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Numerical solutions of Black-Scholes integro-differential equations with convergence analysis

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Abstract: Stochastic integro-differential equations are obtained when we consider prices jump in financial modelling. In this paper, these equations are solved numerically by applying the two-dimensional Tau method with ordinary bases. Next, the numerical solutions of the equations above are investigated by the ordinary bases to the Hermitian one. Moreover, we provide an error analysis for the Tau method with ordinary bases. Also, we will prove that the errors of the approximate solutions decay exponentially in weighted L^2-norm. At the end, we will provide some numerical examples which show the efficiency and accuracy of the method.

Key words: Tau method, stochastic integro-differential Black-Scholes equation, European option pricing problem, Hermitian polynomial

1. Introduction

Actually, a call (put) option gives the owner the right, but not the obligation, to buy (sell) an underlying asset at a specified strike price and on a specified date. The price indicated in the contract is called ”exercise price” or ”strike price” and the date is called ”the maturity date of the option” or ”expiration date”.
There are two styles of options (call or put), i.e. European and American. The European option can be exercised only at the expiration date while an American option can be exercised on or at any time before the expiration date.

An option has different significances some of which are as follows:

1. With regard to a call option, there is a joint stock request with an investment which is less than what is needed for the stock itself.
2. The maximum loss of a buyer is predetermined, i.e. it is the same as the option price.
3. An option offers financial leverage.
4. An option provides opportunities for investors and presents a combination of risk and return which would otherwise be inaccessible.
5. Buying options are like buying insurance. The cost of the insurance is equal to the price of the put options.

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Fischer Black, Myron Scholes, and Robert Merton proposed the Black–Scholes model in 1970. This model was revolutionary in its impact on pricing financial derivatives whose underlying asset was a stock. Then it was possible to price financial derivatives by using the closed-form solution. This model had a great impact on the pricing method and hedging option risk. Moreover, it played a significant role in achieving financial engineering success in the $1980s$ and $1990s$.

The assumptions of the model above are as follows:

1. The stock price satisfies $\frac{ds}{s} = \mu dt + \sigma dz$, where z is a stochastic variable (Brownian motion). μ and σ are constants. μ is the expected return of the stock and σ is the stock volatility.
2. Securities are sold in the short-term, and the revenue is completely gained.
3. No transaction costs.
4. No dividend paid.
5. There are no risk-free arbitrage opportunities.
6. The securities trading is continuous.
7. The risk-free interest rate r is constant, and it is the same for all maturities.

This model was formulated as follows:

$$v_t + \frac{1}{2}\sigma^2 x^2 v_{xx} + rxv_x - rv = 0, \tag{1.1}$$

where $v(x,t)$ is the option price with the underlying asset x in time t. T is the maturity date, r is the rate of return and σ is the underlying asset velocity. Empirical evidences show that the assumptions of the Black–Scholes model are in contradiction with market realities. Therefore, the generalized model assumptions have been used in many cases. In order to make the Black–Scholes model more suitable for the market, some of its limitations have been omitted, and it was generalized. However, this model is nowadays only used as the basic one to define other models. For example, the stochastic volatility model [1, 2] was formulated by deleting the assumption (1), transaction cost model [3, 4] was formulated by deleting the assumption (3), the stochastic interest rate model [5] was formulated by deleting the assumption (7), and fractional Black–Scholes model [6–8] was formulated by substituting Brownian motion by fractional Brownian motion, and by deleting the assumption (2). Moreover, considering prices jump in the market, Merton proposed the Black–Scholes equation under jump-diffusion model [9]. Therefore, the differential equation (1.1) turned into the integro-differential equation. Generally, obtaining the analytical solution of the required integro-differential equation seems to be difficult or impossible, so different numerical approximations to the solution of such equations have been proposed. For instance, Briani et al. [10] used the explicit finite difference method for the above-mentioned Black–Scholes integro-differential equation. Cont and Voltchkova [11] proposed an explicit-implicit finite difference method using the notion of viscosity solution. Matache et al. [12] used discontinuous Galerkin method in time and discrete wavelet-Galerkin method in place. Furthermore, Matache et al. [13] solved a partial integro-differential equation (PIDE) using θ-method in time and a discrete wavelet in place. Working on numerical solutions of differential equations [14] and generalized Black–Scholes differential equations is still an interesting area for researches. For example, Patel and Mehra [15] proposed a compact finite difference method for pricing European and American options under jump-diffusion models. Also, Rambeerich and Pantelous [16] solved a system of PIDEs using Lagrange finite element techniques. For more information, we refer the reader to [17–19].

In this paper, we have used the so-called Tau method to introduce a new method for solving a matrix solution for the Black–Scholes equation with jump.

2. Black–Scholes equation with jump

We want to obtain a numerical solution for the following equation:

$$\begin{cases} V_\tau + \frac{1}{2}\sigma^2 s^2 V_{ss} + (r - \lambda\overline{k})sV_s - rV + \lambda\int_0^\infty V(s\eta,\tau)\tilde{\Gamma}_\delta(s\eta,\tau)\tilde{\Gamma}_\delta(\eta)d\eta - V = 0, \\ V(s,0) = (s-k)^+, \end{cases} \tag{2.1}$$

where

$$\tilde{\Gamma}_\delta(\eta) = \frac{1}{\sqrt{2\pi}\delta\eta} exp(-\frac{1}{2}(\frac{log\eta}{\delta})^2), \tag{2.2}$$

and

$$\overline{k} = \int_0^\infty (\eta - 1)\tilde{\Gamma}_\delta(\eta)d\eta. \tag{2.3}$$

With change of variables as follows:

$$x = \frac{1}{\delta}logs, \quad z = \frac{1}{\delta}log\eta, \quad t = T - \tau, \tag{2.4}$$

we define $v(x,t)$ by $V(s,\tau) = v(exp(x\delta), T-t)$. Hence, the equation (2.1) can be written as

$$\begin{cases} v_\tau = \frac{1}{2}\frac{\sigma^2}{\delta^2}v_{xx} + [\frac{(r-\lambda\overline{k})}{\delta} - \frac{\sigma^2}{2\delta}]v_x - (r+\lambda)v + \lambda\int_0^\infty v(x+z,t)\Gamma_\delta(z)dz = 0, \\ v(x,0) = (exp(x\delta) - k)^+. \end{cases} \tag{2.5}$$

If we define

$$a = \frac{1}{2}\frac{\sigma^2}{\delta^2}, \quad b = \frac{(r-\lambda\overline{k})}{\delta} - \frac{\sigma^2}{2\delta}, \quad d = -(r+\lambda), \tag{2.6}$$

then we have

$$\begin{cases} v_\tau = av_{xx} + bv_x + dv + \lambda\int_0^\infty v(x+z,t)\Gamma_\delta(z)dz, \\ v(x,0) = (exp(x\delta) - kexp(-rT))^+. \end{cases} \tag{2.7}$$

3. Tau method

In [20], it was proposed that the Tau method is based on three simple matrices:

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mu = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \iota = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1/2 & 0 & \dots \\ 0 & 0 & 0 & 1/3 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Lemma 3.1 *If* $y_n(x) = \underset{-}{a_n}\,\underset{-}{X}$ *with* $\underset{-}{a_n} = (a_0, a_1, \ldots, a_n, 0, 0, \ldots)$ *and* $\underset{-}{X} = (1, x, x^2, \ldots, x^n, \ldots)^T$, *then*

1. $\frac{d^m}{dx^m} y_n(x) = \underset{-}{a_n}\,\eta^m\,\underset{-}{X}$;
2. $x^m y_n(x) = \underset{-}{a_n}\,\mu^m\,\underset{-}{X}$;
3. $\int_0^x y_n(t)dt = \underset{-}{a_n}\,\iota\,\underset{-}{X}$.

In [21], the two-dimensional linear Fredholm integral equations of the second kind by using the operational Tau method is formulated as:

$$\underset{-}{X} = (1, x, x^2, \ldots, x^n, \ldots)^T, \tag{3.1}$$

$$\underset{-}{T} = (1, t, t^2, \ldots, t^n, \ldots)^T, \tag{3.2}$$

$$C = (c_{ij})_{n\times n} = \begin{pmatrix} c_{00} & c_{01} & \cdots & c_{0n} \\ c_{10} & c_{11} & \cdots & c_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n0} & c_{n1} & \cdots & c_{nn} \end{pmatrix},$$

and

$$\phi(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} c_{ij}x^i t^j = \underset{-}{X}^T\,C\,\underset{-}{T}.$$

Clearly, the partial integro-differential equation (2.7) has two variables, namely t and x. We will apply the two-dimensional Tau method to determine $v(x,t)$. The solution of (2.7) will be as follows:

$$v(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} c_{ij}x^i t^j = \underset{-}{X}^T\,C\,\underset{-}{T}. \tag{3.3}$$

Lemma 3.2 *If* $v(x,t) = \underset{-}{X}^T\,C\,\underset{-}{T}$ *where* $\underset{-}{X}$, C, *and* $\underset{-}{T}$ *are as the above, then:*

1. $v_x(x,t) = \underset{-}{X}\,\eta^T C\,\underset{-}{T}$;
2. $v_{xx}(x,t) = \underset{-}{X}\,\eta^{T^2} C\,\underset{-}{T}$;
3. $v_t(x,t) = \underset{-}{X}\,\eta C\,\underset{-}{T}$;
4. $\int_{-\infty}^{\infty} v(x+z,t)\Gamma_\delta(z)dz = \underset{-}{X}^T\,NC\,\underset{-}{T}$,

where $N = \sum_{i=0}^{n} \frac{\alpha_i}{i!}\eta^{T^i}$, $\quad \alpha_i = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^i exp(\frac{-1}{2}z^2)dz = \begin{cases} \frac{2^k}{\sqrt{\pi}}\Gamma(k+\frac{1}{2}), & i = 2k, \\ 0. & o.w. \end{cases}$

Proof Proof is clear. □

Theorem 3.3 *If we apply the Tau method on the Black–Scholes integro-differential equation, then we will obtain the following Sylvester equation:*

$$LC - C\eta = G.$$

Proof By using (a), (b), (c), and (d) from Lemma 3.2 in (2.7), we obtain:

$$a\,\underline{X}^T\,\eta^{T^2}C\,\underline{T} + b\,\underline{X}^T\,\eta^T C\,\underline{T} + d\,\underline{X}^T\,C\,\underline{T} - \underline{X}^T\,C\eta\,\underline{T} + \lambda\,\underline{X}^T\,NC\,\underline{T} = 0, \tag{3.4}$$

or

$$\underline{X}^T(a\eta^{T^2}C + b\eta^T C + dC - C\eta + \lambda NC)\,\underline{T} = 0.$$

Since $\underline{X}$ and $\underline{T}$ are bases, we have:

$$a\eta^{T^2}C + b\eta^T C + dC - C\eta + \lambda NC = 0,$$

and

$$(a\eta^{T^2} + b\eta^T + dI + \lambda N)C - C\eta = 0.$$

If

$$L = a\eta^{T^2} + b\eta^T + dI + \lambda N,$$

then

$$LC - C\eta = 0. \tag{3.5}$$

We define

$$g(x) = (exp(x\delta) - K)^+.$$

Now, $g(x)$ must be approximated by a polynomial of suitable degree. Therefore, we approximate $g(x)$ by the Taylor polynomial. Thus,

$$g(x) = \sum_{j=0}^{n} \frac{g(0)^(i)}{i!}x^i, \quad G = (g_i)_{n\times 1}, \quad g_i = \frac{g(0)^{(i)}}{i!},$$

and

$$T_1 = (1, T, T^2, ..., T^{n-1})^T.$$

We define T_2 and G_2 matrices as follows:

$$T_2 = (0, 0, 0, ..., T_1),$$

$$G_2 = (0, 0, 0, ..., G).$$

The initial value of $v(x, 0) = (exp(x\delta) - K)^+$ is rewritten as $CT_1 = G$, so

$$CT_2 = G_2. \tag{3.6}$$

By combining (2.5) and (2.6), we obtain:

$$LC - C(\eta - T_2) = G_2. \tag{3.7}$$

We define $\overline{\eta} = \eta - T_2$; thus,

$$LC - C\overline{\eta} = G_2, \tag{3.8}$$

and the proof is completed. □

By solving the Sylvester equation (3.8), we obtain $v(x,t)$ from (2.4).
Then $v(x,T) = \underline{X}^T\, C\, \underline{T_1}$ is the solution of (2.7).

4. Tau method with Hermite base

Since we are using the Gaussian random variable, according to [22], by using Hermite polynomials basis in the Tau method, we obtain option pricing under jump-diffusion models.
Hermite polynomials of order n are defined by

$$H_n(x) = \frac{(-1)^n}{\sqrt{n!}} exp(\frac{x^2}{2}) \frac{d}{dx^n} exp(\frac{-x^2}{2}), \quad n = 0, 1, 2, \ldots \tag{4.1}$$

These polynomials have many properties including:

a. $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad n \geq 1;$

b. $H_n^{'}(x) = 2xH_{n-1(x)}, \quad n \geq 1;$

c. $H_n^{''}(x) - 2xH_n'(x) + 2nH_n(x) = 0;$

d. $exp(-2t^2 + 2tx) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!};$

e. $\int_{-\infty}^{\infty} H_n(x)H_m(x)exp(-x^2)dx = 2^n\sqrt{\pi}n!\delta_{nm}.$

Lemma 4.1 *If* $\underline{H}(x) = (H_0, H_1, \ldots, H_{n-1})^T$, *where* $H_i, \quad (i = 1, 2, \cdots, n-1)$ *are Hermite polynomials then*

1. $\underline{H}'(x) = 2\eta\, \underline{H}(x);$

2. $\underline{H}''(x) = 2(x-1)2\eta\, \underline{H}(x).$

Proof

a. Since $H_i(x), \quad \{i = 0, 1, \cdots, n\}$, are Hermite polynomials, then from property (b) we have

$$\begin{aligned}
\underline{H}'(x) &= (H'_0, H'_1, \ldots, H'_{n-1})^T \\
&= (0, 2H_0, 4H_1, 6H_2, \ldots, 2(n-1)H_{n-2})^T \\
&= 2\begin{pmatrix} 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & \ldots \\ 0 & 2 & 0 & 0 & \ldots \\ 0 & 0 & 3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 0 \\ H_0 \\ H_1 \\ \vdots \\ H_{n-1} \end{pmatrix} \\
&= 2\eta\, \underline{H}(x).
\end{aligned}$$

b. By using property (c), and similar to (a). □

Lemma 4.2 *If X is an ordinary base, i.e. $\underline{X} = (1, x, x^2, \ldots, x^n)^T$ where $\underline{H}(x)$ is defined in the same way as the lemma above, then there exists an invertible lower triangular matrix U so that $\underline{H} = U\,\underline{X}$, and elements of U as the following:*

$$u_{ij} = 2u_{i-1,j-1} - 2(i-2)u_{i-2,j}.$$

Proof By the property (a), the proof is clear. □

Lemma 4.3 *If U and η are defined as before, then $U\eta = 2\eta U$.*

Proof We have

$$\underline{H}'(x) = (H'_0, H'_1, \ldots, H'_{n-1})^T = 2\eta\, \underline{H}(x) = 2\eta U\, \underline{X}, \tag{4.2}$$

on the other hand,

$$\underline{H}'(x) = (U\,\underline{X})' = U\,\underline{X}' = U\eta\,\underline{X}, \tag{4.3}$$

by equating (4.2) and (4.3), we obtain $2\eta U = U\eta$. □

Lemma 4.4 *If $v(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} d_{ij}H_i(x)H_j(t) = \underline{H}(x)^T D\, \underline{H}(t)$ where $D = (d_{ij})_{n\times n}$, then*

1. $v_x(x,t) = \underline{H}(x)^T 2\eta^T D\, \underline{H}(t)$;
2. $v_{xx}(x,t) = \underline{H}(x)^T 4\eta^{T^2} D\, \underline{H}(t)$;
3. $v_t(x,t) = \underline{H}(x)^T D 2\eta\, \underline{H}(t)$;
4. $\int_{-\infty}^{\infty} v(x+z,t)\Gamma_\delta(z)dz = \underline{H}(x)^T \overline{N} D\, \underline{H}(t)$.

where

$$\overline{N} = U^{-T}NU, \quad N = \sum_{i=0}^{n} \frac{\alpha_i}{i!}\eta^{T^i},$$

$$\alpha_i = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^i exp(-\frac{1}{2}z^2)dz = \begin{cases} \frac{2^k}{\sqrt{\pi}}\Gamma(k+\frac{1}{2}), & i = 2k, \\ 0, & o.w. \end{cases}$$

Proof By Lemmas 3.2 and 4.3, it is clear. □

By applying the Tau method with Hermite basis on the Black–Scholes integro-differential equation, we will obtain the following Sylvester equation

$$\overline{LC} - 2\overline{C}\overline{\eta} = G,$$

where

$$\overline{L} = 4a\eta^{T^2} + 2b\eta^T + dI + \lambda\overline{N}.$$

5. Convergence analysis

In this section, we analyze the estimating error of the Tau method for the integro-differential equations. Let us first introduce some notations. We set $\Lambda = \mathbf{R}$ and $\partial_x^k v = \frac{\partial^k v}{(\partial x)^k}$.

Suppose N, r be any positive and nonnegative integer, respectively.

$$\mathcal{P}_N = \{\sum_{j=0}^{N} a_j x^j | \quad a_0, a_1, \cdots, a_N \in \mathbf{R}\},$$

$$\mathbf{H}_\omega^r(\Lambda) = \left\{ v/ \quad \partial_x^k v \in L_\omega^2(\Lambda), \quad 0 \le k \le r \right\},$$

where $w(x)$ is a weight function and

$$\mathcal{H}_N = span < H_0(x), H_1(x), ..., H_N(x) > .$$

$|v|_{\mathbf{H}_\omega^r(\Lambda)} = \|\partial_x^r v\|_{\mathbf{L}_\omega^2(\Lambda)}$ and $\|v\|_{\mathbf{H}_\omega^r(\Lambda)} = (\sum_{k=0}^{r} |v|^2_{\mathbf{H}_\omega^r(\Lambda)})^{\frac{1}{2}}$ are the seminorm and the norm of $\mathbf{H}_\omega^r(\Lambda)$, respectively. Also, (u,v) and $(u,v)_p$ are the inner product of $L^2(\Lambda)$ and $L^P(\Lambda)$. We define the space $\mathbf{H}_\omega^m(\Lambda)$ with the norm $\|v\|_{m,\omega} = \|v\|_{\mathbf{H}_\omega^m(\Lambda)}$ where $m \ge 0$ is any real number [23]. We first present some inequalities which will be used later.

Lemma 5.1 *[24] For any $\varphi \in \mathcal{P}_N$,*

$$|\varphi|_{1,\omega} \le \sqrt{2N}\|\varphi\|_\omega.$$

Lemma 5.2 *[24] For any $v \in H_\omega^2(\Lambda)$,*

$$\|xv\|_\omega \le \|v\|_{1,\omega}.$$

Lemma 5.3 *[24] For any $\varphi \in \mathcal{H}_N$ and nonnegative integer m,*

$$\|\partial_x^m \varphi\| \le (2N+1)^{\frac{m}{2}}\|\varphi\|.$$

The lemma above can be extended to noninteger cases, indeed:

Lemma 5.4 *[24] For any $\varphi \in \mathcal{H}_N$ and $r \geq 0$, we have:*

$$\|\varphi\|_r \leq cN^{\frac{r}{2}}\|\varphi\|.$$

We are now in a position to introduce several orthogonal projection operators. The $L^2(\Lambda)$-orthogonal projection $\mathbf{P}_N : L^2(\Lambda) \longrightarrow \mathcal{H}_N$ is a mapping such that for any $v \in L^2(\Lambda)$,

$$(\mathbf{P}_N v - v, \varphi) = 0, \quad \forall \varphi \in \mathcal{H}_N,$$

or equivalently,

$$\mathbf{P}_N v(x) = \sum_{k=0}^{N} v_k(x) H_k(x).$$

In order to obtain an optimal error estimation in the Tau method, we need the $H^2_\omega(\Lambda)$-orthogonal projection $\mathbf{P}^2_N : H^2_\omega(\Lambda) \longrightarrow \mathcal{P}_N$. That is, for any $v \in H^2_\omega(\Lambda)$,

$$(\partial^2_x(v - P^2_N v), \partial^2_x \varphi)_\omega = 0, \quad \forall \varphi \in \mathcal{P}_N. \tag{5.1}$$

Lemma 5.5 *For any x, y,*

$$(x, y) \leq \|x\|^2 + \frac{1}{4}\|y\|^2.$$

We can also easily prove the following lemma.

Lemma 5.6 *If $\omega = e^{-x^2}$ then*

1. $(\partial^2_x \varphi, \varphi)_\omega = -\|\varphi\|^2_{1,\omega} + 2\|x\varphi\|^2_\omega \leq \|\varphi\|^2_{1,\omega}$;
2. $(\partial_x \varphi, \varphi)_\omega = 0$;
3. $\|\varphi\| \leq \sqrt{\pi}\|\varphi\|_{\omega^{-1}}$;
4. $\|\varphi\|_\omega \leq \sqrt{\pi}\|\varphi\|^2$.

Theorem 5.7 *[24] For any $v \in H^r_\omega(\Lambda)$ and $r \geq 0$,*

$$\|v - P^2_N v\| \leq cN^{\frac{-r}{2}}\|v\|_{r,\omega}.$$

We first consider an estimation error for the following equation [25].

$$\begin{cases} -\partial^2_x v + \lambda^2 v = g(x), & x \in \Lambda, \lambda \in R, \\ \lim_{x\to-\infty} v(x) = d_1, & \lim_{x\to\infty} v(x) = d_2, \end{cases} \tag{5.2}$$

where d_1, d_2 are constants. The Tau method with Hermite bases for (5.2) is to find $v_N \in \mathcal{P}_N$ satisfying the boundary conditions such that for any $\varphi \in \mathcal{P}_{N-2}$,

$$-(\partial^2_x v_N, \varphi) + \lambda^2(v_N, \varphi) = (g, \varphi). \tag{5.3}$$

We next give an error estimate of the scheme (5.2). Assume that the exact solution $v(x)$ is smooth enough. Let $\eta = v - v^*$ and $e = v_N - v^*$, where $v^* = P_N^2 v$. From (5.2) and (5.3), we have

$$-(\partial_x^2 e, \varphi) + \lambda^2(e, \varphi) = \lambda^2(\eta, \varphi). \tag{5.4}$$

According to (5.1), we know $(\partial_x^2 \eta, \varphi) = 0$ for all $\varphi \in \mathcal{P}_{N-2}$.
We take $\varphi = e$, from the above lemma and theorem we get

$$\|e\|_\omega^2 \leq \frac{\lambda^2 c^2 N^{-1}}{4(\lambda^2 - 3 - 3(2N+1))} \|v\|_{2,\omega}^2,$$

and

$$\|v - v_N\|_\omega \leq C N^{-1} \|v\|_{2,\omega}.$$

Now we want to analyze the estimating error for the follow integro-differential equations.

$$\begin{cases} v_t - a v_{xx} + b v_x + c v + \lambda \int_0^\infty v(x+z,t)\Gamma_\delta(z)dz = 0, \\ v(x,0) = (e^{x\delta} - k e^{-rT})^+. \end{cases} \tag{5.5}$$

Theorem 5.8 *Let v and v_N be the exact and approximation solutions to (5.5), respectively. For all $r \geq 2$,*

$$\|e\|_\omega^2 \leq (C_1 N^{-r} + C_2 N^{2-r}) \|v\|_{r,\omega}. \tag{5.6}$$

Proof We have

$$(\partial_t v_N, \varphi) - (\partial_x^2 v_N, \varphi) + b(\partial_x v_N, \varphi) - c(v_N, \varphi) + \lambda\left(\int_{-\infty}^\infty v_N(x+z,t)\Gamma_\delta(z)\eta dz, \varphi\right) = 0. \tag{5.7}$$

Let $\eta = v - v^*$ and $e = v_N - v^*$, where $v^* = P_N^2 v$.
From (5.5) and (5.7), for $\varphi = e$ we conclude that

$$\begin{aligned}(\partial_t \eta, e) + a(\partial_x^2 \eta, e) + b(\partial_x \eta, e) + c(\eta, e) + \lambda\left(\int_{-\infty}^\infty (\eta - e)\Gamma_\delta(z)dz, e\right) - (\partial_t e, e) \\ + a(\partial_x^2 e, e) + b(\partial_x e, e) + c(e, e) = 0.\end{aligned}$$

By Lemma 5.6 (1,2), we have

$$\begin{aligned}(\partial_t e, e) - a(\partial_x^2 e, e) - c(e, e) + \lambda\left(\int_{-\infty}^\infty e\Gamma_\delta(z)dz, e\right) = (\partial_t \eta, e) + b(\partial_x \eta, e) + c(\eta, e) \\ + \lambda\left(\int_{-\infty}^\infty \eta\Gamma_\delta(z)dz, e\right).\end{aligned} \tag{5.8}$$

By using Lemmas 5.1–5.6, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{d_t}\|e\|_\omega^2 + a\|e\|_{1,\omega}^2 - 2a\|xe\|_\omega^2 - c\|e\|_\omega^2 \leq \|\frac{\partial}{\partial t}\eta\|_\omega^2 + \frac{1}{4}\|e\|_\omega^2 + b\|\frac{\partial}{\partial x}\eta\|_\omega^2 + \frac{b}{4}\|e\|_\omega^2 \\ + c\|\eta\|_\omega^2 + \frac{c}{4}\|e\|_\omega^2 + \lambda\|\int_{-\infty}^\infty \eta\Gamma_\delta(z)dz\|_\omega^2 + \frac{\lambda}{4}\|e\|_\omega^2,\end{aligned} \tag{5.9}$$

and

$$-(\frac{1+b+c+\lambda}{4})\|e\|_{\omega}^{2}-a\|e\|_{1,\omega}^{2}+\frac{1}{2}\frac{d}{d_t}\|e\|_{\omega}^{2}c_1N^{-r}\|v\|_{r,\omega}^{2}\leq c_2N^{2-r}\|v\|_{r,\omega}^{2}+\|\frac{\partial}{\partial t}\eta\|_{\omega}^{2}$$
$$+\lambda\|\int_{-\infty}^{\infty}\eta\Gamma_{\delta}(z)dz\|_{\omega}^{2}.$$

By integrating the inequalities, we find that

$$\frac{1}{2}\|e\|_{\omega}^{2}-a\int_0^t\|e\|_{1,\omega}^{2}dt-c_3\int_0^t\|e\|_{\omega}^{2}dt\leq c_1N^{-r}\int_0^t\|v\|_{r,\omega}^{2}+c_2N^{2-r}\int_0^t\|v\|_{r,\omega}^{2}$$
$$+\int_0^t\|\frac{\partial}{\partial t}\eta\|_{\omega}^{2}+\lambda\int_0^t\|\int_{-\infty}^{\infty}\eta\Gamma_{\delta}(z)dz\|_{\omega}^{2}. \tag{5.10}$$

Now by applying the mean value theorem for integrals, we have

$$\frac{1}{2}\|e\|_{\omega}^{2}-at\|e(x,t_0)\|_{1,\omega}^{2}-c_3t\|e(x,t_0)\|_{\omega}^{2}dt\leq c_1N^{-r}t\|v\|_{r,\omega}^{2}+c_2N^{2-r}t\|v\|_{r,\omega}^{2}. \tag{5.11}$$

Thus,

$$\|e\|_{\omega}^{2}\leq(C_1N^{-r}+C_2N^{2-r})\|v\|_{r,\omega}.$$

□

By the above theorem, we conclude that for any fixed t, $\|e\|\to 0$ when N goes to infinity.

6. Numerical examples

The numerical experiments performed in this chapter are related to the option pricing problems described in previous sections.

Example 6.1 *Here we give an example to show the accuracy of the solutions obtained from our proposed method for pricing European options under jump-diffusion models. We solve the following system of equation.*

$$\begin{cases} v_{\tau}=av_{xx}+bv_x+dv+\lambda\int_0^{\infty}v(x+z,t)\Gamma_{\delta}(z)dz,\\ v(x,0)=(e^{x\delta}-ke^{-rT})^{+}.\end{cases} \tag{6.1}$$

The experiments have been performed on an Intel Core 1.7 GHz i5 Computer. We use"lyap"function in MATLAB for the numerical solution of Sylvester equations. We have used an example of [10]. Formula parameters are given in Table 1. Figure 1a shows convergence of the Tau method for this example.

In Table 2, we mention solutions of the Tau method with an ordinary base for European call options with jump-diffusion process and corresponding computational CPU time in seconds. In Table 3, we have the results for the Tau method with Hermite base. In [10], this example solved by an explicit finite difference method and for n = 1024 they obtained call value = 13.2230.

Example 6.2 *We present the following example from [22]. Parameters of the corresponding formula are given in Table 4. In Table 5, we have the results for the Tau method with an ordinary base. Figure 1b shows convergence of the Tau method for this example.*

Example 6.3 *We also show the convergence behavior of equation (6.1) for the parameters of Tables 6 and 7 in the Figures 1c and 1d, respectively. We also have the call values associated with Table 6 in Table 8.*

Table 1. Parameters of the formula associated with Example 6.1.

Interest rate	r	0.05
Volatility	σ	0.2
Intensity of jump	λ	0.1
Standard deviation of jump size	δ	0.8
Expiration date	T	1
Strike price	E	100
Present asset price	$S(0)$	100
Expected return	μ	0

Table 2. Call values associated with Table 1 for ordinary base.

n	14	16	18	20	22
Call value	12.5714	13.1634	13.1659	13.1901	13.2019
$cpu(s)$	0.8988	0.9207	0.9437	0.9797	0.9951

Table 3. Call values associated with Table 1 for Hermite base.

n	14	16	18	20	22
Call value	12.6815	13.0695	13.1927	13.2094	13.2102
$cpu(s)$	0.96	0.98	0.999	1	1.025

Table 4. Parameters of the formula associated with Example 6.2.

Interest rate	r	0.1
Volatility	σ	$\sqrt{0.05}$
Intensity of jump	λ	1
Standard deviation of jump size	δ	$\sqrt{0.05}$
Expiration date	T	1/2
Strike price	E	35
Present asset price	$S(0)$	38
Expected return	μ	-0.025

Table 5. Call values associated with Table 6.

n	14	16	20	Exact value
Call value	6.8147	6.8146	6.8130	6.8066
$cpu(sec)$	0.8978	0.9295	0.9786	
Error	0.0081	0.008	0.0064	

7. Conclusion

In this paper, we proposed the two-dimensional Tau method with two bases (ordinary and Hermitian) for the solution of Black–Scholes integro-differential equations. We proved the convergence of the method and

Table 6. Parameters of the formula associated with Example 6.3.

Interest rate	r	0.05
Volatility	σ	0.03
Intensity of jump	λ	1
Standard deviation of jump size	δ	0.1
Expiration date	T	0.5
Strike price	E	40
Present asset price	$S(0)$	40
Expected return	μ	0.01

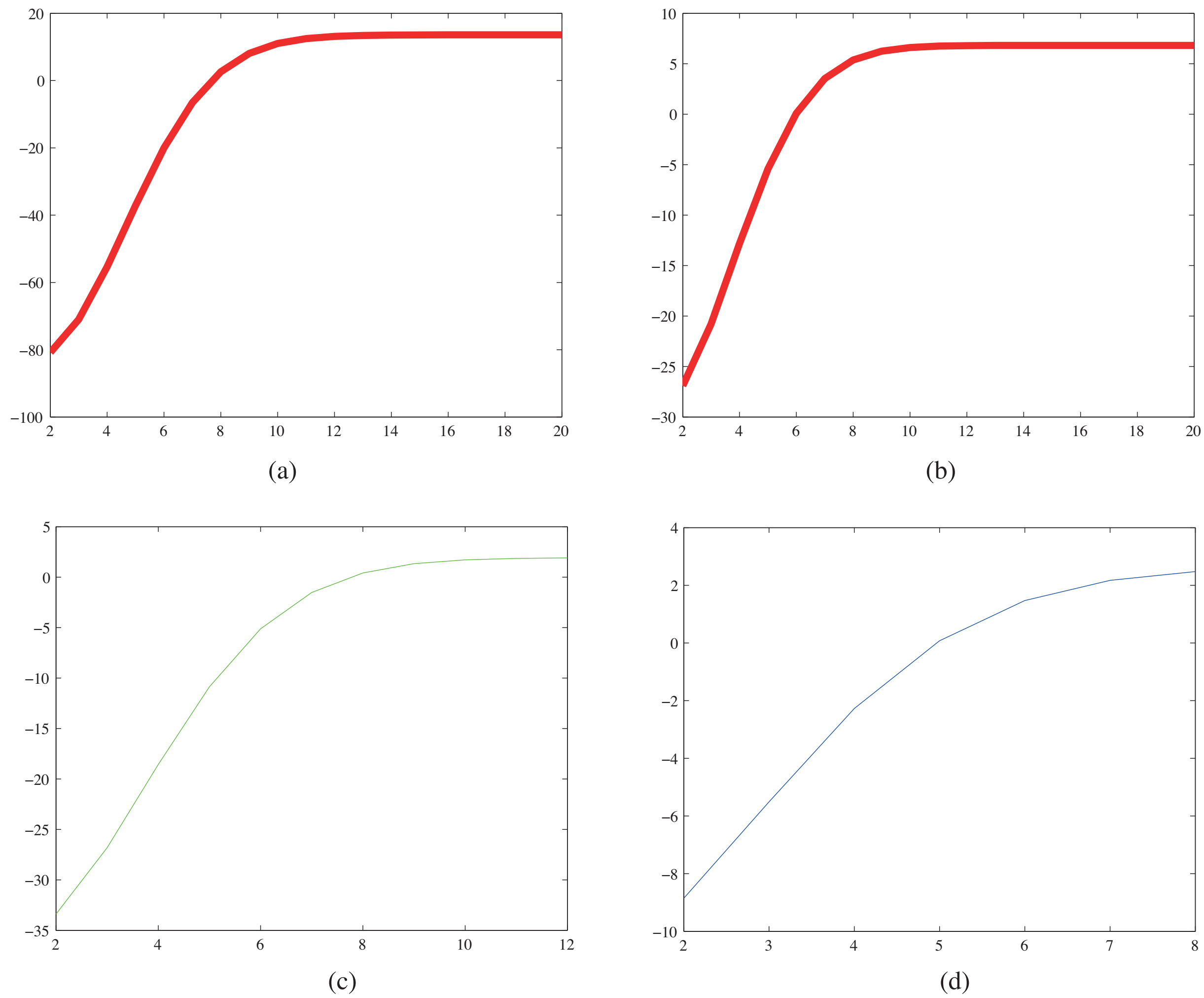

Figure. a) Convergence of the Tau method for Example 6.1, b) Convergence of the Tau method for Example 6.2, c) Convergence of the Tau method for equation (6.1) with parameters in Table 6, d) Convergence of the Tau method for equation (6.1) with parameters in Table 7.

obtained the error estimates in weighted L^2 and uniform norms of the approximated solution. These results were confirmed by some numerical examples. We compared our numerical results with exact values and showed

Table 7. Parameters of the formula associated with Example 6.3.

Interest rate	r	0.05
Volatility	σ	0.03
Intensity of jump	λ	1
Standard deviation of jump size	δ	0.03
Expiration date	T	2
Strike price	E	15
Present asset price	$S(0)$	15
Expected return	μ	0.02

Table 8. Call values associated with Table 6.

n	8	9	10	11	12	13	14	15	16	17	18
Call value	0.3206	1.1912	1.5508	1.6846	1.7298	1.7439	1.7479	1.7490	1.7492	1.7493	1.7493

that the Tau method is efficient in both accuracy and CPU time. Also, we can apply the method used in this paper to various problems arising in finance.

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