Suppose that $u_3 \geq a_3$. If $u_4 \geq a_{14}$, then we let $P = x_1^{a_1}x_2^{u_2}x_3^{u_3-a_{13}}x_4^{u_4-a_{14}}$, so we have that $M - P \in I(C)$ and also $\deg(M) < \deg(P)$ since $a_{13} + a_{14} < a_1$. Similarly if $u_2 \geq a_{42}$, then we let $P = x_1^{a_{21}}x_2^{u_2-a_{42}}x_3^{u_3-a_{13}}x_4^{u_4+a_{34}}$. So we have that $M - P \in I(C)$ and also $\deg(M) \leq \deg(P)$. If both inequalities $u_4 < a_{14}$ and $u_2 < a_{42}$ hold, then condition (3) implies that there exists a monomial N with $1 \in \operatorname{supp}(N)$ such that $M - N \in I(C)$ and also $\deg(M) \leq \deg(N)$.

Suppose now that $u_3 < a_3$. Recall that $M - P \in I(C)$ and G generates $I(C)$. Then M is divided by at least one of the monomials $x_2^{a_{42}}x_3^{a_{13}}$, $x_3^{a_{13}}x_4^{a_{14}}$, and $x_2^{a_{32}}x_4^{a_{34}}$. If M is divided by $x_2^{a_{42}}x_3^{a_{13}}$, then $M = x_2^{a_{42}+p}x_3^{a_{13}+q}x_4^{u_4}$, for some nonnegative integers p and q, so $M - x_1^{a_{21}}x_2^{p}x_3^{q}x_4^{a_{34}+u_4} \in I(C)$ and also $\deg(M) \leq \deg(x_1^{a_{21}}x_2^{p}x_3^{q}x_4^{a_{34}+u_4})$. If M is divided by $x_3^{a_{13}}x_4^{a_{14}}$, then $M = x_2^{u_2}x_3^{a_{13}+p}x_4^{a_{14}+q}$, for some nonnegative integers p and q, and therefore the binomial $M - x_1^{a_1}x_2^{u_2}x_3^{p}x_4^{q} \in I(C)$ and also $\deg(M) < \deg(x_1^{a_1}x_2^{u_2}x_3^{p}x_4^{q})$. Assume that neither $x_2^{a_{32}}x_4^{a_{34}}$ nor $x_3^{a_{13}}x_4^{a_{14}}$ divides M. Then necessarily $x_2^{a_{42}}x_3^{a_{13}}$ divides M. However, M is not divided by any leading monomial of G with respect to $lex - inf$ such that $x_1 >_{lex} x_2 >_{lex} x_3 >_{lex} x_4$. Thus, $m = u_2n_2 + u_3n_3 + u_4n_4$ is in Q, a contradiction to the fact that $m - n_1 \in \mathcal{S}$. Therefore, from Theorem 2.5, C has Cohen–Macaulay tangent cone at the origin. □

Proposition 3.10 *Suppose that $I(C)$ is given as in case 1(b). Assume that C has Cohen–Macaulay tangent cone at the origin and also $a_{42} \leq a_{32}$.*

1. *If $a_{34} < a_{14}$, then $a_3 + a_{13} \leq a_{21} + a_{32} - a_{42} + 2a_{34}$.*
2. *If $a_{14} \leq a_{34}$, then $a_3 + a_{13} \leq a_1 + a_{32} + a_{34} - a_{14}$.*

Proof. (1) Suppose that $a_{34} < a_{14}$ and assume that $a_3 + a_{13} > a_{21} + a_{32} - a_{42} + 2a_{34}$. Since $x_3^{a_3+a_{13}} - x_1^{a_{21}} x_2^{a_{32}-a_{42}} x_4^{2a_{34}} \in I(C)$, we deduce that $x_1^{a_{21}} x_2^{a_{32}-a_{42}} x_4^{2a_{34}} \in I(C)_*$, and therefore $x_2^{a_{32}-a_{42}} x_4^{2a_{34}} \in I(C)_*$ since C has Cohen–Macaulay tangent cone at the origin. However, $G = \{x_1^{a_1} - x_3^{a_{13}} x_4^{a_{14}}, x_2^{a_2} - x_1^{a_{21}} x_3^{a_{23}}, x_3^{a_3} - x_2^{a_{32}} x_4^{a_{34}}, x_4^{a_4} - x_1^{a_{41}} x_2^{a_{42}}, x_1^{a_{21}} x_4^{a_{34}} - x_2^{a_{42}} x_3^{a_{13}}\}$ is a generating set for $I(C)$, so $x_2^{a_{32}-a_{42}} x_4^{2a_{34}}$ must be divided by at least one of the monomials in G, a contradiction.
(2) Suppose that $a_{14} \leq a_{34}$ and let $a_3 + a_{13} > a_1 + a_{32} + a_{34} - a_{14}$. Since $x_3^{a_3+a_{13}} - x_1^{a_1} x_2^{a_{32}} x_4^{a_{34}-a_{14}} \in I(C)$, we deduce that $x_1^{a_1} x_2^{a_{32}} x_4^{a_{34}-a_{14}} \in I(C)_*$ and therefore $x_2^{a_{32}} x_4^{a_{34}-a_{14}} \in I(C)_*$. However, G is a generating set for $I(C)$, so $x_2^{a_{32}} x_4^{a_{34}-a_{14}}$ must be divided by at least one of the monomials in G, a contradiction. □

Theorem 3.11 *Suppose that $I(C)$ is given as in case 1(b) and also that $a_{42} \leq a_{32}$. Then C has Cohen–Macaulay tangent cone at the origin if and only if*

1. *$a_2 \leq a_{21} + a_{23}$,*
2. *$a_{42} + a_{13} \leq a_{21} + a_{34}$, and*
3. *either $a_{34} < a_{14}$ and $a_3 + a_{13} \leq a_{21} + a_{32} - a_{42} + 2a_{34}$ or $a_{14} \leq a_{34}$ and $a_3 + a_{13} \leq a_1 + a_{32} + a_{34} - a_{14}$.*

Proof. ($\Longrightarrow$) From Proposition 3.9 we have that conditions (1) and (2) are true. From Proposition 3.10 condition (3) is also true.
($\Longleftarrow$) From Proposition 3.9 it is enough to consider a monomial $N = x_2^{u_2} x_3^{u_3} x_4^{u_4}$, where $u_2 < a_{42}$, $u_3 \geq a_3$ and $u_4 < a_{14}$, with the following property: there exists at least one monomial P such that $1 \in \mathrm{supp}(P)$ and also $N - P$ is in $I(C)$. Suppose that $u_3 \geq a_3 + a_{13}$ and let M denote either the monomial $x_1^{a_{21}} x_2^{a_{32}-a_{42}} x_4^{2a_{34}}$ when $a_{34} < a_{14}$ or the monomial $x_1^{a_1} x_2^{a_{32}} x_4^{a_{34}-a_{14}}$ when $a_{14} \leq a_{34}$. Let $P = x_2^{u_2} x_3^{u_3-a_3-a_{13}} x_4^{u_4} M$. We have that $N - P \in I(C)$ and

$$\deg(N) = u_2 + u_3 + u_4 \leq u_2 + u_3 + u_4 + \deg(M) - a_3 - a_{13} = \deg(P).$$

It suffices to consider the case where $u_3 - a_3 < a_{13}$. Recall that $G = \{f_1, \ldots, f_5\}$ generates $I(C)$. The binomial $N - P$ belongs to $I(C)$, so $N - P = \sum_{i=1}^{5} H_i f_i$ for some polynomials $H_i \in K[x_1, \ldots, x_4]$ and therefore N is a term in the sum $\sum_{i=1}^{5} H_i f_i$. Note that N is not divided by any of the monomials $x_3^{a_{13}} x_4^{a_{14}}$, $x_2^{a_2}$, $x_2^{a_{32}} x_4^{a_{34}}$, $x_4^{a_4}$, and $x_2^{a_{42}} x_3^{a_{13}}$. Now the monomial N is divided by the monomial $x_3^{a_3}$, so $Q = -x_2^{u_2+a_{32}} x_3^{u_3-a_3} x_4^{u_4+a_{34}}$ is a term in the sum $\sum_{i=1}^{5} H_i f_i$ and should be canceled with another term of the above sum. Remark that $u_2 + a_{32} < a_2$ and $u_4 + a_{34} < a_4$. Thus, $x_2^{a_{42}} x_3^{a_{13}}$ divides $-Q$, so $u_3 - a_3 \geq a_{13}$, a contradiction. □

Proposition 3.12 *Suppose that $I(C)$ is given as in case 1(b) and also that $a_{32} < a_{42}$. If C has Cohen–Macaulay tangent cone at the origin, then*

1. $a_2 \leq a_{21} + a_{23}$,

2. $a_{42} + a_{13} \leq a_{21} + a_{34}$, *and*

3. $a_3 + a_{13} \leq a_1 + a_{32} + a_{34} - a_{14}$.

Proof. By Proposition 3.9 conditions (1) and (2) are true. Suppose first that $a_{14} \leq a_{34}$ and let $a_3 + a_{13} > a_1 + a_{32} + a_{34} - a_{14}$. Since $x_3^{a_3+a_{13}} - x_1^{a_1} x_2^{a_{32}} x_4^{a_{34}-a_{14}} \in I(C)$, we have that $x_2^{a_{32}} x_4^{a_{34}-a_{14}} \in I(C)_*$, but $G = \{f_1, \ldots, f_5\}$ is a generating set for $I(C)$, so $x_2^{a_{32}} x_4^{a_{34}-a_{14}}$ must be divided by at least one of the monomials in G, a contradiction. Suppose now that $a_{14} > a_{34}$. Note that $x_1^{a_1} x_2^{a_{32}} - x_3^{a_3+a_{13}} x_4^{a_{14}-a_{34}} \in I(C)$. If $a_3 + a_{13} + a_{14} - a_{34} > a_1 + a_{32}$, then $x_1^{a_1} x_2^{a_{32}} \in I(C)_*$ and therefore $x_2^{a_{32}} \in I(C)_*$, a contradiction. Thus, $a_3 + a_{13} \leq a_1 + a_{32} + a_{34} - a_{14}$. □

Theorem 3.13 *Suppose that $I(C)$ is given as in case 1(b) and also that $a_{32} < a_{42}$. Assume that $a_{14} \leq a_{34}$. Then C has Cohen–Macaulay tangent cone at the origin if and only if*

1. $a_2 \leq a_{21} + a_{23}$,

2. $a_{42} + a_{13} \leq a_{21} + a_{34}$, *and*

3. $a_3 + a_{13} \leq a_1 + a_{32} + a_{34} - a_{14}$.

Proof. ($\Longrightarrow$) By Proposition 3.12 conditions (1), (2), and (3) are true.
($\Longleftarrow$) From Proposition 3.9 it is enough to consider a monomial $N = x_2^{u_2} x_3^{u_3} x_4^{u_4}$, where $u_2 < a_{42}$, $u_3 \geq a_3$, and $u_4 < a_{14}$, with the following property: there exists at least one monomial P such that $1 \in \mathrm{supp}(P)$ and also $N - P$ is in $I(C)$. Suppose that $u_3 \geq a_3 + a_{13}$. Let $P = x_1^{a_1} x_2^{u_2+a_{32}} x_3^{u_3-a_3-a_{13}} x_4^{u_4+a_{34}-a_{14}}$. We have that $N - P \in I(C)$ and

$$\deg(N) = u_2 + u_3 + u_4 \leq u_2 + u_3 + u_4 + a_1 + a_{32} + a_{34} - a_{14} - a_3 - a_{13} = \deg(P).$$

It suffices to assume that $u_3 - a_3 < a_{13}$. Since the binomial $N - P$ belongs to $I(C)$, we have that $N - P = \sum_{i=1}^{5} H_i f_i$ for some polynomials $H_i \in K[x_1, \ldots, x_4]$, so N is a term in the sum $\sum_{i=1}^{5} H_i f_i$. Then $T = -x_2^{u_2+a_{32}} x_3^{u_3-a_3} x_4^{u_4+a_{34}}$ is a term in the sum $\sum_{i=1}^{5} H_i f_i$ and it should be canceled with another term of the above sum. Remark that $u_2 + a_{32} < a_2$ and $u_4 + a_{34} < a_4$. Thus, $x_2^{a_{42}} x_3^{a_{13}}$ divides $-T$, so $u_3 - a_3 \geq a_{13}$, a contradiction. □

Example 3.14 Consider $n_1 = 1199$, $n_2 = 2051$, $n_3 = 2352$, and $n_4 = 3032$. The toric ideal $I(C)$ is minimally generated by the set

$$G = \{x_1^{16} - x_3^3 x_4^4, x_2^{19} - x_1^7 x_3^{13}, x_3^{16} - x_2^8 x_4^7, x_4^{11} - x_1^9 x_2^{11}, x_1^7 x_4^7 - x_2^{11} x_3^3\}.$$

Here $a_1 = 16$, $a_{32} = 8$, $a_{42} = 11$, $a_{14} = 4$, $a_{34} = 7$, $a_{13} = 3$, and $a_3 = 16$. Note that $a_2 = 19 < 20 = a_{21} + a_{23}$ and $a_{42} + a_{13} = 14 = a_{21} + a_{34}$. We have that $a_3 + a_{13} = 19 < 27 = a_1 + a_{32} + a_{34} - a_{14}$. Thus, C has a Cohen–Macaulay tangent cone at the origin.

Remark 3.15 Suppose that $I(C)$ is given as in case 2(a).

(1) It holds that $a_1 > a_{12} + a_{13}$, $a_2 > a_{23} + a_{24}$ and $a_4 < a_{41} + a_{42}$.

(2) If $a_{34} \leq a_{24}$, then $x_2^{a_2+a_{12}} - x_1^{a_{41}} x_3^{2a_{23}} x_4^{a_{24}-a_{34}} \in I(C)$.

(3) If $a_{13} \leq a_{23}$, then the binomial $x_2^{a_2+a_{12}} - x_1^{a_1} x_3^{a_{23}-a_{13}} x_4^{a_{24}}$ belongs to $I(C)$.

Proposition 3.16 *Suppose that $I(C)$ is given as in case 2(a). Then C has Cohen–Macaulay tangent cone at the origin if and only if*

1. $a_3 \leq a_{31} + a_{34}$,
2. $a_{12} + a_{34} \leq a_{41} + a_{23}$, *and*
3. *for every monomial $M = x_2^{u_2} x_3^{u_3} x_4^{u_4}$, where $u_2 \geq a_2$, $u_3 < a_{13}$ and $u_4 < a_{34}$, with $u_2 n_2 + u_3 n_3 + u_4 n_4 \in n_1 + \mathcal{S}$, there exists a monomial N with $1 \in \operatorname{supp}(N)$ such that $M - N \in I(C)$ and also $\deg(M) \leq \deg(N)$.*

Proof. In this case $I(C)$ is minimally generated by the set

$$G = \{f_1 = x_1^{a_1} - x_2^{a_{12}} x_3^{a_{13}}, f_2 = x_2^{a_2} - x_3^{a_{23}} x_4^{a_{24}}, f_3 = x_3^{a_3} - x_1^{a_{31}} x_4^{a_{34}},$$

$$f_4 = x_4^{a_4} - x_1^{a_{41}} x_2^{a_{42}}, f_5 = x_1^{a_{41}} x_3^{a_{23}} - x_2^{a_{12}} x_4^{a_{34}}\}.$$

Suppose that C has Cohen–Macaulay tangent cone at the origin. Since $I(C)$ is generated by the indispensable binomials, every binomial f_i, $1 \leq i \leq 5$, is indispensable of $I(C)$. In particular the binomials f_3 and f_5 are indispensable of $I(C)$. Therefore, the inequalities $a_3 \leq a_{31} + a_{34}$ and $a_{12} + a_{34} \leq a_{41} + a_{23}$ hold. By Theorem 1.1, condition (3) is also true.

To prove the converse statement, from Theorem 2.6 (i), it is enough to consider a monomial $M = x_2^{u_2} x_3^{u_3} x_4^{u_4}$, where $u_2 \geq a_2$, $u_3 < a_3$, and $u_4 < a_4$, with the following property: there exists at least one monomial P such that $1 \in \operatorname{supp}(P)$ and also $M - P$ is in $I(C)$. If $u_3 \geq a_{13}$, then we let $P = x_1^{a_1} x_2^{u_2 - a_{12}} x_3^{u_3 - a_{13}} x_4^{u_4}$. We have that $M - P \in I(C)$ and also $\deg(M) < \deg(P)$. Similarly, if $u_4 \geq a_{34}$, then we let $P = x_1^{a_{41}} x_2^{u_2 - a_{12}} x_3^{u_3 + a_{23}} x_4^{u_4 - a_{34}}$, so we have that $M - P \in I(C)$ and also $\deg(M) \leq \deg(P)$. If both conditions $u_3 < a_{13}$ and $u_4 < a_{34}$ hold, then condition (3) implies that there exists a monomial N with $1 \in \operatorname{supp}(N)$ such that $M - N \in I(C)$ and also $\deg(M) \leq \deg(N)$. Therefore, from Theorem 2.6, C has Cohen–Macaulay tangent cone at the origin. □

The proof of the next proposition is similar to that of Proposition 3.10 and therefore it is omitted.

Proposition 3.17 *Suppose that $I(C)$ is given as in case 2(a). Assume that C has Cohen–Macaulay tangent cone at the origin and also $a_{34} \leq a_{24}$.*

1. *If $a_{23} < a_{13}$, then $a_2 + a_{12} \leq a_{41} + 2a_{23} + a_{24} - a_{34}$.*
2. *If $a_{13} \leq a_{23}$, then $a_2 + a_{12} \leq a_1 + a_{23} - a_{13} + a_{24}$.*

Theorem 3.18 *Suppose that $I(C)$ is given as in case 2(a) and also that $a_{34} \leq a_{24}$. Then C has Cohen–Macaulay tangent cone at the origin if and only if*

1. $a_3 \leq a_{31} + a_{34}$,

2. $a_{12} + a_{34} \le a_{41} + a_{23}$, *and*

3. *either* $a_{23} < a_{13}$ *and* $a_2 + a_{12} \le a_{41} + 2a_{23} + a_{24} - a_{34}$ *or* $a_{13} \le a_{23}$ *and* $a_2 + a_{12} \le a_1 + a_{23} - a_{13} + a_{24}$.

Proof. ($\Longrightarrow$) From Proposition 3.16 we have that conditions (1) and (2) are true. From Proposition 3.17 condition (3) is also true.
($\Longleftarrow$) From Proposition 3.16 it is enough to consider a monomial $N = x_2^{u_2} x_3^{u_3} x_4^{u_4}$, where $u_2 \ge a_2$, $u_3 < a_{13}$, and $u_4 < a_{34}$, with the following property: there exists at least one monomial P such that $1 \in \text{supp}(P)$ and also $N - P$ is in $I(C)$. Suppose that $u_2 \ge a_2 + a_{12}$ and let M denote either the monomial $x_1^{a_{41}} x_3^{2a_{23}} x_4^{a_{24}-a_{34}}$ when $a_{23} < a_{13}$ or the monomial $x_1^{a_1} x_3^{a_{23}-a_{13}} x_4^{a_{24}}$ when $a_{13} \le a_{23}$. Let $P = x_2^{u_2-a_2-a_{12}} x_3^{u_3} x_4^{u_4} M$. We have that $N - P \in I(C)$ and

$$\deg(N) = u_2 + u_3 + u_4 \le u_2 + u_3 + u_4 + \deg(M) - a_2 - a_{12} = \deg(P).$$

It suffices to consider the case that $u_2 - a_2 < a_{12}$. Since the binomial $N - P$ belongs to $I(C)$, we have that $N - P = \sum_{i=1}^{5} H_i f_i$ for some polynomials $H_i \in K[x_1, \dots, x_4]$. Now the monomial N is divided by the monomial $x_2^{a_2}$, so $Q = -x_2^{u_2-a_2} x_3^{u_3+a_{23}} x_4^{u_4+a_{24}}$ is a term in the sum $\sum_{i=1}^{5} H_i f_i$ and should be canceled with another term of the above sum. Remark that $u_3 + a_{23} < a_3$ and $u_4 + a_{24} < a_4$. Thus, $x_2^{a_{12}} x_4^{a_{34}}$ divides $-Q$, so $u_2 - a_2 \ge a_{12}$, a contradiction. □

Proposition 3.19 *Suppose that* $I(C)$ *is given as in case 2(a) and also that* $a_{24} < a_{34}$. *If* C *has Cohen–Macaulay tangent cone at the origin, then*

1. $a_3 \le a_{31} + a_{34}$,

2. $a_{12} + a_{34} \le a_{41} + a_{23}$, *and*

3. $a_2 + a_{12} \le a_1 + a_{23} - a_{13} + a_{24}$.

Proof. From Proposition 3.16 we have that conditions (1) and (2) are true. Suppose first that $a_{13} \le a_{23}$. Assume that $a_2 + a_{12} > a_1 + a_{23} - a_{13} + a_{24}$. Since $x_2^{a_2+a_{12}} - x_1^{a_1} x_3^{a_{23}-a_{13}} x_4^{a_{24}} \in I(C)$, we deduce that $x_1^{a_1} x_3^{a_{23}-a_{13}} x_4^{a_{24}} \in I(C)_*$ and therefore $x_3^{a_{23}-a_{13}} x_4^{a_{24}} \in I(C)_*$. However, $G = \{x_1^{a_1} - x_2^{a_{12}} x_3^{a_{13}}, x_2^{a_2} - x_3^{a_{23}} x_4^{a_{24}}, x_3^{a_3} - x_1^{a_{31}} x_4^{a_{34}}, x_4^{a_4} - x_1^{a_{41}} x_2^{a_{42}}, x_1^{a_{41}} x_3^{a_{23}} - x_2^{a_{12}} x_4^{a_{34}}\}$ is a generating set for $I(C)$, so $x_3^{a_{23}-a_{13}} x_4^{a_{24}}$ must be divided by at least one of the monomials in G, a contradiction.
Suppose now that $a_{13} > a_{23}$. Note that $x_1^{a_1} x_4^{a_{24}} - x_2^{a_2+a_{12}} x_3^{a_{13}-a_{23}} \in I(C)$. Assume that $a_2 + a_{12} + a_{13} - a_{23} > a_1 + a_{24}$; then $x_1^{a_1} x_4^{a_{24}} \in I(C)_*$ and therefore $x_4^{a_{24}} \in I(C)_*$. However, G is a generating set for $I(C)$, so $x_4^{a_{24}}$ must be divided by at least one of the monomials in G, a contradiction. Thus, $a_2 + a_{12} + a_{13} - a_{23} \le a_1 + a_{24}$. □

Theorem 3.20 *Suppose that* $I(C)$ *is given as in case 2(a) and also that* $a_{24} < a_{34}$. *Assume that* $a_{13} \le a_{23}$. *Then* C *has Cohen–Macaulay tangent cone at the origin if and only if*

1. $a_3 \le a_{31} + a_{34}$,

2. $a_{12} + a_{34} \le a_{41} + a_{23}$, *and*

3. $a_2 + a_{12} \le a_1 + a_{23} - a_{13} + a_{24}$.

Proof. ($\Longrightarrow$) From Proposition 3.19 we have that conditions (1), (2), and (3) are true.
($\Longleftarrow$) From Proposition 3.16 it is enough to consider a monomial $N = x_2^{u_2}x_3^{u_3}x_4^{u_4}$, where $u_2 \geq a_2$, $u_3 < a_{13}$, and $u_4 < a_{34}$, with the following property: there exists at least one monomial P such that $1 \in \operatorname{supp}(P)$ and also $N - P$ is in $I(C)$. Suppose that $u_2 \geq a_2 + a_{12}$. Let $P = x_1^{a_1}x_2^{u_2-a_2-a_{12}}x_3^{u_3+a_{23}-a_{13}}x_4^{u_4+a_{24}}$. We have that $N - P \in I(C)$ and

$$\deg(N) = u_2 + u_3 + u_4 \leq u_2 + u_3 + u_4 + a_1 + a_{23} - a_{13} + a_{24} - a_2 - a_{12} = \deg(P).$$

It suffices to assume that $u_2 - a_2 < a_{12}$. Since the binomial $N - P$ belongs to $I(C)$, we have that $N - P = \sum_{i=1}^{5} H_i f_i$ for some polynomials $H_i \in K[x_1, \ldots, x_4]$. Then $T = -x_2^{u_2-a_2}x_3^{u_3+a_{23}}x_4^{u_4+a_{24}}$ is a term in the sum $\sum_{i=1}^{5} H_i f_i$ and it should be canceled with another term of the above sum. Thus, $x_2^{a_{12}}x_4^{a_{34}}$ divides $-T$, so $u_2 - a_2 \geq a_{12}$, a contradiction. $\square$

Example 3.21 Consider $n_1 = 627$, $n_2 = 1546$, $n_3 = 1662$, and $n_4 = 3377$. The toric ideal $I(C)$ is minimally generated by the set

$$G = \{x_1^{18} - x_2^3x_3^4, x_2^{25} - x_3^7x_4^8, x_3^{11} - x_1^{13}x_4^3, x_4^{11} - x_1^5x_2^{22}, x_1^5x_3^7 - x_2^3x_4^3\}.$$

Here $a_{24} = 8$, $a_{34} = 3$, $a_{12} = 3$, $a_1 = 18$, $a_2 = 25$, and $a_{13} = 4 < 7 = a_{23}$. Note that $a_3 = 11 < 14 = a_{31} + a_{34}$ and $a_{12} + a_{34} = 6 < 12 = a_{41} + a_{23}$. We have that $a_2 + a_{12} = 28 < 29 = a_1 + a_{23} - a_{13} + a_{24}$. Thus, C has a Cohen–Macaulay tangent cone at the origin. Remark that $x_2^{28} - x_1^5x_3^{14}x_4^5 \in I(C)$, but $\deg(x_2^{28}) = 28 > 24 = \deg(x_1^5x_3^{14}x_4^5)$.

Remark 3.22 Suppose that $I(C)$ is given as in case 2(b).
(1) It holds that $a_1 > a_{12} + a_{13}$, $a_4 < a_{41} + a_{43}$ and $a_{24} + a_{13} < a_{41} + a_{32}$.
(2) If $a_{24} \leq a_{34}$, then $x_3^{a_3+a_{13}} - x_1^{a_{41}}x_2^{2a_{32}}x_4^{a_{34}-a_{24}} \in I(C)$.
(3) If $a_{12} \leq a_{32}$, then the binomial $x_3^{a_3+a_{13}} - x_1^{a_1}x_2^{a_{32}-a_{12}}x_4^{a_{34}}$ belongs to $I(C)$.

Proposition 3.23 *Suppose that $I(C)$ is given as in case 2(b). Then C has Cohen–Macaulay tangent cone at the origin if and only if*

1. $a_2 \leq a_{21} + a_{24}$ *and*

2. *for every monomial $x_2^{u_2}x_3^{u_3}x_4^{u_4}$, where $u_2 < a_{12}$, $u_3 \geq a_3$, and $u_4 < a_{24}$, with $u_2n_2+u_3n_3+u_4n_4 \in n_1+\mathcal{S}$, there exists a monomial N with $1 \in \operatorname{supp}(N)$ such that $M - N \in I(C)$ and also $\deg(M) \leq \deg(N)$.*

Proof. In this case $I(C)$ is minimally generated by the set

$$G = \{f_1 = x_1^{a_1} - x_2^{a_{12}}x_3^{a_{13}}, f_2 = x_2^{a_2} - x_1^{a_{21}}x_4^{a_{24}}, f_3 = x_3^{a_3} - x_2^{a_{32}}x_4^{a_{34}},$$

$$f_4 = x_4^{a_4} - x_1^{a_{41}}x_3^{a_{43}}, f_5 = x_1^{a_{41}}x_2^{a_{32}} - x_3^{a_{13}}x_4^{a_{24}}\}.$$

Suppose that C has Cohen–Macaulay tangent cone at the origin. Since $I(C)$ is generated by the indispensable binomials, every binomial f_i, $1 \leq i \leq 5$, is indispensable of $I(C)$. In particular, the binomial f_2 is indispensable of $I(C)$. Therefore, the inequality $a_2 \leq a_{21} + a_{24}$ holds. By Theorem 1.1 condition (2) is also true.

Conversely, from Theorem 2.5 (iii), it is enough to consider a monomial $M = x_2^{u_2}x_3^{u_3}x_4^{u_4}$, where $u_2 < a_2$, $u_3 \geq 0$, and $u_4 < a_4$, with the following property: there exists at least one monomial P such that $1 \in \operatorname{supp}(P)$ and also $\deg_{\mathcal{S}}(M) = \deg_{\mathcal{S}}(P)$. Suppose that $u_3 \geq a_3$. If $u_2 \geq a_{12}$, then we let $P = x_1^{a_1}x_2^{u_2-a_{12}}x_3^{u_3-a_{13}}x_4^{u_4}$, so we have that $M - P \in I(C)$ and also $\deg(M) < \deg(P)$. Similarly, if $u_4 \geq a_{24}$, then we let $P = x_1^{a_{41}}x_2^{u_2+a_{32}}x_3^{u_3-a_{13}}x_4^{u_4-a_{24}}$, so we have that $M - P \in I(C)$ and also $\deg(M) < \deg(P)$. If both conditions $u_2 < a_{12}$ and $u_4 < a_{24}$ hold, then condition (2) implies that there exists a monomial N with $1 \in \operatorname{supp}(N)$ such that $M - N \in I(C)$ and also $\deg(M) \leq \deg(N)$. Suppose now that $u_3 < a_3$. Then M is divided by at least one of the monomials $x_2^{a_{12}}x_3^{a_{13}}$, $x_3^{a_{13}}x_4^{a_{24}}$, and $x_2^{a_{32}}x_4^{a_{34}}$. If M is divided by $x_2^{a_{12}}x_3^{a_{13}}$, then $M = x_2^{a_{12}+p}x_3^{a_{13}+q}x_4^{u_4}$, for some nonnegative integers p and q, so $M - x_1^{a_1}x_2^{p}x_3^{q}x_4^{u_4} \in I(C)$ and also $\deg(M) < \deg(x_1^{a_1}x_2^{p}x_3^{q}x_4^{u_4})$. If M is divided by $x_3^{a_{13}}x_4^{a_{24}}$, then $M = x_2^{u_2}x_3^{a_{13}+p}x_4^{a_{24}+q}$, for some nonnegative integers p and q, and therefore the binomial $M - x_1^{a_{41}}x_2^{u_2+a_{32}}x_3^{p}x_4^{q} \in I(C)$ and also $\deg(M) \leq \deg(x_1^{a_{41}}x_2^{u_2+a_{32}}x_3^{p}x_4^{q})$. Assume that neither $x_2^{a_{12}}x_3^{a_{13}}$ nor $x_3^{a_{13}}x_4^{a_{24}}$ divides M. Then necessarily $x_2^{a_{32}}x_4^{a_{34}}$ divides M. However, M is not divided by any leading monomial of G with respect to $lex-inf$ such that $x_1 >_{lex} x_2 >_{lex} x_3 >_{lex} x_4$. Thus, $m = u_2n_2 + u_3n_3 + u_4n_4$ is in Q, a contradiction to the fact that $m - n_1 \in \mathcal{S}$. By Theorem 2.5 C has Cohen–Macaulay tangent cone at the origin. □

The proof of the following proposition is similar to that of Proposition 3.10 and therefore it is omitted.

Proposition 3.24 *Suppose that $I(C)$ is given as in case 2(b). Assume that C has Cohen–Macaulay tangent cone at the origin and also $a_{24} \leq a_{34}$.*

1. *If $a_{32} < a_{12}$, then $a_3 + a_{13} \leq a_{41} + 2a_{32} + a_{34} - a_{24}$.*
2. *If $a_{12} \leq a_{32}$, then $a_3 + a_{13} \leq a_1 + a_{32} - a_{12} + a_{34}$.*

Theorem 3.25 *Suppose that $I(C)$ is given as in case 2(b) and also that $a_{24} \leq a_{34}$. Then C has Cohen–Macaulay tangent cone at the origin if and only if*

1. *$a_2 \leq a_{21} + a_{24}$ and*
2. *either $a_{32} < a_{12}$ and $a_3 + a_{13} \leq a_{41} + 2a_{32} + a_{34} - a_{24}$ or $a_{12} \leq a_{32}$ and $a_3 + a_{13} \leq a_1 + a_{32} - a_{12} + a_{34}$.*

Proof. ($\Longrightarrow$) From Proposition 3.23 we have that condition (1) is true. From Proposition 3.24 condition (3) is also true.
($\Longleftarrow$) From Proposition 3.23 it is enough to consider a monomial $N = x_2^{u_2}x_3^{u_3}x_4^{u_4}$, where $u_2 < a_{12}$, $u_3 \geq a_3$, and $u_4 < a_{24}$, with the following property: there exists at least one monomial P such that $1 \in \operatorname{supp}(P)$ and also $N - P$ is in $I(C)$. Suppose that $u_3 \geq a_3 + a_{13}$ and let M denote either the monomial $x_1^{a_{41}}x_2^{2a_{32}}x_4^{a_{34}-a_{24}}$ when $a_{32} < a_{12}$ or the monomial $x_1^{a_1}x_2^{a_{32}-a_{12}}x_4^{a_{34}}$ when $a_{12} \leq a_{32}$. Let $P = x_2^{u_2}x_3^{u_3-a_3-a_{13}}x_4^{u_4}M$. We have that $N - P \in I(C)$ and

$$\deg(N) = u_2 + u_3 + u_4 \leq u_2 + u_3 + u_4 + \deg(M) - a_3 - a_{13} = \deg(P).$$

It suffices to consider the case where $u_3 - a_3 < a_{13}$. Since the binomial $N - P$ belongs to $I(C)$, we have that $N - P = \sum_{i=1}^{5} H_i f_i$ for some polynomials $H_i \in K[x_1, \ldots, x_4]$. Now the monomial N is divided by the monomial $x_3^{a_3}$, so $Q = -x_2^{u_2+a_{32}}x_3^{u_3-a_3}x_4^{u_4+a_{34}}$ is a term in the sum $\sum_{i=1}^{5} H_i f_i$ and it should be canceled with another term of the above sum. Remark that $u_2 + a_{32} < a_2$ and $u_4 + a_{34} < a_4$. Thus, $x_3^{a_{13}} x_4^{a_{24}}$ divides $-Q$, so $u_3 - a_3 \geq a_{13}$, a contradiction. □

Proposition 3.26 *Suppose that $I(C)$ is given as in case 2(b) and also that $a_{34} < a_{24}$. If C has Cohen–Macaulay tangent cone then*

1. $a_2 \leq a_{21} + a_{24}$ *and*
2. $a_3 + a_{13} \leq a_1 + a_{32} - a_{12} + a_{34}$.

Theorem 3.27 *Suppose that $I(C)$ is given as in case 2(b) and also that $a_{34} < a_{24}$. Assume that $a_{12} \leq a_{32}$. Then C has Cohen–Macaulay tangent cone at the origin if and only if*

1. $a_2 \leq a_{21} + a_{24}$ *and*
2. $a_3 + a_{13} \leq a_1 + a_{32} - a_{12} + a_{34}$.

Example 3.28 Consider $n_1 = 813$, $n_2 = 1032$, $n_3 = 1240$, and $n_4 = 1835$. The toric ideal $I(C)$ is minimally generated by the set

$$G = \{x_1^{16} - x_2^9 x_3^3, x_2^{14} - x_1^{11} x_4^3, x_3^{16} - x_2^5 x_4^8, x_4^{11} - x_1^5 x_3^{13}, x_1^5 x_2^5 - x_3^3 x_4^3\}.$$

Here $a_{13} = a_{24} = 3$, $a_{34} = 8$, $a_{41} = 5$, $a_3 = 16$, and $a_{32} = 5 < 9 = a_{12}$. Note that $a_2 = 14 = a_{21} + a_{24}$. We have that $a_3 + a_{13} = 19 < 20 = a_{41} + 2a_{32} + a_{34} - a_{24}$. Consequently, C has a Cohen–Macaulay tangent cone at the origin.

Theorem 3.29 *Suppose that $I(C)$ is given as in case 3(a). Then C has Cohen–Macaulay tangent cone at the origin if and only if $a_2 \leq a_{21} + a_{23}$ and $a_3 \leq a_{31} + a_{34}$.*

Proof. In this case $I(C)$ is minimally generated by the set

$$G = \{f_1 = x_1^{a_1} - x_2^{a_{12}} x_4^{a_{14}}, f_2 = x_2^{a_2} - x_1^{a_{21}} x_3^{a_{23}}, f_3 = x_3^{a_3} - x_1^{a_{31}} x_4^{a_{34}},$$

$$f_4 = x_4^{a_4} - x_2^{a_{42}} x_3^{a_{43}}, f_5 = x_1^{a_{31}} x_2^{a_{42}} - x_3^{a_{23}} x_4^{a_{14}}\}.$$

If $a_2 \leq a_{21} + a_{23}$ and $a_3 \leq a_{31} + a_{34}$, then we have, from Theorem 2.10 in [1], that the curve C has Cohen–Macaulay tangent cone at the origin. Conversely, suppose that C has Cohen–Macaulay tangent cone at the origin. Since $I(C)$ is generated by the indispensable binomials, every binomial f_i, $1 \leq i \leq 5$, is indispensable of $I(C)$. In particular the binomials f_2 and f_3 are indispensable of $I(C)$. If there exists a monomial $N \neq x_1^{a_{31}} x_4^{a_{34}}$ such that $g = x_3^{a_3} - N$ belongs to $I(C)$, then we can replace f_3 in S by the binomials g and $N - x_1^{a_{31}} x_4^{a_{34}} \in I(C)$, a contradiction to the fact that f_3 is indispensable. Thus, $N = x_1^{a_{31}} x_4^{a_{34}}$ and therefore, from Theorem 1.1, we have that $a_3 \leq a_{31} + a_{34}$. Similarly, we get that $a_2 \leq a_{21} + a_{23}$. □

Remark 3.30 Suppose that $I(C)$ is given as in case 3(b).

(1) It holds that $a_1 > a_{12} + a_{14}$, $a_2 > a_{23} + a_{24}$, $a_3 < a_{31} + a_{32}$, and $a_4 < a_{41} + a_{43}$.

(2) If $a_{43} \leq a_{23}$, then $x_2^{a_2 + a_{12}} - x_1^{a_{31}} x_3^{a_{23} - a_{43}} x_4^{2a_{24}} \in I(C)$.

(3) If $a_{14} \leq a_{24}$, then the binomial $x_2^{a_2 + a_{12}} - x_1^{a_1} x_3^{a_{23}} x_4^{a_{24} - a_{14}}$ belongs to $I(C)$.

Proposition 3.31 *Suppose that $I(C)$ is given as in case 3(b). Then C has Cohen-0Macaulay tangent cone at the origin if and only if*

1. $a_{12}+a_{43}\leq a_{31}+a_{24}$ *and*
2. *for every monomial $M=x_2^{u_2}x_3^{u_3}x_4^{u_4}$, where $u_2\geq a_2$, $u_3<a_{43}$, and $u_4<a_{14}$, with $u_2n_2+u_3n_3+u_4n_4\in n_1+\mathcal{S}$, there exists a monomial N with $1\in\operatorname{supp}(N)$ such that $M-N\in I(C)$ and also $\deg(M)\leq\deg(N)$.*

Proof. In this case $I(C)$ is minimally generated by the set

$$G=\{f_1=x_1^{a_1}-x_2^{a_{12}}x_4^{a_{14}},f_2=x_2^{a_2}-x_3^{a_{23}}x_4^{a_{24}},f_3=x_3^{a_3}-x_1^{a_{31}}x_2^{a_{32}},$$

$$f_4=x_4^{a_4}-x_1^{a_{41}}x_3^{a_{43}},f_5=x_1^{a_{31}}x_4^{a_{24}}-x_2^{a_{12}}x_3^{a_{43}}\}.$$

Suppose that C has Cohen–Macaulay tangent cone at the origin. Since $I(C)$ is generated by the indispensable binomials, every binomial f_i, $1\leq i\leq 5$, is indispensable of $I(C)$. In particular the binomial f_5 is indispensable of $I(C)$. Therefore, the inequality $a_{12}+a_{43}\leq a_{31}+a_{24}$ holds. By Theorem 1.1 condition (2) is also true.

Conversely, from Theorem 2.6 (i), it is enough to consider a monomial $M=x_2^{u_2}x_3^{u_3}x_4^{u_4}$, where $u_2\geq a_2$, $u_3<a_3$, and $u_4<a_4$, with the following property: there exists at least one monomial P such that $1\in\operatorname{supp}(P)$ and also $M-P$ is in $I(C)$. If $u_3\geq a_{43}$, then we let $P=x_1^{a_{31}}x_2^{u_2-a_{12}}x_3^{u_3-a_{43}}x_4^{u_4+a_{24}}$, so we have that $M-P\in I(C)$ and also $\deg(M)\leq\deg(P)$. Similarly, if $u_4\geq a_{14}$, then we let $P=x_1^{a_1}x_2^{u_2-a_{12}}x_3^{u_3}x_4^{u_4-a_{14}}$, so we have that $M-P\in I(C)$ and also $\deg(M)<\deg(P)$. If both conditions $u_3<a_{43}$ and $u_4<a_{14}$ hold, then condition (2) implies that there exists a monomial N with $1\in\operatorname{supp}(N)$ such that $M-N\in I(C)$ and also $\deg(M)\leq\deg(N)$. Therefore, from Theorem 2.6, C has Cohen–Macaulay tangent cone at the origin. □

The proof of the next proposition is similar to that of Proposition 3.10 and therefore it is omitted.

Proposition 3.32 *Suppose that $I(C)$ is given as in case 3(b). Assume that C has Cohen–Macaulay tangent cone at the origin and also $a_{43}\leq a_{23}$.*

1. *If $a_{24}<a_{14}$, then $a_2+a_{12}\leq a_{31}+2a_{24}+a_{23}-a_{43}$.*
2. *If $a_{14}\leq a_{24}$, then $a_2+a_{12}\leq a_1+a_{23}+a_{24}-a_{14}$.*

Theorem 3.33 *Suppose that $I(C)$ is given as in case 3(b) and $a_{43}\leq a_{23}$. Then C has Cohen–Macaulay tangent cone at the origin if and only if*

1. $a_{12}+a_{43}\leq a_{31}+a_{24}$ *and*
2. *either $a_{24}<a_{14}$ and $a_2+a_{12}\leq a_{31}+2a_{24}+a_{23}-a_{43}$ or $a_{14}\leq a_{24}$ and $a_2+a_{12}\leq a_1+a_{23}+a_{24}-a_{14}$.*

Proof. ($\Longrightarrow$) From Proposition 3.31 we have that conditions (1) and (2) are true. From Proposition 3.32 condition (3) is also true.
($\Longleftarrow$) From Proposition 3.31 it is enough to consider a monomial $N=x_2^{u_2}x_3^{u_3}x_4^{u_4}$, where $u_2\geq a_2$, $u_3<a_{43}$, and $u_4<a_{14}$, with the following property: there exists at least one monomial P such that $1\in\operatorname{supp}(P)$ and also $N-P$ is in $I(C)$. Suppose that $u_2\geq a_2+a_{12}$ and let M denote either the monomial $x_1^{a_{31}}x_3^{a_{23}-a_{43}}x_4^{2a_{24}}$

when $a_{24} < a_{14}$ or the monomial $x_1^{a_{31}}x_3^{a_{23}}x_4^{a_{24}-a_{14}}$ when $a_{14} \leq a_{24}$. Let $P = x_2^{u_2-a_2-a_{12}}x_3^{u_3}x_4^{u_4}M$. We have that $N - P \in I(C)$ and

$$\deg(N) = u_2 + u_3 + u_4 \leq u_2 + u_3 + u_4 + \deg(M) - a_2 - a_{12} = \deg(P).$$

It suffices to consider the case where $u_2 - a_2 < a_{12}$. Since the binomial $N - P$ belongs to $I(C)$, we have that $N - P = \sum_{i=1}^{5} H_i f_i$ for some polynomials $H_i \in K[x_1, \ldots, x_4]$. Now the monomial N is divided by the monomial $x_2^{a_2}$, so $Q = -x_2^{u_2-a_2}x_3^{u_3+a_{23}}x_4^{u_4+a_{24}}$ is a term in the sum $\sum_{i=1}^{5} H_i f_i$ and it should be canceled with another term of the above sum. Remark that $u_3 + a_{23} < a_3$ and $u_4 + a_{24} < a_4$. Thus, $x_2^{a_{12}}x_3^{a_{43}}$ divides $-Q$, so $u_2 - a_2 \geq a_{12}$, a contradiction. □

Proposition 3.34 *Suppose that $I(C)$ is given as in case 3(b) and also that $a_{23} < a_{43}$. If C has Cohen–Macaulay tangent cone at the origin then*

1. $a_{12} + a_{43} \leq a_{31} + a_{24}$ *and*
2. $a_2 + a_{12} \leq a_1 + a_{23} + a_{24} - a_{14}$.

Theorem 3.35 *Suppose that $I(C)$ is given as in case 3(b) and also that $a_{23} < a_{43}$. Assume that $a_{14} \leq a_{24}$. Then C has Cohen–Macaulay tangent cone at the origin if and only if*

1. $a_{12} + a_{43} \leq a_{31} + a_{24}$ *and*
2. $a_2 + a_{12} \leq a_1 + a_{23} + a_{24} - a_{14}$.

4. Families of monomial curves supporting Rossi's problem

In this section, we give some examples showing how the criteria given in the previous section can be used to give families of monomial curves supporting Rossi's conjecture in $\mathbb{A}^4(K)$.

Example 4.1 Consider the family $n_1 = m^3 + m^2 - m$, $n_2 = m^3 + 2m^2 + m - 1$, $n_3 = m^3 + 3m^2 + 2m - 2$, and $n_4 = m^3 + 4m^2 + 3m - 2$ for $m \geq 2$ given in [1]. Let C_m be the corresponding monomial curve. The toric ideal $I(C_m)$ is generated by the set

$$S_m = \{x_1^{m+3} - x_3x_4^{m-1}, x_2^{m+2} - x_1^{m+2}x_4, x_3^m - x_1x_2^m, x_4^m - x_2^2x_3^{m-1},$$

$$x_1^{m+2}x_3^{m-1} - x_2^m x_4^{m-1}\}.$$

Thus, we are in case 1(a) of Remark 3.3 and it is sufficient to consider the binomial $x_2^{m+2} - x_1^{m+2}x_4$, which guarantees that C_m has Cohen–Macaulay tangent cone. For each fixed m, by using the technique given in [6], we can construct a new family of monomial curves having Cohen–Macaulay tangent cone. For $m = 2$, we have the symmetric semigroup generated by $n_1 = 10$, $n_2 = 17$, $n_3 = 22$, and $n_4 = 28$. The corresponding monomial curve is C_2 and $I(C_2)$ is minimally generated by the set

$$S_2 = \{x_1^5 - x_3x_4, x_2^4 - x_1^4x_4, x_3^2 - x_1x_2^2, x_4^2 - x_2^2x_3, x_1^4x_3 - x_2^2x_4\}.$$

By the method given in [6], the semigroup generated by $m_1 = 10+6t$, $m_2 = 17+9t$, $m_3 = 22+6t$, and $m_4 = 28+12t$ (for t a nonnegative integer) is symmetric, whenever $\gcd(10+6t, 17+9t, 22+6t, 28+12t) = 1$. Moreover, the toric ideal of the corresponding monomial curve is minimally generated by the set

$$\{x_1^{t+5} - x_3^{t+1}x_4, x_2^4 - x_1^4x_4, x_3^{t+2} - x_1^{t+1}x_2^2, x_4^2 - x_2^2x_3, x_1^4x_3 - x_2^2x_4\}$$

by the construction in [6]. Here, if $t=1$, then $m_1 < m_2 < m_3 < m_4$, and we are in case 1(a) again. From the binomial $x_2^4 - x_1^4x_4$, we deduce that the corresponding monomial curve has Cohen–Macaulay tangent cone. If $t \geq 1$, then $m_2 > m_3$. In this case, we interchange them to get the semigroup generated by $m_1' = 10+6t$, $m_2' = 22+6t$, $m_3' = 17+9t$, and $m_4' = 28+12t$ with $m_1' < m_2' < m_3' < m_4'$. Thus, the toric ideal of the corresponding monomial curve is generated by the set

$$\{x_1^{t+5} - x_2^{t+1}x_4, x_2^{t+2} - x_1^{t+1}x_3^2, x_3^4 - x_1^4x_4, x_4^2 - x_2x_3^2, x_1^4x_2 - x_3^2x_4\}.$$

Now we are in case 3(a) of Remark 3.3 and the binomials $x_3^4 - x_1^4x_4$ and $x_2^{t+2} - x_1^{t+1}x_3^2$ guarantee that the corresponding monomial curve has Cohen–Macaulay tangent cone. In this way, we can construct infinitely many families of Gorenstein monomial curves having Cohen–Macaulay tangent cones. In other words, the corresponding local rings have nondecreasing Hilbert functions supporting Rossi's problem.

In the literature, there are no examples of noncomplete intersection Gorenstein monomial curve families supporting Rossi's problem, although their tangent cones are not Cohen–Macaulay. The next example gives a family of monomial curves with the above property.

Example 4.2 Consider the family $n_1 = 2m+1$, $n_2 = 2m+3$, $n_3 = 2m^2+m-2$, and $n_4 = 2m^2+m-1$, where $m \geq 4$ is an integer. Let C_m be the corresponding monomial curve. The toric ideal $I(C_m)$ is minimally generated by the binomials

$$x_1^{m+1} - x_2x_3, x_2^m - x_1x_4, x_3^2 - x_2^{m-1}x_4, x_4^2 - x_1^mx_3, x_1^mx_2^{m-1} - x_3x_4.$$

Thus, we are in Case 2(b) of Remark 3.3. Consider the binomial $x_2^m - x_1x_4$. Since $m \geq 4$, we have, from Theorem 3.23, that the tangent cone of C_m is not Cohen–Macaulay. It is enough to show that the Hilbert function of $K[x_1,x_2,x_3,x_4]/I(C_m)_*$ is nondecreasing. By a standard basis computation, $I(C_m)_*$ is generated by the set

$$\{x_2x_3, x_3^2, x_1x_4, x_3x_4, x_4^2, x_2^mx_4, x_1^{m+2}x_3, x_2^{2m+1}\}.$$

Let

$$J_0 = I(C_m)_*, J_1 = \langle x_3^2, x_1x_4, x_3x_4, x_4^2, x_2^mx_4, x_1^{m+2}x_3, x_2^{2m+1}\rangle,$$

$$J_2 = \langle x_3^2, x_3x_4, x_4^2, x_2^mx_4, x_1^{m+2}x_3, x_2^{2m+1}\rangle, J_3 = \langle x_3^2, x_4^2, x_2^mx_4, x_1^{m+2}x_3, x_2^{2m+1}\rangle.$$

Note that $J_i = \langle J_{i+1}, q_i\rangle$, where $q_0 = x_2x_3$, $q_1 = x_1x_4$, and $q_2 = x_3x_4$. We apply [3, Proposition 2.2] to the ideal J_i for $0 \leq i \leq 2$, so

$$p(J_i) = p(J_{i+1}) - t^2 p(J_{i+1} : q_i). \tag{1}$$

In this case, we have $J_1 : (x_2x_3) = \langle x_3, x_4, x_1^{m+2}, x_2^{2m}\rangle$, $J_2 : (x_1x_4) = \langle x_3, x_4, x_2^m\rangle$, and $J_3 : (x_3x_4) = \langle x_3, x_4, x_1^{m+2}, x_2^m\rangle$. Since

$$K[x_1,x_2,x_3,x_4]/\langle x_3^2, x_1^{m+2}x_3, x_4^2, x_2^mx_4, x_2^{2m+1}\rangle$$

is isomorphic to

$$K[x_1, x_3]/\langle x_3^2, x_1^{m+2}x_3\rangle \otimes K[x_2, x_4]/\langle x_4^2, x_2^m x_4, x_2^{2m+1}\rangle,$$

we obtain

$$p(J_3) = (1-t)^3(1+t-t^{m+3})(1+2t+\cdots+2t^m+t^{m+1}+\cdots+t^{2m}).$$

Substituting all these recursively in Eq. (1), we obtain that the Hilbert series of $K[x_1, x_2, x_3, x_4]/J_0$ is

$$\frac{1+3t+t^2+t^3+\cdots+t^m+t^{m+2}+t^{m+4}+t^{m+5}+\cdots+t^{2m}}{1-t}.$$

Since the numerator does not have any negative coefficients, the Hilbert function is nondecreasing. In this way, we have shown that the Hilbert function of the local ring corresponding to the noncomplete intersection Gorenstein monomial curve C_m is nondecreasing for $m \geq 4$.

References

[1] Arslan F, Mete P. Hilbert functions of Gorenstein monomial curves. Proceedings of the American Mathematical Society 2007; 135: 1993-2002.

[2] Barucci V, Fröberg R. Associated graded rings of one-dimensional analytically irreducible rings. Journal of Algebra 2006; 304: 349-358.

[3] Bayer D, Stillman M. Computation of Hilbert functions. Journal of Symbolic Computation 1992; 14: 31-50.

[4] Bresinsky H. Symmetric semigroups of integers generated by 4 elements. Manuscripta Mathematica 1975; 17 (3): 205-219.

[5] Garcia A. Cohen-Macaulayness of the associated graded of a semigroup rings. Communications in Algebra 1982; 10: 393-415.

[6] Gimenez P, Srinivasan H. A note on Gorenstein monomial curves. Bulletin of the Brazilian Mathematical Society, New Series 2014; 4: 671-678.

[7] Herzog J. Generators and relations of abelian semigroups and semigroup rings. Manuscripta Mathematica 1970; 3: 175-193.

[8] Herzog J. When is a regular sequence super regular? Nagoya Mathematical Journal 1981; 83: 183-195.

[9] Herzog J, Stamate DI. On the defining equations of the tangent cone of a numerical semigroup ring. Journal of Algebra 2014; 418: 8-28.

[10] Jafari R, Zarzuela S. On monomial curves obtained by gluing. Semigroup Forum 2014; 2: 397-416.

[11] Katsabekis A, Ojeda I. An indispensable classification of monomial curves in $\mathbb{A}^4(K)$. Pacific Journal of Mathematics 2014; 268: 95-116.

[12] Kunz E. The value semigroup of one dimensional Gorenstein ring. Proceedings of the American Mathematical Society 1970; 25: 748-751.

[13] Molinelli S, Tamone G. On the Hilbert function of certain rings of monomial curves. Journal of Pure and Applied Algebra 1995; 101: 191-206.

[14] Oneto A, Strazzanti F, Tamone G. One-dimensional Gorenstein local rings with decreasing Hilbert function. Journal of Algebra 2017; 489: 91-114.

[15] Patil DP, Tamone G. CM defect and Hilbert functions of monomial curves. Journal of Pure and Applied Algebra 2011; 215 (7): 1539-1551.

[16] Pisón Casares P. The short resolution of a lattice ideal. Proceedings of the American Mathematical Society 2003; 131: 1081-1091.

[17] Rossi M. Hilbert functions of Cohen-Macaulay local rings. Commutative Algebra and its Connections to Geometry, Contemporary Mathematics 2011; 555: 173-200.

[18] Şahin M, Şahin N. On pseudo symmetric monomial curves. Communications in Algebra 2018; 46 (6): 2561-2573.

[19] Sharifan L, Zaare-Nahandi R. Minimal free resolution of the associated graded ring of monomial curves of generalized arithmetic sequences. Journal of Pure and Applied Algebra 2009; 213: 360-369.

[20] Sturmfels B. Gröbner Bases and Convex Polytopes. University Lecture Series Vol. 8. Providence, RI, USA: American Mathematical Society, 1996.