On image summand coinvariant modules and kernel summand invariant modules

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Abstract: In this paper we introduce the concept of im-summand coinvariance and im-small coinvariance; that is, a module M over a right perfect ring is said to be im-summand (im-small) coinvariant if, for any endomorphism φ of P such that $\operatorname{Im}\varphi$ is a direct summand (a small submodule) of P, $\varphi(\ker\nu) \subseteq \ker\nu$, where (P,ν) is the projective cover of M. We first give some fundamental properties of im-summand coinvariant modules and im-small coinvariant modules, and we prove that, for modules M and N over a right perfect ring such that N is a small epimorphic image of M, M is N-im-summand coinvariant if and only if M is (im-coclosed) N-projective. Moreover, we introduce ker-summand invariance and ker-essential invariance as the dual concept of im-summand coinvariance and im-small coinvariance, respectively, and show that, for modules M and N such that N is isomorphic to an essential submodule of M, M is N-ker-summand invariant if and only if M is (ker-closed) N-injective.

Key words: Ker-summand invariant, im-summand coinvariant, quasi-injective, quasi-projective, pseudo-injective, pseudo-projective, perfect ring

1. Preliminaries

In 1961, Johnson and Wong [13] showed that quasi-injective modules are fully invariant submodules of their injective hulls. After that, Dickson and Fuller [5] considered that a module is invariant under any automorphism of its injective hull. Such a module is called automorphism invariant. In 2013, Er et al. [7] proved that a module is automorphism invariant if and only if it is pseudo-injective, and Lee and Zhou [19] showed that for an extending module M, M is automorphism invariant iff M is quasi-injective. Moreover, Singh and Srivastava [22] introduced a dual notion of an automorphism invariant module and proved that a lifting module over a right perfect ring is dual automorphism invariant if and only if it is quasi-projective. After that, Guil Asensio et al. [9] showed that a module over a right perfect ring is dual automorphism invariant if and only if it is pseudo-projective. In this paper, we consider relationships between several relative injectivities and the invariance for certain homomorphisms in their injective hulls, and dually study several relative projectivities from the viewpoint of the dual invariant in their projective covers.

We consider associative rings R with identity and all modules considered are unitary right R-modules. The notations $N \subseteq M$, $N \subseteq_e M$, and $N \subseteq_\oplus M$ mean that N is a submodule of M, an essential submodule of M, and a direct summand of M, respectively. We will denote by $E(M)$ the injective hull of a module M. A

submodule A of a module M is said to be *small* in M if $A + B \neq M$ for any proper submodule B of M and we write $A \ll M$ in this case. Let A and B be submodules of M with $A \subseteq B$. A is said to be a *coessential* submodule of B in M if $B/A \ll M/A$ and we denote it by $A \subseteq_c^M B$ in this case. B is said to be a *coclosed* submodule in M if, whenever $A \subseteq_c^M B$, then $B = A$, and we denote it by $B \subseteq_{cc} M$ in this case. A module M is called *lifting* if, for any submodule X of M, there exists a direct summand A of M such that $A \subseteq_c^M X$. Let $\{M_i \mid i \in I\}$ be a family of modules. The direct sum decomposition $M = \oplus_I M_i$ is said to be *exchangeable* if, for any direct summand X of M, there exist $M_i' \subseteq M_i$ $(i \in I)$ such that $M = X \oplus (\oplus_I M_i')$. A module M is said to have the *(finite) internal exchange property* (or briefly (F)IEP) if any (finite) direct sum decomposition $M = \oplus_I M_i$ is exchangeable. It is well known that any projective module over a right perfect ring is a lifting module with IEP and any injective module satisfies IEP (cf. [2, Theorems 1.2.17 and 1.2.19], [20, Theorems 1.21 and 4.41]). For the dual notions to the above notions we refer to [20, 24].

Let A and B be modules. An epimorphism $f : A \to B$ is called a *small epimorphism* if $\ker f$ is small in A. A is said to be *(epi-)B-projective* if, for any module X, any homomorphism (epimorphism) $f : A \to X$, and any epimorphism $g : B \to X$, there exists a homomorphism $h : A \to B$ such that $gh = f$. A is said to be *quasi-(pseudo)-projective* if A is (epi-)A-projective. A is said to be *radical B-projective* if, for any module X, any homomorphism $f : A \to X$, and any epimorphism $g : B \to X$, there exists a homomorphism $h : A \to B$ such that Im $(f - gh) \ll X$ (cf. [11, 16]), and equivalently, for any epimorphism $g : B \to X$ and any homomorphism $f : A \to X$, there exist a small epimorphism $\rho : X \to Y$ for some module Y, a homomorphism $h : A \to B$ such that $\rho g h = \rho f$ ([17, Proposition 1.2]). A is said to be *im-summand (im-coclosed, im-small,* resp.) *B-projective* if, for any module X, any homomorphism $f : A \to X$ with Im$f \subseteq_\oplus X$ (Im$f \subseteq_{cc} X$, Im$f \ll X$, resp.), and any epimorphism $g : B \to X$, there exists a homomorphism $h : A \to B$ such that $gh = f$. Note that there exist modules A and B such that A is epi-B-projective but A is not im-small B-projective (see [14, Example 2.7]). Let A and B be modules. A is said to be *(mono-)B-injective* if, for any module X, any homomorphism (monomorphism) $f : X \to A$, and any monomorphism $g : X \to B$, there exists a homomorphism $h : B \to A$ such that $hg = f$. A is said to be *quasi (pseudo)-injective* if A is (mono-)A-injective. A is said to be *B-ejective* if, for any submodule X of B and any homomorphism $f : X \to A$, there exist an essential submodule X' of X and a homomorphism $g : B \to A$ such that $g|_{X'} = f|_{X'}$ (see [1]). A is said to be *ker-summand (ker-closed, essentially,* resp.) *B-injective* if, for any submodule X of B and any homomorphism $f : X \to A$ with $\ker f \subseteq_\oplus X$ ($\ker f$ is closed in X, $\ker f \subseteq_e X$, resp.), there exists a homomorphism $g : B \to A$ such that $g|_X = f$ (see [6, 20, 24]).

A module M is called *dual automorphism-invariant* if, for any small submodules K_1, K_2 of M and any small epimorphism $f : M/K_1 \to M/K_2$, there exists an endomorphism g of M such that $f\pi_1 = \pi_2 g$, where $\pi_i : M \to M/K_i$ $(i = 1, 2)$ is the natural epimorphism ([22]). Guil Asensio et al. [9] called a dual automorphism-invariant module over a right perfect ring as *automorphism coinvariant* and proved that over a right perfect ring, a module M is automorphism coinvariant if and only if M is pseudo-projective. For the notion of automorphism invariant modules we refer to [7, 23]. Note that a module M is automorphism invariant if and only if it is pseudo-injective (see [7, Theorem 16]).

In this work, we introduce N-im-small coinvariant (N-ker-essential invariant) modules and N-im-summand coinvariant (N-ker-summand invariant) modules for any module N. Let M and N be two modules. Assume that (P, p) and (Q, q) are projective covers of M and N, respectively. M is called *N-im-small coinvariant* if for any homomorphism $\varphi : P \to Q$ with Im$\varphi \ll Q$, $\varphi(\ker p) \subseteq \ker q$. M is called *N-im-*

summand coinvariant if for any homomorphism $\varphi : P \to Q$ with $\text{Im}\varphi \subseteq_{\oplus} Q$, $\varphi(\ker p) \subseteq \ker q$. We can see that these coinvariances do not depend on the way of taking projective covers. M is called *im-small (im-summand) coinvariant* if M is M-im-small (M-im-summand) coinvariant. Dually, M is called N-*ker-essential invariant* if for any homomorphism $\varphi : E(N) \to E(M)$ with $\ker \varphi \subseteq_e E(N)$, $\varphi(N) \subseteq M$. M is called N-*ker-summand invariant* if for any homomorphism $\varphi : E(N) \to E(M)$ with $\ker \varphi \subseteq_{\oplus} E(N)$, $\varphi(N) \subseteq M$. M is called *ker-essential (ker-summand) invariant* if M is M-ker-essential (M-ker-summand) invariant.

In Section 2, we first give some fundamental properties of im-summand coinvariant modules and im-small coinvariant modules over right perfect rings and prove that, for modules M and N over a right perfect ring such that N is a small epimorphic image of M, M is N-im-summand coinvariant if and only if M is (im-coclosed) N-projective. This immediately follows that a module M over a right perfect ring is im-summand coinvariant if and only if M is quasi-projective. In Section 3, we consider ker-summand invariance and ker-essential invariance for modules over any ring as the dual concept of im-summand coinvariance and im-small coinvariance, respectively. In addition, using some fundamental properties of them, we show that for modules M and N such that N is isomorphic to an essential submodule of M, M is N-ker-summand invariant if and only if M is (ker-closed) N-injective. In particular, a module M is ker-summand invariant if and only if M is quasi-injective.

For undefined terminologies, the reader is referred to [2, 3, 20, 24, 25].

2. Im-small coinvariant and im-summand coinvariant modules

We first give some fundamental properties of im-small coinvariant modules and im-summand coinvariant modules.

Proposition 2.1 *Let M, N, M_i $(i \in I)$, and N_j $(j \in J)$ be modules over a right perfect ring. Then:*

(1) *If M is N-im-small coinvariant, then M is N/X-im-small coinvariant and X-im-small coinvariant for any submodule X of N.*

(2) *If M is N-im-summand coinvariant, then M is N/X-im-summand coinvariant for any submodule X of N. Moreover, for any coclosed submodule X of N, M is X-im-summand coinvariant.*

(3) *If M/S is N-im-small (N-im-summand, resp.) coinvariant for some $S \ll M$, then M is N-im-small (N-im-summand, resp.) coinvariant.*

(4) *If M is N-im-small (N-im-summand, resp.) coinvariant, then M' is N-im-small (N-im-summand, resp.) coinvariant for any direct summand M' of M.*

(5) *If M_i is N_j-im-small coinvariant for any $i \in I, j \in J$, then $\oplus_I M_i$ is $\oplus_J N_j$-im-small coinvariant.*

Proof (1) Let X be a submodule of N and let (P, p), (Q, q), and (Q', q') be the projective covers of M, N, and N/X, respectively. Let $\varphi : P \to Q'$ be a homomorphism with $\text{Im}\varphi \ll Q'$. Since Q is projective, there exists a homomorphism $f : Q \to Q'$ such that $q'f = \nu q$, where $\nu : N \to N/X$ is the natural epimorphism. By $\ker q' \ll Q'$ and νq is onto, f is an epimorphism. Hence, there exists a monomorphism $g : Q' \to Q$ such that $fg = 1_{Q'}$. Since M is N-im-small coinvariant, we see $g\varphi(\ker p) \subseteq \ker q$. Thus, $\varphi(\ker p) = fg\varphi(\ker p) \subseteq f(\ker q)$. By $q'f(\ker q) = \nu q(\ker q) = 0$, $f(\ker q) \subseteq \ker q'$ and hence $\varphi(\ker p) \subseteq \ker q'$.

Thus, M is N/X-im-small coinvariant. Next we show that M is X-im-small coinvariant. Let (Q'', q'') be the projective cover of X and let $\psi : P \to Q''$ be a homomorphism with $\operatorname{Im}\psi \ll Q''$. Since Q'' is projective, there exists a homomorphism $h : Q'' \to Q$ such that $qh = q''$. As M is N-im-small coinvariant, $h\psi(\ker p) \subseteq \ker q$ and hence $\psi(\ker p) \subseteq \ker(qh) = \ker q''$. Thus, M is X-im-small coinvariant.

(2) We can see that M is N/X-im-summand coinvariant for any submodule X of N by a similar proof of (1). Let X be a coclosed submodule of N and let (P, p) and (Q, q) be projective covers of M and N, respectively. Since Q is lifting, by [3, 3.2 (7)], there exists a direct summand T of Q such that $X = q(T)$. Then $(T, q|_T)$ is the projective cover of X. For any homomorphism $f : P \to T$ with $\operatorname{Im}f \subseteq_{\oplus} T$, since M is N-im-summand coinvariant, $f(\ker p) \subseteq \ker q$. On the other hand, since $f(\ker p) \subseteq T$, we see $f(\ker p) \subseteq \ker q \cap T = \ker q|_T$. Thus, M is X-im-summand coinvariant.

(3) We prove only for the case of N-im-small coinvariant. Let S be a small submodule of M and let (P, p), (Q, q) be the projective covers of M, N, respectively. By $S \ll M$, $(P, \nu p)$ is the projective cover of M/S, where $\nu : M \to M/S$ is the natural epimorphism. Let $\varphi : P \to Q$ be a homomorphism with $\operatorname{Im}\varphi \ll Q$. Since M/S is N-im-small coinvariant, $\varphi(\ker \nu p) \subseteq \ker q$ and hence $\varphi(\ker p) \subseteq \ker q$. Thus, M is N-im-small coinvariant.

(4) We prove only for the case of N-im-small coinvariant. Let $M = M' \oplus M''$ and let (P', p'), (P'', p''), (Q, q) be the projective covers of M', M'', N, respectively. Let $\varphi : P' \to Q$ be a homomorphism with $\operatorname{Im}\varphi \ll Q$. Put $P = P' \oplus P''$ and $p = p' \oplus p''$. Then (P, p) is the projective cover of M. Since M is N-im-small coinvariant, $(\varphi \oplus 0)(\ker p) \subseteq \ker q$. Hence, $\varphi(\ker p') \subseteq \ker q$. Thus, M' is N-im-small coinvariant.

(5) First we show if each M_i is N-im-small coinvariant, then so is $\oplus_I M_i$. Let (P_i, p_i) and (Q, q) be the projective covers of M_i $(i \in I)$ and N, respectively. Put $M = \oplus_I M_i$, $P = \oplus_I P_i$, and $p = \oplus_I p_i$. Then (P, p) is the projective cover of M. Let $\varphi : P \to Q$ with $\operatorname{Im}\varphi \ll Q$. Since each M_i is N-im-small coinvariant, $(\varphi|_{P_i})(\ker p_i) \subseteq \ker q$. Hence, $\varphi(\ker p) = \varphi(\oplus_I \ker p_i) = \sum_I \varphi(\ker p_i) \subseteq \ker q$.

Next we show if M is N_j-im-small coinvariant for any $j \in J$, then M is $\oplus_J N_j$-im-small coinvariant. Let (P, p) and (Q_j, q_j) be the projective covers of M and N_j $(j \in J)$, respectively. Put $N = \oplus_J N_j$, $Q = \oplus_J Q_j$, and $q = \oplus_J q_j$. Then (Q, q) is the projective cover of N. Let $\pi_k : Q = \oplus_J Q_j \to Q_k$ $(k \in J)$ be the projection and $\varphi : P \to Q$ with $\operatorname{Im}\varphi \ll Q$. Since M is N_j-im-small coinvariant, $(\pi_j \varphi)(\ker p) \subseteq \ker q_j$. Hence, $\varphi(\ker p) \subseteq \sum_J \pi_j \varphi(\ker p) \subseteq \sum_J \ker q_j = \ker q$. $\qquad\square$

Let $M = A \oplus B$ and let $\varphi : A \to B$ be a homomorphism. Then $\langle A \overset{\varphi}{\to} B \rangle = \{a - \varphi(a) \mid a \in A\}$ is a submodule of M, which is isomorphic to A, and we note that $M = A \oplus B = \langle A \overset{\varphi}{\to} B \rangle \oplus B$.

Lemma 2.2 *Let M_1, M_2 be modules and put $M = M_1 \oplus M_2$. If $M = X \oplus M_1'' \oplus M_2''$ for some $X \subseteq M$ and $M_i'' \subseteq M_i$ $(i = 1, 2)$, then there exist $M_i' \subseteq M_i$ $(i = 1, 2)$ and a homomorphism $\alpha_i : M_i' \to M_j''$ $(i \neq j)$ such that $M_i = M_i' \oplus M_i''$ and $X = \langle M_1' \overset{\alpha_i}{\to} M_2'' \rangle \oplus \langle M_2' \overset{\alpha_i}{\to} M_1'' \rangle$.*

Proof Let $M = X \oplus M_1'' \oplus M_2''$, where $M_i'' \subseteq M_i$, and put $M_i = A_i \oplus M_i''$ $(i = 1, 2)$. Let $p : M = A_1 \oplus A_2 \oplus M_1'' \oplus M_2'' \to A_1 \oplus A_2$ and $q : M = A_1 \oplus A_2 \oplus M_1'' \oplus M_2'' \to M_1'' \oplus M_2''$ be the projections. Then we can define a homomorphism $f : p(X) \to q(X)$ by $f(p(x)) = -q(x)$, where $x \in X$, and so we see

$$X = \langle p(X) \overset{f}{\to} q(X) \rangle = \langle A_1 \oplus A_2 \overset{f}{\to} M_1'' \oplus M_2'' \rangle = \langle A_1 \overset{f|_{A_1}}{\to} M_1'' \oplus M_2'' \rangle \oplus \langle A_2 \overset{f|_{A_2}}{\to} M_1'' \oplus M_2'' \rangle.$$

Let $\pi_i : M_1'' \oplus M_2'' \to M_i''$ be the projection and let $\beta_i : \langle A_i \overset{\pi_i f}{\to} M_i'' \rangle \to A_i$ be the natural isomorphism $(i = 1, 2)$. Then we obtain $\langle A_i \overset{f|_{A_i}}{\to} M_1'' \oplus M_2'' \rangle = \langle\langle A_i \overset{\pi_i f}{\to} M_i'' \rangle \overset{\pi_j f \beta_i}{\to} M_j'' \rangle$ $(i \neq j)$. Put $M_i' = \langle A_i \overset{\pi_i f}{\to} M_i'' \rangle$, $\alpha_i = \pi_j f \beta_i$ $(i \neq j)$ and then $M_i = M_i' \oplus M_i''$ $(i = 1, 2)$ and $X = \langle M_1' \overset{\alpha_1}{\to} M_2'' \rangle \oplus \langle M_2' \overset{\alpha_2}{\to} M_1'' \rangle$. □

Proposition 2.3 *Let R be a right perfect ring and let $M_1, M_2, \ldots, M_m$, $N_1, N_2, \ldots, N_n$ be modules. If M_i is N_j-im-summand coinvariant and N_j-im-small coinvariant $(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$, then $\oplus_{i=1}^m M_i$ is $\oplus_{j=1}^n N_j$-im-summand coinvariant.*

Proof It is enough to show the case of $m = n = 2$, by Proposition 2.1(5).

First we show that if M is N_i-im-summand coinvariant and N_i-im-small coinvariant $(i = 1, 2)$, then M is $N_1 \oplus N_2$-im-summand coinvariant. Let (P, p) and (Q_i, q_i) be the projective covers of M and N_i $(i = 1, 2)$, respectively, and put $Q = Q_1 \oplus Q_2$, $q = q_1 \oplus q_2$ and let $\varphi : P \to Q$ be a homomorphism with $\varphi(P) \subseteq_\oplus Q$. Since Q satisfies FIEP, there exists $Q_i'' \subseteq Q_i$ $(i = 1, 2)$ such that $Q = \varphi(P) \oplus Q_1'' \oplus Q_2''$. By Lemma 2.2, there exist a direct summand Q_i' of Q_i and a homomorphism $\alpha_i : Q_i' \to Q_j''$ $(i \neq j)$ such that $Q_i = Q_i' \oplus Q_i''$ and $\varphi(P) = \langle Q_1' \overset{\alpha_1}{\to} Q_2'' \rangle \oplus \langle Q_2' \overset{\alpha_2}{\to} Q_1'' \rangle$. Let $\pi_i : \varphi(P) = \langle Q_1' \overset{\alpha_1}{\to} Q_2'' \rangle \oplus \langle Q_2' \overset{\alpha_2}{\to} Q_1'' \rangle \to \langle Q_i' \overset{\alpha_i}{\to} Q_j'' \rangle$ $(i \neq j)$, $s_i' : Q = Q_1' \oplus Q_2' \oplus Q_1'' \oplus Q_2'' \to Q_i'$, $s_i'' : Q = Q_1' \oplus Q_2' \oplus Q_1'' \oplus Q_2'' \to Q_i''$ $(i = 1, 2)$ be the projections. Since M is N_i-im-summand coinvariant and $s_i'\pi_i\varphi(P) = Q_i' \subseteq_\oplus Q_i$, we see

$$s_i'\pi_i\varphi(\ker p) \subseteq \ker q_i \quad \cdots (i)$$

for $i = 1, 2$. As Q_j'' is lifting, there exists a decomposition $Q_j'' = Q_{j1}'' \oplus Q_{j2}''$ such that $Q_{j1}'' \subseteq_c^{Q_j''} s_j''\pi_i\varphi(P)$ $(i \neq j)$. Let $t_{ij} : Q_i'' = Q_{i1}'' \oplus Q_{i2}'' \to Q_{ij}''$ be the projections $(i, j = 1, 2)$. By $s_j''\pi_i\varphi(P) = Q_{j1}'' \oplus (s_j''\pi_i\varphi(P) \cap Q_{j2}'')$, we see $t_{j1}(s_j''\pi_i\varphi(P)) = Q_{j1}'' \subseteq_\oplus Q_j$ and $t_{j2}(s_j''\pi_i\varphi(P)) = s_j''\pi_i\varphi(P) \cap Q_{j2}'' \ll Q_{j2}''$ $(i \neq j)$. Since M is N_j-im-summand coinvariant we obtain $t_{j1}s_j''\pi_i\varphi(\ker p) \subseteq \ker q_j$ $(i \neq j)$. On the other hand, since M is N_j-im-small coinvariant, we see $t_{j2}s_j''\pi_i\varphi(\ker p) \subseteq \ker q_j$ $(i \neq j)$. Hence, we obtain

$$s_j''\pi_i\varphi(\ker p) \subseteq t_{j1}s_j''\pi_i\varphi(\ker p) \oplus t_{j2}s_j''\pi_i\varphi(\ker p) \subseteq \ker q_j \quad \cdots (ii).$$

By (i), (ii), $\pi_i\varphi(\ker p) \subseteq \ker q$ $(i = 1, 2)$, since $\pi_i\varphi(\ker p) \subseteq Q_i' \oplus Q_j''$ $(i \neq j)$. Thus, we obtain

$$\varphi(\ker p) \subseteq \pi_1\varphi(\ker p) \oplus \pi_2\varphi(\ker p) \subseteq \ker q.$$

Therefore, M is $N_1 \oplus N_2$-im-summand coinvariant.

Next we prove that if M_i is N-im-summand coinvariant and N-im-small coinvariant $(i = 1, 2)$, then $M_1 \oplus M_2$ is N-im-summand coinvariant. Let (P_i, p_i) and (Q, q) be the projective covers of M_i $(i = 1, 2)$ and N, respectively, and put $P = P_1 \oplus P_2$, $p = p_1 \oplus p_2$ and let $\varphi : P \to Q$ be a homomorphism with $\varphi(P) \subseteq_\oplus Q$. Since $\varphi(P)$ is projective, $\ker \varphi$ is a direct summand of P. Since P satisfies FIEP, there exists $P_i'' \subseteq P_i$ $(i = 1, 2)$ such that $P = \ker \varphi \oplus P_1'' \oplus P_2''$. By Lemma 2.2, there exist a direct summand P_i' of P_i and a homomorphism $\beta_i : P_i' \to P_j''$ $(i \neq j)$ such that $P_i = P_i' \oplus P_i''$ and $\ker \varphi = \langle P_1' \overset{\beta_1}{\to} P_2'' \rangle \oplus \langle P_2' \overset{\beta_2}{\to} P_1'' \rangle$. Let $u_i' : P = P_1' \oplus P_2' \oplus P_1'' \oplus P_2'' \to P_i'$ be the projection $(i = 1, 2)$. Then $u_i'|_{\langle P_i' \overset{\beta_i}{\to} P_j'' \rangle}$ is an isomorphism

from $\langle P_i' \xrightarrow{\beta_i} P_j'' \rangle$ to P_i' $(i \neq j)$. Put $\varepsilon_i = (u_i'|_{\langle P_i' \xrightarrow{\beta_i} P_j'' \rangle})^{-1} \oplus 1_{P_i''}$. Since M_i is N-im-summand coinvariant and $\varphi \varepsilon_i(P_i) = \varphi(P_i'') \subseteq_\oplus Q$, we see

$$\varphi \varepsilon_i(\ker p_i) \subseteq \ker q \quad \cdots (iii).$$

As P_j'' is lifting, there exists a decomposition $P_j'' = P_{j1}'' \oplus P_{j2}''$ such that $P_{j1}'' \subseteq_c^{P_j''} \beta_i(P_i')$ $(i \neq j)$. Let $v_{ij} : P_i'' = P_{i1}'' \oplus P_{i2}'' \to P_{ij}''$ be the projections $(i, j = 1, 2)$. Since M_i is N-im-summand coinvariant and $\varphi v_{j1} \beta_i(u_i'|_{P_i})(P_i) = \varphi(P_{j1}'') \subseteq_\oplus Q$, we see $\varphi v_{j1} \beta_i u_i'(\ker p_i) \subseteq \ker q$. On the other hand, by $v_{j2} \beta_i(P_i') = \beta_i(P_i') \cap P_{j2}'' \ll P_{j2}''$, $\varphi v_{j2} \beta_i(u_i'|_{P_i})(P_i) = \varphi(\beta_i(P_i') \cap P_{j2}'') \ll Q$. Since M_i is N-im-small coinvariant, we see $\varphi v_{j2} \beta_i(u_i'|_{P_i})(\ker p_i) \subseteq \ker q$. Hence,

$$\varphi \beta_i u_i'(\ker p_i) = \varphi(v_{j1} + v_{j2}) \beta_i u_i'(\ker p_i) \subseteq \ker q \quad \cdots (iv).$$

For any $k_i \in \ker p_i$, we express k_i in $P_i = P_i' \oplus P_i''$ as $k_i = k_i' + k_i''$, where $k_i' \in P_i'$ and $k_i'' \in P_i''$. By (iii) and (iv), $\varphi(k_i) = \varphi(k_i' + k_i'') = \varphi(k_i' - \beta_i(k_i') + k_i'' + \beta_i(k_i')) = \varphi \varepsilon_i(k_i) + \varphi \beta_i u_i'(k_i) \in \ker q$ and hence $\varphi(\ker p_i) \subseteq \ker q$. Thus, we obtain

$$\varphi(\ker p) = \varphi(\ker p_1 \oplus \ker p_2) \subseteq \ker q.$$

Therefore, $M_1 \oplus M_2$ is N-im-summand coinvariant. □

Now we consider a connection between im-small coinvariance and im-small projectivity over a right perfect ring. First we give a useful lemma.

Lemma 2.4 *(cf. [23, Theorem 27]) Let M and N be modules, (P, p) and (Q, q) projective covers of M and N, respectively, and $\varphi : P \to Q$ a homomorphism and $\pi : N \to N/q\varphi(\ker p)$ the natural epimorphism. Then we can define a homomorphism $f : M \to N/q\varphi(\ker p)$ by $f(p(\alpha)) = \pi q \varphi(\alpha)$, where $\alpha \in P$. If f is lifted to a homomorphism $g : M \to N$, then $\varphi(\ker p) \subseteq \ker q$.*

Proof Let $\varphi : P \to Q$ be a homomorphism. Then we can define a homomorphism $f : M \to N/q\varphi(\ker p)$ by $f(p(\alpha)) = \pi q \varphi(\alpha)$, where $\alpha \in P$ and $\pi : N \to N/q\varphi(\ker p)$ is the natural epimorphism. If f is lifted to a homomorphism $g : M \to N$, then $\pi g = f$. Since P is projective, there exists a homomorphism $\psi : P \to Q$ such that $q\psi = gp$. Then $q\psi(\ker p) = gp(\ker p) = 0$ and so $\psi(\ker p) \subseteq \ker q$. For any $\alpha \in P$, $\pi q \varphi(\alpha) = f(p(\alpha)) = \pi g p(\alpha) = \pi q \psi(\alpha)$ and so $q(\varphi - \psi)(\alpha) \in \ker \pi = q\varphi(\ker p)$. Hence, there exists $k \in \ker p$ such that $q(\varphi - \psi)(\alpha) = q\varphi(k)$. By $\psi(k) \in \ker q$, $q\psi(k) = 0$ and hence $q(\varphi - \psi)(\alpha) = q(\varphi - \psi)(k)$. Thus, $P = \ker(q(\varphi - \psi)) + \ker p = \ker(q(\varphi - \psi))$, so we see $q\varphi = q\psi$. Hence, $q\varphi(\ker p) = q\psi(\ker p) \subseteq q(\ker q) = 0$. Therefore, $\varphi(\ker p) \subseteq \ker q$. □

Proposition 2.5 *Let M and N be modules over a right perfect ring. Then M is N-im-small coinvariant if and only if M is im-small N-projective.*

Proof Let (P, p) and (Q, q) be projective covers of M and N, respectively.

$(\Rightarrow)$ Let $f : M \to X$ be a homomorphism with $\mathrm{Im} f \ll X$ and $g : N \to X$ an epimorphism. Since P is projective, there exists a homomorphism $\varphi : P \to Q$ such that $gq\varphi = fp$. As Q is lifting, there exists a decomposition $Q = K \oplus Q'$ such that $K \subseteq_c^Q \ker gq$. Let $\pi : Q = K \oplus Q' \to Q'$ be the projection. Suppose $Q' = \pi\varphi(P) + T$ for some $T \subseteq Q'$. By $gq(\pi\varphi(P)) = gq\varphi(P) = fp(P) = f(M) \ll X$,

$gq(Q') = gq(\pi\varphi(P)) + gq(T) = gq(T)$, so $Q' = \ker(gq|_{Q'}) + T = T$ and hence $\pi\varphi(P) \ll Q'$. Since M is N-im-small coinvariant, $\pi\varphi(\ker p) \subseteq \ker q$. Thus, we can define a homomorphism $h : M \to N$ by $h(p(\alpha)) = q\pi\varphi(\alpha)$, where $\alpha \in P$. For any $m = p(\alpha) \in M$, $gh(m) = gh(p(\alpha)) = gq\pi\varphi(\alpha)$. By $gq\varphi(\alpha) = gq\pi\varphi(\alpha)$, we see $gh(m) = gq\varphi(\alpha) = fp(\alpha) = f(m)$.

($\Leftarrow$) Let $\varphi : P \to Q$ be a homomorphism with $\varphi(P) \ll Q$ and let $\pi : N \to N/q\varphi(\ker p)$ be the natural epimorphism. Then we can define a homomorphism $f : M \to N/q\varphi(\ker p)$ by $f(p(\alpha)) = \pi q\varphi(\alpha)$, where $\alpha \in P$. By $\varphi(P) \ll Q$, $\mathrm{Im} f \ll N/q\varphi(\ker p)$. Since M is im-small N-projective, there exists a homomorphism $g : M \to N$ such that $\pi g = f$. By Lemma 2.4, we obtain $\varphi(\ker p) \subseteq \ker q$. $\qquad\square$

Remark 2.6 Let R be a right perfect ring and M and N be two R-modules. Assume that (P, p) and (Q, q) are projective covers of M and N, respectively. Then, by the similar proof of Proposition 2.5,
(1) M is N-projective if and only if for every homomorphism $\varphi : P \to Q$, $\varphi(\ker p) \subseteq \ker q$.
(2) If M is epi-N-projective, then for every isomorphism $\varphi : P \to Q$, $\varphi(\ker p) \subseteq \ker q$. The converse of this fact is not true. In fact, $\mathbb{Z}/4\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ is not epi-$(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z})_{\mathbb{Z}/4\mathbb{Z}}$-projective. However, since $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ are projective covers of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, respectively, there are no isomorphisms between the projective cover of $\mathbb{Z}/2\mathbb{Z}$ and of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Hence, the converse does not hold.

Next we consider a connection between im-summand coinvariance and im-coclosed projectivity over a right perfect ring.

Theorem 2.7 *Let M and N be modules over a right perfect ring. Then M is N-im-summand coinvariant if and only if M is im-coclosed N-projective.*

Proof Let (P, p) and (Q, q) be projective covers of the modules M and N, respectively.

($\Rightarrow$) Let f be a homomorphism from M to some module X such that $f(M)$ is coclosed in X and let g be an epimorphism from N to X. Since Q is lifting, there exists a decomposition $Q = K \oplus Q'$ such that $K \subseteq_c^Q \ker gq$. Then $gq|_{Q'} : Q' \to X$ is a small epimorphism. Since Q' is also lifting, there exists a decomposition $Q' = Q_1 \oplus Q_2$ such that $Q_1 \subseteq_c^{Q'} (gq|_{Q'})^{-1}(f(M))$. By [3, 3.2(7)], $gq(Q_1) \subseteq_c^X f(M)$. As $f(M)$ is coclosed in X, we see $gq(Q_1) = f(M)$. Since P is projective, there exists a homomorphism $\varphi : P \to Q_1$ such that $(gq|_{Q_1})\varphi = fp$. By $\ker gq|_{Q_1} \ll Q_1$, φ is onto. We see $\varphi(\ker p) \subseteq \ker q$. Hence, we can define a homomorphism $h : M \to N$ by $h(p(\alpha)) = q\varphi(\alpha)$. Then $gh = f$.

($\Leftarrow$) Let $\varphi : P \to Q$ be a homomorphism with $\varphi(P) \subseteq_\oplus Q$. By $q\varphi(\ker p) \ll N$, the natural map $\pi : N \to N/q\varphi(\ker p)$ is a small epimorphism. Hence, $\pi q\varphi(P) \subseteq_{cc} N/q\varphi(\ker p)$ by [3, 3.7(5)]. Now we define a homomorphism $f : M \to N/q\varphi(\ker p)$ by $f(p(\alpha)) = \pi q\varphi(\alpha)$ for every $\alpha \in P$. Since $f(M) \subseteq_{cc} N/q\varphi(\ker p)$, there exists a homomorphism $g : M \to N$ such that $\pi g = f$. By Lemma 2.4, we obtain $\varphi(\ker p) \subseteq \ker q$. $\qquad\square$

Proposition 2.8 *Let M and N be modules over a right perfect ring. Then M is X-im-summand coinvariant for any submodule X of N if and only if M is N-projective.*

Proof ($\Leftarrow$) By Theorem 2.7 and [20, Proposition 4.31].

($\Rightarrow$) Let $f : M \to N/K$ be a homomorphism and let $\pi_N : N \to N/K$ be the natural epimorphism, where K is any submodule of N. Then we can denote $\mathrm{Im} f = A/K$ for some submodule A of N with

$K \subseteq A$. Let (P, p) and (Q, q) be projective covers of M and A, respectively. Since Q is lifting, there exists a decomposition $Q = T \oplus Q'$ such that $T \subseteq_c^Q \ker \pi_A q$, where $\pi_A : A \to A/K$ is the natural epimorphism. We denote $\pi_A q|_{Q'}$ by q', and then (Q', q') is the projective cover of A/K. Since P is projective, there exists an epimorphism $g : P \to Q'$ such that $fp = q'g$. Since M is A-im-summand coinvariant by the assumption, we see $\iota g(\ker p) \subseteq \ker q$, where ι is the injection from Q' to $Q = T \oplus Q'$. Hence, we can define a homomorphism $\varphi : M \to N$ by $\varphi(p(\alpha)) = q\iota g(\alpha)$, where $\alpha \in P$. Then $\pi_N \varphi = f$. □

The following corollary is immediate from Propositions 2.1 and 2.5 and Theorem 2.7.

Corollary 2.9 *Let M, N, M_i $(i \in I)$, and N_j $(j \in J)$ be modules over a right perfect ring. Then:*

(1) *If M is im-small N-projective, then M is im-small N/X-projective and im-small X-projective for any submodule X of N.*

(2) *If M is im-coclosed N-projective, then M is im-coclosed N/X-projective for any submodule X of N. Moreover, for any coclosed submodule X of N, M is im-coclosed X-projective.*

(3) *If M/S is im-small (im-coclosed, resp.) N-projective for some $S \ll M$, then M is im-small (im-coclosed, resp.) N-projective.*

(4) *If M is im-small (im-coclosed, resp.) N-projective, then M' is im-small (im-coclosed, resp.) N-projective for any direct summand M' of M.*

(5) *If M_i is im-small N_j-projective for any $i \in I, j \in J$, then $\oplus_I M_i$ is im-small $\oplus_J N_j$-projective.*

Example 2.10 (1) An N-im-summand coinvariant module is not necessarily N-im-small coinvariant. Let
$R = \begin{pmatrix} K & K & K & K \\ 0 & K & 0 & K \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & K \end{pmatrix}$, where K is any field. Then the ring R is Artinian, that is, right perfect. Let
$M = (0, K, K, K)/(0, 0, K, K)$ and $N = (K, K, K, K)$. Since N is indecomposable lifting, N/X is also indecomposable lifting for any submodule X of N, so N/X is lifting. A homomorphism $f : M \to N/X$ with $\text{Im} f \subseteq_{cc} N/X$ is only the zero map, because M is not isomorphic to N/X for any submodule X of N. Hence, M is im-coclosed N-projective. On the other hand, the inclusion map $\iota : M \to N/(0, 0, K, K)$ cannot be lifted to a homomorphism from M to N. Since $\text{Im} \iota$ is small in $N/(0, 0, K, K)$, M is not im-small N-projective. Therefore, M is N-im-summand coinvariant but not N-im-small coinvariant by Theorem 2.7 and Proposition 2.5.

(2) Let R be the ring $\begin{pmatrix} K & 0 & K \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix}$, where K is any field. Then R/J is im-small R-projective, but not epi-R-projective. Therefore, the im-small N-projectivity does not imply the epi-N-projectivity.

According to the above example, in general, an N-im-summand coinvariant module M need not be N-im-small coinvariant for a module M over a right perfect ring. However, if N is a small epimorphic image of M, the following holds.

Proposition 2.11 *Let R be a right perfect ring and let M, N be modules. Suppose that there exists a small epimorphism from M to N. If M is N-im-summand coinvariant, then M is N-im-small coinvariant.*

Proof Let M be an N-im-summand coinvariant module and let $f : M \to N$ be a small epimorphism. Since f is a small epimorphism, we can take (P, p) and (P, fp) as the projective covers of M and N, respectively.

Let $\varphi : P \to P$ be an endomorphism with $\varphi(P) \ll P$. By $\varphi(P) \ll P$, $P = (1 - \varphi)(P) + \varphi(P) = (1 - \varphi)(P)$, so we see that $1 - \varphi$ is onto. Since M is N-im-summand coinvariant, $(1 - \varphi)(\ker p) \subseteq \ker fp$. By $\ker p \subseteq \ker fp$, $\varphi(\ker p) \subseteq \ker fp$. Therefore, M is N-im-small coinvariant. □

In the proof of Proposition 2.11, for any $k \in \ker(1 - \varphi)$, $k = \varphi(k) \in \varphi(P)$. Thus, $\ker(1 - \varphi) \subseteq \mathrm{Im}\varphi \ll P$. On the other hand, $\ker(1 - \varphi)$ is a direct summand of P by $P/\ker(1 - \varphi) \simeq (1 - \varphi)(P) = P$. Thus, $1 - \varphi$ is a monomorphism. Hence, by the similar proof of Proposition 2.11, we obtain the following:

Corollary 2.12 *Let M be a module over a right perfect ring and consider the following conditions:*

(1) *M is im-summand coinvariant,*

(2) *M is automorphism coinvariant,*

(3) *M is im-small coinvariant.*

Then $(1) \Rightarrow (2) \Rightarrow (3)$ holds.

Now we show that the implication in Corollary 2.12 is not reversible.

Example 2.13 Put $G_{\mathbb{Z}/4\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Then G is an im-small G-projective $\mathbb{Z}/4\mathbb{Z}$-module but not epi-G-projective, because $(\mathbb{Z}/2\mathbb{Z})_{\mathbb{Z}/4\mathbb{Z}}$ is im-small $(\mathbb{Z}/4\mathbb{Z})_{\mathbb{Z}/4\mathbb{Z}}$-projective but not (epi-)$(\mathbb{Z}/4\mathbb{Z})_{\mathbb{Z}/4\mathbb{Z}}$-projective. Hence, G is not automorphism coinvariant by [9, Theorem 2.3]. Also, G is im-small coinvariant by Proposition 2.5. In addition, it is known that there exists an automorphism coinvariant right R-module that is not quasi-projective (and hence not im-summand coinvariant by Theorem 2.17) over a noncommutative perfect ring R (see [10, Example 5.1]). Thus, in general, the converse of the above corollary does not hold over a noncommutative perfect ring R.

Proposition 2.14 *Let M and N be modules over a right perfect ring. If M is im-summand N-projective, then M is radical N-projective.*

Proof Let $f : M \to X$ be a homomorphism and let $g : N \to X$ be an epimorphism. Let Y be a supplement of $\mathrm{Im}f$ in X. By $\mathrm{Im}f \cap Y$ is small in X, the natural epimorphism $\rho : X \to X/(\mathrm{Im}f \cap Y)$ is a small epimorphism. Then $X/(\mathrm{Im}f \cap Y) = \mathrm{Im}f/(\mathrm{Im}f \cap Y) \oplus Y/(\mathrm{Im}f \cap Y)$ and so $\mathrm{Im}\rho f = \mathrm{Im}f/(\mathrm{Im}f \cap Y)$ is a direct summand of $X/(\mathrm{Im}f \cap Y)$. Since M is im-summand N-projective, there exists a homomorphism $h : M \to N$ such that $\rho f = \rho g h$. Therefore, M is radical N-projective. □

Proposition 2.15 *Let R be a right perfect ring and let M, N be modules. Then M is N-im-summand coinvariant and N-im-small coinvariant if and only if M is N-projective.*

Proof By Propositions 2.5 and 2.14 and [17, Proposition 1.3]. □

The following is obtained by Propositions 2.1 and 2.15 and Remark 2.6.

Corollary 2.16 *Let R be a right perfect ring, N a module, and $S \ll M$. If M/S is N-projective then M is N-projective.*

Theorem 2.17 *Let R be a right perfect ring, let M be a module, and let N be a small epimorphic image of M. Then the following conditions are equivalent:*

(a) *M is N-projective,*

(b) *M is im-coclosed N-projective,*

(c) *M is N-im-summand coinvariant.*

Proof By Theorem 2.7 and Propositions 2.11 and 2.15. □

Next we show that M is quasi-projective if and only if M is im-summand M-projective, if and only if M is im-summand coinvariant, for any module M over a right perfect ring. We first need to give the following lemma:

Lemma 2.18 *Let M be an im-summand M-projective module and let (P, ν) be the projective cover of M. For any decomposition $P = P_1 \oplus P_2$, $M = \nu(P_1) \oplus \nu(P_2)$.*

Proof Let M be im-summand M-projective, let (P, ν) be the projective cover of M, and let $P = P_1 \oplus P_2$. Let $p_i : P = P_1 \oplus P_2 \to P_i$ be the projection $(i = 1, 2)$. Given $\overline{\nu(x_1)} = \overline{\nu(x_2)} \in [\nu(P_1)/\nu p_1(\ker \nu)] \cap [(\nu(P_2) + \nu p_1(\ker \nu))/\nu p_1(\ker \nu)]$. By $\nu(x_1 - x_2) \in \nu p_1(\ker \nu)$, there exists $k \in \ker \nu$ such that $\nu(x_1 - x_2) = \nu(p_1(k))$. Then $x_1 - x_2 - p_1(k) \in \ker \nu$ and so $x_1 - p_1(k) \in p_1(\ker \nu)$. By $x_1 \in p_1(\ker \nu)$, we see $[\nu(P_1)/\nu p_1(\ker \nu)] \cap [(\nu(P_2) + \nu p_1(\ker \nu))/\nu p_1(\ker \nu)] = 0$. Now we define $f : M \to M/\nu p_1(\ker \nu)$ by $f(\nu(\alpha)) = \overline{\nu p_1(\alpha)}$, where $\alpha \in P$. Then $\mathrm{Im} f = \nu(P_1)/\nu p_1(\ker \nu) \subseteq_{\oplus} M/\nu p_1(\ker \nu)$. Since M is im-summand M-projective, there exists an endomorphism g of M such that $\pi g = f$, where $\pi : M \to M/\nu p_1(\ker \nu)$ is the natural epimorphism. By Lemma 2.4, we obtain $p_1(\ker \nu) \subseteq \ker \nu$.

By $\nu(P_1) \cap \nu(P_2) \subseteq \nu p_1(\ker \nu) \subseteq \nu(\ker \nu) = 0$, we obtain that $M = \nu(P_1) \oplus \nu(P_2)$. □

Theorem 2.19 *Let M be a module over a right perfect ring. Then M is im-summand M-projective if and only if M is im-summand coinvariant.*

Proof ($\Leftarrow$) By Theorem 2.7.

($\Rightarrow$) Let (P, p) be the projective cover of M and let φ be an endomorphism of P with $\mathrm{Im} \varphi \subseteq_{\oplus} P$. Put $P = \varphi(P) \oplus P'$. By Lemma 2.18, we see $M = p\varphi(P) \oplus p(P')$. Then $M/p\varphi(\ker p) = (p\varphi(P)/p\varphi(\ker p)) \oplus ((p(P') + p\varphi(\ker p))/p\varphi(\ker p))$. Now we define the homomorphism $f : M \to M/p\varphi(\ker p)$ by $f(p(\alpha)) = \pi p\varphi(\alpha)$, where $\alpha \in P$ and $\pi : M \to M/p\varphi(\ker p)$ is the natural epimorphism. Since M is im-summand M-projective and $p\varphi(P)/p\varphi(\ker p) \subseteq_{\oplus} M/p\varphi(\ker p)$, we obtain $\varphi(\ker p) \subseteq \ker p$ by Lemma 2.4. Thus, M is im-summand coinvariant. □

Corollary 2.20 *Let M be a module over a right perfect ring. Then the following conditions are equivalent:*

(a) *M is quasi-projective,*

(b) *M is im-coclosed M-projective,*

(c) *M is im-summand M-projective,*

(d) *M is im-summand coinvariant.*

If M is lifting, then (a)–(d) are equivalent to:

(e) *M is automorphism coinvariant.*

Proof By Proposition 2.19, Theorem 2.17, [9, Theorem 2.3], and [14, Corollary 2.6]. □

By [3, Exercises 4.45(8)] (cf.[4, Example 2.3 and Theorem 2.5]), there is a pseudo-projective module over a noncommutative two-sided perfect ring but not quasi-projective. Hence, Corollary 2.20 $(a) \Leftrightarrow (e)$ does not hold in general.

3. Ker-essential invariant and ker-summand invariant modules

In this section, we first give some fundamental properties of ker-essential invariant modules and ker-summand invariant modules.

Proposition 3.1 *Let M, N, M_i $(i \in I)$, and N_j $(j \in J)$ be modules. Then:*

(1) *If M is N-ker-essential invariant, then M is X-ker-essential invariant and N/X-ker-essential invariant for any submodule X of N.*

(2) *If M is N-ker-summand invariant, then M is X-ker-summand invariant for any submodule X of N.*

(3) *If A is N-ker-essential (N-ker-summand, resp.) invariant for some essential submodule A of M, then M is N-ker-essential (N-ker-summand, resp.) invariant.*

(4) *If M is N-ker-essential (N-ker-summand, resp.) invariant, then M' is N-ker-essential (N-ker-summand, resp.) invariant for any direct summand M' of M.*

(5) *If M_i is N_j-ker-essential invariant $(i \in I, j \in J)$, then $\Pi_I M_i$ is $\oplus_J N_j$-ker-essential invariant.*

Proof (1) Let X be a submodule of N and let $f : E(X) \to E(M)$ be a homomorphism with $\ker f \subseteq_e E(X)$. Let Y be a complement of X in N. Then $E(N) = E(X) \oplus E(Y)$. Now we define $f^* : E(N) = E(X) \oplus E(Y) \to E(M)$ by $f^*(a + b) = f(a)$, where $a \in E(X)$ and $b \in E(Y)$. Then $\ker f^* = \ker f \oplus E(Y) \subseteq_e E(N)$. Since M is N-ker-essential invariant, we see $f(X) \subseteq f^*(N) \subseteq M$.

Next we show that M is N/X-ker-essential invariant. Let $g : E(N/X) \to E(M)$ be a homomorphism with $\ker g \subseteq_e E(N/X)$ and let $\pi : N \to N/X$ be the natural epimorphism. Since $E(N/X)$ is injective, there exists a homomorphism $h : E(N) \to E(N/X)$ such that $h|_N = \pi$. By $\ker gh = h^{-1}(\ker g) \subseteq_e E(N)$, $g(N/X) = gh(N) \subseteq M$, so M is N/X-ker-essential invariant.

(2) We can see by the similar proof of (1).

(3) Obvious.

(4) We prove only for the case of N-ker-essential invariant. Let $M = M' \oplus M''$ and let $f : E(N) \to E(M')$ be a homomorphism with $\ker f \subseteq_e E(N)$. Since M is N-ker-essential invariant, $f(N) \subseteq M = M' \oplus M''$. By $f(N) \subseteq E(M')$, $f(N) \subseteq M'$.

(5) First we show if each M_i is N-ker-essential invariant, then so is a direct product $\Pi_I M_i$. Since $\Pi_I E(M_i)$ is injective, we can take $E(\Pi_I M_i)$ as a direct summand of $\Pi_I E(M_i)$. Let $\iota : E(\Pi_I M_i) \to \Pi_I E(M_i)$ be the injection and $p_k : \Pi_I E(M_i) \to E(M_k)$ be the projection $(k \in I)$. Let $f : E(N) \to E(\Pi_I M_i)$ be a homomorphism with $\ker f \subseteq_e E(N)$. Then $\ker p_i \iota f$ is essential in $E(N)$. Since each M_i is N-ker-essential invariant, $p_i \iota f(N) \subseteq M_i$. Hence, $f(N) = \iota f(N) \subseteq \Pi_I p_i \iota f(N) \subseteq \Pi_I M_i$. Therefore, $\Pi_I M_i$ is N-ker-essential invariant.

Next we show that if M is N_j-ker-essential invariant for any $j \in J$, then M is $\oplus_J N_j$-ker-essential invariant. Let $f : E(\oplus_J N_j) \to E(M)$ be a homomorphism with $\ker f \subseteq_e E(\oplus_J N_j)$. For any $l \in J$, $E(\oplus_J N_j) = E(N_l) \oplus E(\oplus_{J-\{l\}} N_j)$ and $\ker(f|_{E(N_l)}) = \ker f \cap E(N_l) \subseteq_e E(N_l)$. Since M is N_l-ker-essential invariant, $f(N_l) = f|_{E(N_l)}(N_l) \subseteq M$. Hence, $f(\oplus_J N_j) = \sum_J f(N_j) \subseteq M$. □

A module M is said to be *extending* if, for any submodule X of M, there exists a direct summand N of M such that $X \subseteq_e N$. It is well known that any injective module is extending with FIEP.

Proposition 3.2 *Let $M_1, M_2, \ldots, M_m$, $N_1, N_2, \ldots, N_n$ be modules. If M_i is N_j-ker-summand invariant and N_j-ker-essential invariant $(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$, then $(M_1 \oplus M_2 \oplus \cdots \oplus M_m)$ is $(N_1 \oplus N_2 \oplus \cdots \oplus N_n)$-ker-summand invariant.*

Proof It is enough to show the case of $m = n = 2$ by Proposition 3.1(5).

First we show that if M is N_i-ker-summand invariant and N_i-ker-essential invariant $(i = 1, 2)$, then M is $N_1 \oplus N_2$-ker-summand invariant. Put $E_i = E(N_i)$ $(i = 1, 2)$, $E = E_1 \oplus E_2$, and let $\varphi : E \to E(M)$ be a homomorphism with $\ker \varphi \subseteq_\oplus E$. Since E satisfies FIEP, there exists $E_i'' \subseteq E_i$ $(i = 1, 2)$ such that $E = \ker \varphi \oplus E_1'' \oplus E_2''$. By Lemma 2.2, there exist a direct summand E_i' of E_i and a homomorphism $\alpha_i : E_i' \to E_j''$ $(i \neq j)$ such that $E_i = E_i' \oplus E_i''$ and $\ker \varphi = \langle E_1' \overset{\alpha_1}{\to} E_2'' \rangle \oplus \langle E_2' \overset{\alpha_2}{\to} E_1'' \rangle$. Let $f_i : E_i = E_i' \oplus E_i'' \to \langle E_i' \overset{\alpha_i}{\to} E_j'' \rangle \oplus E_i''$ be the natural isomorphism $(i = 1, 2)$. Then $\ker(\varphi f_i) = E_i' \subseteq_\oplus E_i$ $(i = 1, 2)$. Since M is N_i-ker-summand invariant, we see

$$\varphi f_i(N_i) \subseteq M \quad \cdots (i).$$

As E_i' is extending, there exists a decomposition $E_i' = E_{i1}' \oplus E_{i2}'$ such that $\ker \alpha_i \subseteq_e E_{i1}'$. Let $p_i : E_i = E_i' \oplus E_i'' \to E_i'$ and $q_{ij} : E_i' = E_{i1}' \oplus E_{i2}' \to E_{ij}'$ $(j = 1, 2)$ be the projections. By $\ker \alpha_1 \subseteq_e E_{11}'$, $\ker \varphi \alpha_1 q_{11} p_1 = E_{12}' \oplus E_1'' \oplus \ker(\varphi \alpha_1|_{E_{11}'}) \subseteq_e E_1$. Since M is N_1-ker-essential invariant, $\varphi \alpha_1 q_{11} p_1(N_1) \subseteq M$. On the other hand, by $\ker \varphi \alpha_1 q_{12} p_1 = \ker q_{12} p_1 = E_{11}' \oplus E_1'' \subseteq_\oplus E_1$, we see $\varphi \alpha_1 q_{12} p_1(N_1) \subseteq M$, since M is N_1-ker-summand invariant. Thus, we see

$$\varphi \alpha_1 p_1(N_1) \subseteq \varphi \alpha_1 q_{11} p_1(N_1) + \varphi \alpha_1 q_{12} p_1(N_1) \subseteq M \quad \cdots (ii).$$

For any $n_1 \in N_1$, we express n_1 in $E_1 = E_1' \oplus E_1''$ as $n_1 = n_1' + n_1''$, where $n_1' \in E_1'$ and $n_1'' \in E_1''$. By (i) and (ii), $\varphi(n_1) = \varphi(n_1' - \alpha_1(n_1') + n_1'' + \alpha_1(n_1')) = \varphi f_1(n_1) + \varphi \alpha_1 p_1(n_1) \in M$, so we see $\varphi(N_1) \subseteq M$. Similarly we obtain $\varphi(N_2) \subseteq M$. Thus, $\varphi(N_1 \oplus N_2) \subseteq M$.

Next we show that if M_i is N-ker-summand invariant and N-ker-essential invariant $(i = 1, 2)$, then $M_1 \oplus M_2$ is N-ker-summand invariant. Put $F_i = E(M_i)$ $(i = 1, 2)$, $F = F_1 \oplus F_2$, and let $\varphi : E(N) \to F$ be a homomorphism with $\ker \varphi \subseteq_\oplus E(N)$. As $\varphi(E(N))$ is injective, it is a direct summand of F. Since F satisfies FIEP, there exists $F_i'' \subseteq F_i$ $(i = 1, 2)$ such that $F = \varphi(E(N)) \oplus F_1'' \oplus F_2''$. By Lemma 2.2, there

exist a direct summand F_i' of F_i and a homomorphism $\beta_i : F_i' \to F_j''$ $(i \neq j)$ such that $F_i = F_i' \oplus F_i''$ and $\varphi(E(N)) = \langle F_1' \xrightarrow{\beta_1} F_2'' \rangle \oplus \langle F_2' \xrightarrow{\beta_2} F_1'' \rangle$. Let $\pi_i : \varphi(E(N)) = \langle F_1' \xrightarrow{\beta_1} F_2'' \rangle \oplus \langle F_2' \xrightarrow{\beta_2} F_1'' \rangle \to \langle F_i' \xrightarrow{\beta_i} F_j'' \rangle$ $(i \neq j)$ and $s_i' : F = F_1' \oplus F_2' \oplus F_1'' \oplus F_2'' \to F_i'$ $(i = 1, 2)$ be the projections. Then $s_i'|_{\langle F_i' \xrightarrow{\beta_i} F_j'' \rangle}$ is an isomorphism from $\langle F_i' \xrightarrow{\beta_i} F_j'' \rangle$ to F_i' $(i \neq j)$. Since M_i is N-ker-summand invariant and $\ker s_i' \pi_i \varphi \subseteq_\oplus E(N)$, we see

$$s_i' \pi_i \varphi(N) \subseteq M_i \cdots (iii).$$

As F_i' is extending, there exists a decomposition $F_i' = F_{i1}' \oplus F_{i2}'$ such that $\ker \beta_i \subseteq_e F_{i1}'$. Let $t_{ij} : F_i' = F_{i1}' \oplus F_{i2}' \to F_{ij}'$ be the projections $(i, j = 1, 2)$. By $\ker \beta_1 \subseteq_e F_{11}'$, $\ker(\beta_1 t_{11} s_1' \pi_1 \varphi) \subseteq_e E(N)$. Since M_2 is N-ker-essential invariant, we see $\beta_1 t_{11} s_1' \pi_1 \varphi(N) \subseteq M_2$. On the other hand, by $\ker(\beta_1 t_{12} s_1' \pi_1 \varphi) \subseteq_\oplus E(N)$, $\beta_1 t_{12} s_1' \pi_1 \varphi(N) \subseteq M_2$ since M_2 is N-ker-summand invariant. Hence,

$$\beta_1 s_1' \pi_1 \varphi(N) \subseteq \beta_1 t_{11} s_1' \pi_1 \varphi(N) + \beta_1 t_{12} s_1' \pi_1 \varphi(N) \subseteq M_2 \ \cdots (iv).$$

For any $n \in N$, there exists $x_1' \in F_1'$ such that $\pi_1 \varphi(n) = x_1' - \beta_1(x_1')$. By (iii), $x_1' = s_1'(x_1' - \beta_1(x_1')) = s_1' \pi_1 \varphi(n) \in s_1' \pi_1 \varphi(N) \subseteq M_1$. In addition, by (iv), $\beta_1(x_1') = \beta_1 s_1' \pi_1 \varphi(n) \in \beta_1 s_1' \pi_1 \varphi(N) \subseteq M_2$. Thus, $\pi_1 \varphi(n) = x_1' - \beta_1(x_1') \in M_1 \oplus M_2$. Similarly, we see $\pi_2 \varphi(n) \in M_1 \oplus M_2$. Hence, $\varphi(n) = \pi_1 \varphi(n) + \pi_2 \varphi(n) \in M_1 \oplus M_2$. Therefore, we obtain $\varphi(N) \subseteq M_1 \oplus M_2$. $\square$

Now we consider a connection between essential injectivity and ker-essential invariance.

Proposition 3.3 *Let M and N be two modules. Then M is essentially N-injective if and only if M is N-ker-essential invariant.*

Proof $(\Rightarrow)$ Let $\varphi : E(N) \to E(M)$ be a homomorphism with $\ker \varphi \subseteq_e E(N)$. Then $\varphi^{-1}(M) \subseteq_e E(N)$. Put $f = \varphi|_{\varphi^{-1}(M) \cap N}$. By $\ker f = \ker \varphi \cap N \subseteq_e E(N) \cap N = N$, we see $\ker f \subseteq_e \varphi^{-1}(M) \cap N$. Since M is essentially N-injective, there exists a homomorphism $g : N \to M$ such that $g|_{\varphi^{-1}(M) \cap N} = f$. Since $E(M)$ is injective, there exists a homomorphism $\psi : E(N) \to E(M)$ such that $\psi|_N = g$. Clearly $\psi(N) \subseteq M$. We claim that $M \cap (\varphi - \psi)(N) = 0$. Let $m \in M \cap (\varphi - \psi)(N)$. Then there exists $n \in N$ such that $m = (\varphi - \psi)(n)$. Hence, $\varphi(n) = m + \psi(n) \in M$, so we see $n \in \varphi^{-1}(M) \cap N$ and hence $\varphi(n) = f(n) = \psi(n)$. Thus, $m = 0$. By $M \subseteq_e E(M)$, $(\varphi - \psi)(N) = 0$. Therefore, we obtain $\varphi(N) = \psi(N) \subseteq M$.

$(\Leftarrow)$ Let X be a submodule of N and let $f : X \to M$ be a homomorphism with $\ker f \subseteq_e X$. Since $E(M)$ is injective, there exists a homomorphism $\varphi : E(N) \to E(M)$ such that $\varphi|_X = f$. Since $E(N)$ is extending, there exists a decomposition $E(N) = T \oplus Q$ such that $X \subseteq_e T$. Let $\pi : E(N) = T \oplus Q \to T$ be the projection map. Put $\psi = \varphi \pi$. By $\ker f \subseteq_e X \subseteq_e T$, $\ker f \oplus Q \subseteq_e E(N)$. By $\ker f \oplus Q \subseteq \ker \psi$, $\ker \psi \subseteq_e E(N)$. Since M is N-ker-essential invariant, we obtain $\psi(N) \subseteq M$. Put $h = \psi|_N$. Then it is easy to check that $h|_X = f$. Therefore, M is essentially N-injective. $\square$

Recall that a module M is said to be *ker-summand (ker-closed) N-injective* if, for any submodule X of N and any homomorphism $f : X \to M$ with $\ker f \subseteq_\oplus X$ ($\ker f$ is closed in X), there exists a homomorphism $g : N \to M$ such that $g|_X = f$.

Proposition 3.4 *Let M and N be modules and consider the following conditions:*

(1) M is ker-closed N-injective,

(2) M is N-ker-summand invariant,

(3) M is ker-summand N-injective.

Then $(1) \Rightarrow (2) \Rightarrow (3)$ holds.

Proof $(1) \Rightarrow (2)$: Let $\varphi : E(N) \to E(M)$ be a homomorphism with $\ker \varphi \subseteq_{\oplus} E(N)$. Put $f = \varphi|_{\varphi^{-1}(M) \cap N}$. We claim that $\ker f = \ker \varphi \cap N$ is a closed submodule of $\varphi^{-1}(M) \cap N$. Assume that there exists a submodule T of $\varphi^{-1}(M) \cap N$ such that $\ker \varphi \cap N \subsetneq_e T$. By $\ker \varphi \cap N \subseteq T \subseteq N$, we see $\ker \varphi \cap N = \ker \varphi \cap T$. Let A be a complement of $\ker \varphi \cap N$ in $\varphi^{-1}(M) \cap N$, and then $T \cap A = 0$. Since $(\ker \varphi \cap N) \oplus A \subseteq_e \varphi^{-1}(M) \cap N \subseteq_e N$, we can write $E(N) = \ker \varphi \oplus E(A)$. Let x be an element of $T \setminus (\ker \varphi \cap N) = T \setminus (\ker \varphi \cap T)$ and let $x = k + a$, where $k \in \ker \varphi$ and $a \in E(A)$. Then $a \neq 0$. Since $A \subseteq_e E(A)$, there exists an element r of R such that ar is a nonzero element of A. Then $kr = xr - ar \in \ker \varphi \cap N = \ker \varphi \cap T$, so $0 \neq ar = xr - kr \in A \cap T = 0$, a contradiction. Hence, $\ker f = \ker \varphi \cap N$ is closed in $\varphi^{-1}(M) \cap N$. Since M is ker-closed N-injective, there exists a homomorphism $g : N \to M$ such that $g|_{\varphi^{-1}(M) \cap N} = f$. Therefore, we see $\varphi(N) \subseteq M$ by the same proof as Proposition 3.3.

$(2) \Rightarrow (3)$: Let X be a submodule of N and let $f : X \to M$ be a homomorphism with $\ker f \subseteq_{\oplus} X$. Put $X = \ker f \oplus X'$. Let Y be a complement of X in N. Then $\ker f \oplus X' \oplus Y = X \oplus Y \subseteq_e N$, so $E(N) = E(\ker f) \oplus E(X') \oplus E(Y)$. Since $E(M)$ is injective and $X' \subseteq_e E(X')$, there exists a monomorphism $g : E(X') \to E(M)$ such that $g|_{X'} = f|_{X'}$. Define $g^* : E(N) = E(\ker f) \oplus E(X') \oplus E(Y) \to E(M)$ by $g^*(a_1 + a_2 + a_3) = g(a_2)$, where $a_1 \in E(\ker f)$, $a_2 \in E(X')$, and $a_3 \in E(Y)$. Then $\ker g^* = E(\ker f) \oplus E(Y)$ is a direct summand of $E(N)$. Since M is N-ker-summand invariant, $g^*(N) \subseteq M$. Hence, $g^*|_N$ is a homomorphism from N to M. For any $x \in X$, we express x as $x = k + x'$ in $X = \ker f \oplus X'$, where $k \in \ker f$ and $x' \in X'$. Then $g^*(x) = g^*(k + x') = g(x') = f(x') = f(k + x') = f(x)$. Thus, M is ker-summand N-injective. $\square$

The authors do not know whether the converse of Proposition 3.4 holds or not.

Proposition 3.5 *Let M and N be modules. Then M is N/X-ker-summand invariant for any submodule X of N if and only if M is N-injective.*

Proof $(\Leftarrow)$ By Proposition 3.4 and [20, Proposition 1.3].

$(\Rightarrow)$ Let $f : K \to M$ be a homomorphism, where K is any submodule of N. Then we can define the monomorphism $\overline{f} : K/\ker f \to M$ by $\overline{f}(k + \ker f) = f(k)$, where $k \in K$. Since $E(N/\ker f)$ is extending, there exists a decomposition $E(N/\ker f) = T \oplus Q$ such that $K/\ker f \subseteq_e T$. Then T is the injective hull of $K/\ker f$. Since $E(M)$ is injective, there exists a monomorphism $g : T \to E(M)$ such that $g|_{K/\ker f} = \overline{f}$. Since M is $(N/\ker f)$-ker-summand invariant by the assumption, $g\pi(N/\ker f) \subseteq M$, where π is the projection from $E(N/\ker f) = T \oplus Q$ to T. Let $\eta : N \to N/\ker f$ be the natural epimorphism. Then for any $k \in K$, $g\pi\eta(k) = g\pi(k + \ker f) = g(k + \ker f) = \overline{f}(k + \ker f) = f(k)$. Therefore, $g\pi\eta|_K = f$. $\square$

Example 3.6 An N-ker-summand invariant module is not necessarily N-ker-essential invariant. Let $R =$

$$\begin{pmatrix} K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ K & 0 & K & 0 \\ K & K & 0 & K \end{pmatrix}, \ M = (K, K, 0, 0)/(K, 0, 0, 0), \text{ and } N = (K, K, K, K), \text{ where } K \text{ is any field. Since } N$$

is indecomposable extending (which is called *uniform*), X is indecomposable for any submodule X of N. A homomorphism $f : X \to M$ with $\ker f$ closed in X is only the zero map because M is not isomorphic to X for any submodule X of N. Hence, M is ker-closed N-injective. On the other hand, the natural epimorphism $f : (K, K, 0, 0) \to M$ cannot be extended to a homomorphism from N to M. Since $\ker f$ is essential in $(K, K, 0, 0)$, M is not essentially N-injective. Therefore, M is N-ker-summand invariant but not N-ker-essential invariant by Propositions 3.3 and 3.4.

Next we consider a connection between ker-summand invariance and ker-summand injectivity. We first give the following proposition:

Proposition 3.7 *If M is N-ker-summand invariant, then it is N-ejective.*

Proof Let X be a submodule of N and let $f : X \to M$ be a homomorphism. Since $E(N)$ is extending, there exists a decomposition $E(N) = E_1 \oplus E_1'$ such that $\ker f \subseteq_e E_1$. By [21, Lemma 2.2], $\ker f \oplus (E_1' \cap X) \subseteq_e X$. As E_1' is extending, there exists a decomposition $E_1' = E_2 \oplus E_3$ such that $E_1' \cap X \subseteq_e E_2$. Since $f|_{E_1' \cap X}$ is monic and $E_1' \cap X \subseteq_e E_2$, there exists a monomorphism $g : E_2 \to E(M)$ such that $g|_{E_1' \cap X} = f|_{E_1' \cap X}$. Now we define a homomophism φ from $E(N)$ to $E(M)$ by $\varphi(x_1 + x_2 + x_3) = g(x_2)$, where $x_i \in E_i$ $(i = 1, 2, 3)$. Then $\ker \varphi = E_1 \oplus E_3 \subseteq_\oplus E(N)$ and hence $\varphi(N) \subseteq M$. For any $k + x \in \ker f \oplus (E_1' \cap X)$, $\varphi(k + x) = g(x) = f(x) = f(k + x)$. Thus, M is N-ejective. $\qquad\square$

Corollary 3.8 *Let N be a module and let $X \subseteq_e M$. If X is N-injective, then M is N-injective.*

Proof Let $X \subseteq_e M$ and let X be N-injective. By Propositions 3.1 and 3.3, M is essentially N-injective. On the other hand, by Propositions 3.1, 3.4, and 3.7, M is N-ejective. Thus, M is N-injective by [18, Proposition 3]. $\qquad\square$

Proposition 3.9 *Let M and N be modules. Suppose that there exists a monomorphism $f : N \to M$ such that $\mathrm{Im} f \subseteq_e M$. If M is N-ker-summand invariant, then M is N-ker-essential invariant.*

Proof Let M be an N-ker-summand invariant module and let $f : N \to M$ be a monomorphism with $\mathrm{Im} f \subseteq_e M$. By $\mathrm{Im} f \subseteq_e M$, there exists an isomorphism $\alpha : E(M) \to E(N)$ such that $\alpha|_{f(N)} = f^{-1}$. Let $\varphi : E(N) \to E(M)$ be a homomorphism with $\ker \varphi \subseteq_e E(N)$. Given $k \in \ker(1_{E(M)} - \varphi\alpha)$. If $k \neq 0$, then there exists $r \in R$ such that $0 \neq kr \in \ker \varphi\alpha$, by $\ker \varphi\alpha \subseteq_e E(M)$. Then $0 = (1_{E(M)} - \varphi\alpha)(kr) = kr - \varphi\alpha(kr) = kr$, a contradiction. Thus, we see that $1_{E(M)} - \varphi\alpha$ is a monomorphism.

Since M is N-ker-summand invariant, $(1_{E(M)} - \varphi\alpha)(\alpha^{-1}(N)) \subseteq M$. By $\alpha^{-1}(N) = f(N) \subseteq M$, $\varphi(N) = \varphi\alpha(\alpha^{-1}(N)) \subseteq M$. Thus, M is N-ker-essential invariant. $\qquad\square$

In the proof of Proposition 3.9, by $\ker \varphi\alpha \subseteq_e E(M)$, we see $(1_{E(M)} - \varphi\alpha)(E(M)) \subseteq_e E(M)$. Since $(1_{E(M)} - \varphi\alpha)(E(M)) \simeq E(M)$ is injective, $(1_{E(M)} - \varphi\alpha)(E(M)) = E(M)$. Hence, $1_{E(M)} - \varphi\alpha$ is onto. Thus, by Proposition 3.3 and the similar proof of Proposition 3.9, we obtain the following:

Corollary 3.10 *If M is an automorphism invariant module, then it is essentially M-injective.*

Remark 3.11 (1) From Corollary 3.10, we see that the "extending" condition on M in [15, Proposition 2.4] can be removed. Also, there exists a mono-N-injective module M with $N \not\cong M$ such that M is not essentially N-injective by [15, Example 2.5].

(2) By [12, Lemma 2] and [7, Theorem 16], we see that the converse in the above does not hold in general.

Proposition 3.12 *Let M and N be modules. Then M is N-ker-summand invariant and N-ker-essential invariant if and only if M is N-injective.*

Proof By Propositions 3.3, 3.4, and 3.7 and [18, Proposition 3]. □

Theorem 3.13 *Let M be a module and let N be a module that is isomorphic to an essential submodule of M. Then the following conditions are equivalent:*

(a) *M is N-injective,*

(b) *M is ker-closed N-injective,*

(c) *M is N-ker-summand invariant.*

Proof By Propositions 3.4, 3.9, and 3.12. □

Theorem 3.14 *Let M be a module. Then the following conditions are equivalent:*

(a) *M is quasi-injective,*

(b) *M is ker-closed M-injective,*

(c) *M is ker-summand M-injective,*

(d) *M is ker-summand invariant.*

Proof $(a) \Leftrightarrow (b) \Leftrightarrow (d)$: By Theorem 3.13.

$(b) \Rightarrow (c)$ is clear.

$(c) \Rightarrow (d)$: Let f be an endomorphism of $E(M)$ with $\ker f \subseteq_\oplus E(M)$. Put $E(M) = \ker f \oplus E$. By $f^{-1}(M) \subseteq_e E(M)$, we see $M \cap f^{-1}(M) \subseteq_e E(M)$. Let $0 \neq m \in M \cap f^{-1}(M)$ and express m in $E(M) = \ker f \oplus E$ as $m = k + n$, where $k \in \ker f$ and $n \in E$. In the case of $k = 0$ we see $m = n \in M \cap f^{-1}(M) \cap E$. If $k \neq 0$, then there exists $r \in R$ such that $0 \neq kr \in M \cap \ker f$ by $M \cap \ker f = \ker f \cap (M \cap f^{-1}(M)) \subseteq_e \ker f$. By $M \cap f^{-1}(M) \cap E \subseteq_e E$, if $nr \notin M \cap f^{-1}(M) \cap E$, then there exists $r' \in R$ such that $0 \neq nrr' \in M \cap f^{-1}(M) \cap E$. Thus, we see

$$(M \cap \ker f) \oplus (M \cap f^{-1}(M) \cap E) \subseteq_e M \cap f^{-1}(M) \subseteq_e E(M).$$

Put $X = (M \cap \ker f) \oplus (M \cap f^{-1}(M) \cap E)$. Then $\ker(f|_X)$ is a direct summand of X. By (c), there exists an endomorphism g of M such that $g|_X = f|_X$. By $X \subseteq \ker((f - g)|_M) \subseteq M \cap f^{-1}(M)$, $\ker((f - g)|_M) \subseteq_e$

$M \cap f^{-1}(M)$. By [7, Theorem 16] and Proposition 3.10, M is essentially M-injective and hence there exists an endomorphism h of M such that $h|_{M \cap f^{-1}(M)} = (f - g)|_{M \cap f^{-1}(M)}$.

Given $m = (f - h - g)(n) \in (f - (h + g))(M) \cap M$. Then $n \in f^{-1}(M) \cap M$ and hence $m = (f-g)(n) - h(n) = h(n) - h(n) = 0$. We see $(f - (h+g))(M) \cap M = 0$. By $M \subseteq_e E(M)$, $(f - (h+g))(M) = 0$. Thus, $f(M) = (h + g)(M) \subseteq M$. □

Corollary 3.15 *Let M be an extending module. Then the following conditions are equivalent:*

(a) *M is automorphism invariant,*

(b) *M is ker-summand invariant,*

(c) *M is quasi-injective.*

Proof By [7, Theorem 16], [8, Theorem 5.9], and Theorem 3.14. □

By [7, Example 9], there is an automorphism invariant module that is not quasi-injective. Finally, we will give another such example. Let $R = \begin{pmatrix} K & K & K \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix}$, where K is any field. Then the right R-module $M = (K, K, K)$ is not extending. Put $N_1 = (0, K, 0)$ and $N_2 = (0, 0, K)$. Then we see $E(N_1) = M/N_2 = \overline{(K, K, 0)}$, $E(N_2) = M/N_2 = \overline{(K, 0, K)}$, and $E(M) = E(N_1) \oplus E(N_2)$. Since there is no nonzero homomorphism between $E(N_1)$ and $E(E_2)$, M is automorphism invariant, but M is not quasi-injective since it is not extending. Hence, the condition "extending" in the above corollary is not superfluous.

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